International Review of Fuzzy Mathematics
Vol. 3 No. 2 (December, 2018)

Received: 15rd April 2017 Revised: 10th May 2017 Accepted: 17th January 2018

Jin Han Park & Jin Keun Park

GENERALIZED FUZZY SEMI-CLOSED SETS AND GENERALIZATIONS OF FUZZY CONTINUOUS FUNCTIONS

ABSTRACT: In this paper we de.ne and study another various generalizations of fuzzy continuous functions by using the concept of generalized fuzzy semi closed sets. A comparative study regarding the mutual inter-relations among these functions along with those functions obtained in [4] is made. Finally, we have introduced and studied the notions of gfsconnectedness, gfs-extremally disconnectedness and gfs-compactness.

Key Words: *Fuzzy topology; generalized fuzzy semi closed set; generalized fuzzy semi continuous functions; gfs-connected set; gfs-extremally disconnected space; gfs-compact space.*

2000 Mathematics Subject Classification: *54C08, 54C10*

1. INTRODUCTION

Fuzzy topological space (in short, fts) and fuzzy continuity in fts's were introduced by Chang [5] early in 1968. In 1981, Azad [1] introduced and studied the concept of fuzzy semi open sets which is weaker than fuzzy open sets and used it to define fuzzy semi continuity in fts's. Recently, using the concept of generalized fuzzy closed sets, Balasubramanian and Sundaram [4] have introduced certain types of near-fuzzy continuous functions between fuzzy topological spaces, i.e., generalized fuzzy continuous, fuzzy *gc*-irresolute, strongly *gf*-continuous and perfectly *gf*-continuous functions etc. They also introduced the notions of *gf*-connectedness, *gf*-extremally disconnectedness and *gf*-compactness and studied properties of those notions under above-mentioned functions.

In this paper, we study another generalizations of fuzzy continuous functions and their applications. Section 2 is devoted to generalized fuzzy semi closed sets and study their properties. In Section 3 we introduce generalized fuzzy semi continuous functions and their properties by using generalized fuzzy semi closure *gs*-Cl. In Section 4 we introduce fuzzy *gsc*-irresolute. functions and study their properties, whereas in Section 5 we introduce and study strongly *gfs*-continuous and perfectly *gfs*-continuous functions and investigate inter-relations among these functions and those functions de.ned in [3]. In Sections 6 and 7, using the concept of generalized fuzzy semi closed (open) set, we introduce and study the notions of *gfs*-connectedness, *gfs*-extremally disconnectedness and *gfs*-compactness, respectively.

2. GENERALIZED FUZZY SEMI CLOSED SETS IN FUZZY TOPOLOGY

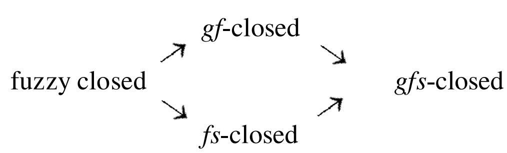
Definition 2.1 A fuzzy set A in a fts X is said to be

(i) generalized fuzzy closed [4] (in short, *gf*-closed) if $\mathrm{Cl}(A) \le U$ whenever $A \le U$ U and U is fuzzy open,

(ii) fuzzy semi-closed [1] (in short, *fs*-closed) if $A \ge \mathrm{Int}(\mathrm{Cl}(A))$.

Definition 2.2 A fuzzy set A in a fts X is called generalized fuzzy semi-closed (in short, *gfs*-closed) if $s\mathrm{Cl}(A) \le U$ whenever $A \le U$ and U is fuzzy open. A fuzzy set A is called generalized fuzzy semi-open (in short, *gfs*-open) if its complement $1 - A$ is *gfs*-closed.

From De.nitions 2.1 and 2.2, we obtain the following diagram:



The reverse implications need not be true as seen in the following examples. Also, Example 2.4 shows that *fs*-closed (resp. *fs*-open) and *gs*-closed (resp. *gfs*-open) sets are independent each other.

Example 2.3 Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A_1, A_2, A_1 \wedge A_2, A_1 \vee A_2\}$, where $A_1(a) = 1$, $A_1(b) = 0.2$; $A_2(a) = 0.3$, $A_2(b) = 0.7$. Define fuzzy sets A_3 as follows: $A_3(a) = 0.5$ and $A_3(b) = 0.7$. Then we have $A3$ is a *gfs*-closed set but not *gf*-closed in (X, τ).

Example 2.4 Let (X, τ) be a fts given in Example 2.3. Define fuzzy sets A_3 and A_4 in X as follow: $A_3(a) = 0.5$, $A_3(b) = 0.7$; $A_4(a) = 0$, $A_4(b) = 1$. Then we have A_3 is *fs*-closed set but not *gf*-closed in (X, τ) and A_4 is a *gf*-closed set but not *fs*-closed in (X, τ).

In general, the intersection and union of two *gfs*-closed (resp. *gfs*-open) sets are not *gfs*-closed (resp. *gfs*-open) as following examples show.

Example 2.5 Let $X = \{a, b, c\}$ and $\tau = \{0_X, 1_X, A_1\}$, where $A_1(a) = 1$, $A_1(b) = 0$, $A_1(c) = 0$. Define fuzzy sets A_2 and A_3 in X as follows: $A_2(a) = 1$, $A_2(b) = 1$, $A_2(c) = 0$; $A_3(a) = 1$, $A_3(b) = 0$, $A_3(c) = 1$. Then we have A_2 and A_3 are *gfs*-closed sets but $A_2 \wedge A_3$ is not *gfs*-closed in (X, τ).

Example 2.6 Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A_1, A_2, A_1 \wedge A_2, A_1 \vee A_2\}$, where $A_1(a) = 1$, $A_1(b) = 0.2$; $A_2(a) = 0.3$, $A_2(b) = 0.7$. Define fuzzy sets A_3 and A_4 in X as follows: $A_3(a) = 0.2$, $A_3(b) = 0.5$; $A_4(a) = 0.7$, $A_4(b) = 0.3$. Then we have A_3 and A_4 are *gfs*-closed sets but $A_2 \vee A_3$ is not *gfs*-closed in (X, τ).

Theorem 2.7 *A fuzzy set A in X is gfs-open if and only if* $F \leq s\mathrm{Int}(A)$ *whenever F is fuzzy* closed set in X and F ≤ A.

Proof Let A be a *gfs*-open set in X and F be a fuzzy closed set such that $F \leq A$. Then $1 - F$ is fuzzy open and $1 - A \leq 1 - F$. Since $1 - A$ is *gfs*-closed, $1 - s\mathrm{Int}(A) = s\mathrm{Cl}(1 - A) \leq 1 - F$ which implies $F \leq s\mathrm{Int}(A)$.

Conversely, Suppose that A is a fuzzy set such that $F \leq s\mathrm{Int}(A)$ whenever F is a fuzzy closed set and $F \leq A$. We want to show that $1 - A$ is a *gfs*-closed set. So, let $1 - A \leq U$ where U is fuzzy open. Then since $1 - A \leq U$, $1 - U \leq A$. By assumption we have $1 - U \leq s\mathrm{Int}(A)$, i.e. $1 - s\mathrm{Int}(A) \leq U$. Hence $s\mathrm{Cl}\,(1 - A) \leq U$, which implies that $1 - A$ is a *gfs*-closed set.

Theorem 2.8 *Let A, B be fuzzy sets in a fts X. Then the following hold:*

(i) *If A is gfs-closed in X and* $A \leq B \leq s\mathrm{Cl}(A)$*, then B is gfs-closed.*

(ii) *If A is gfs-open in X and* $s\mathrm{Int}(A) \leq B \leq A$*, then B is gfs-open.*

Proof (i): Let U be fuzzy open in X and $B \leq U$. Since $A \leq B \leq U$ and A is *gfs* closed, $s\mathrm{Cl}(A) \leq U$. But $s\mathrm{Cl}(B) \leq s\mathrm{Cl}(A)$ since $s\mathrm{Cl}(B) \leq s\mathrm{Cl}(s\mathrm{Cl}(A)) = s\mathrm{Cl}(A)$ and thus $s\mathrm{Cl}(B) \leq U$. Hence B is *gfs*-closed.

(ii): It follows from (i) and Theorem 2.7.

Definition 2.9 A function $f : X \to Y$ is said to be

(i) fuzzy continuous [5] (in short, *f*-continuous) if the inverse image of every fuzzy closed set in Y is fuzzy closed in X,

(ii) strongly fuzzy semi closed (in short, strongly *fs*-closed) if the image of every *fs*-closed set in X is fuzzy closed in Y.

Theorem 2.10 *If A is a gfs-closed set in X and if $f : X \to Y$ is f-continuous and strongly fs-closed, then $f(A)$ is gfs-closed in Y.*

Proof Let B be a fuzzy open set in Y such that $f(A) \leq B$. Then $A \leq f^{-1}(B)$. Since A is *gfs*-closed and $f^{-1}(B)$ is fuzzy open, $s\text{Cl}(A) \leq f^{-1}(B)$, i.e. $f(s\text{Cl}(A)) \leq B$. Also since f is strongly *fs*-closed, $f(s\text{Cl}(A))$ is *fs*-closed and thus $s\text{Cl}(f(A)) \leq B$. Hence $f(A)$ is *gfs*-closed.

However, under strongly *fs*-closed and *f*-continuous function, the image of *gfs*-open set need not be *gfs*-open.

Example 2.11 Let $X = \{a\}$, $Y = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A\}$ and $\tau_2 = \{0_Y, 1_Y, B_1, B_2\}$ where A, B_1 and B_2 are fuzzy sets in X (and γ) defined by $A(a) = 0.5$; $B_1(a) = 1$, $B_1(b) = 0.5$, $B_1(c) = 1$ and $B_2(a) = 1$, $B_2(b) = 0$, $B_2(c) = 1$. Define a function $f : (X, \tau_1) \to (Y, \tau_2)$ by $f(a) = b$. Clearly, f is *f*-continuous and strongly fuzzy semi-closed. Now we shall show that image of *gfs*-open set is not *gf*-open. Consider a fuzzy set A_1 in X defined by $A_1(a) = 0.8$. Then A_1 is *gfs*-open in (X, τ_1) but $f(A_1)$ is not *gfs*-open in (Y, τ_2).

3. GENERALIZED FUZZY SEMI-CONTINUOUS FUNCTIONS AND THEIR PROPERTIES

Definition 3.1 [4] A function $f : X \to Y$ is called generalized fuzzy continuous (in short, *gf*-continuous) if the inverse image of every fuzzy closed set in Y is *gf*-closed in X.

Definition 3.2 A function $f : X \to Y$ is called generalized fuzzy semi-continuous (in short, *gfs*-continuous) if the inverse image of every fuzzy closed set in Y is *gfs*-closed in X.

Every *f*-continuous function is *gf*-continuous and every *gf*-continuous function is *gfs*continuous. However, the converses are not true as Example 3.3 in [4] and the following example show.

Example 3.3 Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$, where $A_1(a) = 0.4$, $A_1(b) = 0.7$, $A_1(c) = 0.3$; $A_2(a) = 0.7$, $A_2(b) = 0.7$, $A_2(c) = 0.8$. Let $f : (X, \tau_1) \to (X, \tau_2)$ be the identity function. Then f is *gfs*-continuous but not *gf*-continuous since $f^{-1}(1 - A_2)$ is not *gf*-closed in (X, τ_1) for fuzzy closed set $1 - A_2$ in (X, τ_2).

Theorem 3.4 *For a function $f : X \to Y$, the following are equivalent:*

(i) *f is gfs-continuous.*

(ii) *The inverse image of each fuzzy open set in Y is gfs-open in X.*

To obtain properties of *gfs*-continuity, using *gfs*-closed sets, we introduce the notion of generalized fuzzy semi closure operator *gs*-Cl. So, we define the *gs*-Cl(A) for any fuzzy set A in a fts X as follows:

$$gs\text{-Cl}(A) = \wedge \{B \mid A \leq B \text{ and } B \text{ is } gfs\text{-closed}\}.$$

Theorem 3.5 *If $f : X \to Y$ is gfs-continuous, then $f(gs\text{-Cl}(A)) \leq \text{Cl}(f(A))$ for any fuzzy set A in X.*

Proof Let A be any fuzzy set in X. Then $A \leq f^{-1}(f(A)) \leq f^{-1}(\text{Cl}(A))$. Since f is *gfs* continuous, $gs\text{-Cl}(A) \leq f^{-1}(\text{Cl}(A))$ and hence $f(gs\text{-Cl}(A)) \leq \text{Cl}(f(A))$.

The converse of Theorem 3.5 need not be true as seen from the following example.

Example 3.6 Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$ where A_1 and A_2 are fuzzy sets in X defined by $A_1(a) = 1$, $A_1(b) = A_1(c) = 0$; $A_2(a) = A_2(b) = 1$, $A_2(c) = 1$. Consider a function $f : (X, \tau_1) \to (X, \tau_2)$ defined by $f(a) = c$, $f(b) = b$, $f(c) = a$. Then for fuzzy set A_1, $f(gs\text{-Cl}(A_1)) \leq \text{Cl}(f(A_1))$, but f is not *gfs*-continuous. (Since $1 - A_2$ is a fuzzy closed in (X, τ_2) but $f^{-1}(1 - A_2)$ is not *gfs*-closed in (X, τ_1)).

Definition 3.7 [4] A fts X is called fuzzy $T_{1/2}$ if every *gf*-closed set in X is fuzzy closed.

Theorem 3.8 *Let $f : X \to Y$ be gfs-continuous and $g : Y \to Z$ be gf-continuous. If Y is fuzzy $T_{1/2}$, then the composition $g \circ f : X \to Z$ is gfs-continuous.*

In Theorem 3.8, the condition that Y is a fuzzy $T_{1/2}$ space can not be omitted as shown in our next example.

Example 3.9 Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$, $\tau_2 = \{0_X, 1_X, A_2, A_3\}$ and $\tau_3 = \{0_X, 1_X, A_4\}$ where A_1, A_2, A_3 and A_4 are fuzzy sets in X defined as follows:

$$A_1(a) = 1,\ A_1(b) = 1,\ A_1(c) = 0; \quad A_2(a) = 0,\ A_2(b) = 1,\ A_2(c) = 1;$$

$$A_3(a) = 1,\ A_3(b) = 0,\ A_3(c) = 0; \quad A_4(a) = 1,\ A_4(b) = 0,\ A_4(c) = 1.$$

Let $f : (X, \tau_1) \to (X, \tau_2)$ be a function defined by $f(a) = f(c) = c$, $f(b) = b$ and $g : (X, \tau_2) \to (X, \tau_3)$ be the identity. Then f and g are *gfs*-continuous but $g \circ f$ is not *gfs*-continuous; for $1 - A_4$ is fuzzy closed in (X, τ_3), $f^{-1}(g^{-1}(1 - A_4))$ is not *gfs*-closed in (X, τ_1). Further (X, τ_2) is not fuzzy $T_{1/2}$.

Theorem 3.10 *If $f : X \to Y$ is gfs-continuous and $g : Y \to Z$ is fuzzy continuous, then the composition $g \circ f : X \to Z$ is gfs-continuous.*

The following example shows that the composition of any two *gfs*-continuous functions need not be *gfs*-continuous.

Example 3.11 Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$, $\tau_2 = \{0_X, 1_X, A_2\}$ and $\tau_3 = \{0_X, 1_X, A_3\}$ where A_1, A_2 and A_3 are fuzzy sets in X defined as follows:

$$A_1(a) = 0.7,\ A_1(b) = 0.6,\ A_1(c) = 0.7;$$

$$A_2(a) = 0.2,\ A_2(b) = 0.2,\ A_2(c) = 0.2;$$

$$A_3(a) = 0.3,\ A_3(b) = 0.4,\ A_3(c) = 0.3.$$

Let $f : (X, \tau_1) \to (X, \tau_2)$ be a function defined by $f(a) = f(b) = f(c) = b$ and $g : (X, \tau_2) \to (X, \tau_3)$ be the identity function. Then f and g are *gfs*-continuous but $g \circ f$ is not *gfs* continuous; for $1 - A_3$ is fuzzy closed in (X, τ_3), $f^{-1}(g^{-1}(1 - A_3))$ is not *gfs*-closed in (X, τ_1). Hence $g \circ f$ is not *gfs*-continuous.

4. FUZZY *GSC*-IRRESOLUTE FUNCTIONS AND THEIR PROPERTIES

Definition 4.1 [3] A function $f : X \to Y$ is called fuzzy *gc*-irresolute if the inverse image of every *gf*-closed set in Y is *gf*-closed in X.

Definition 4.2 A function $f : X \to Y$ is called fuzzy *gsc*-irresolute if the inverse image of every *gfs*-closed set in Y is *gfs*-closed in X.

Every fuzzy *gsc*-irresolute function is *gfs*-continuous but the converse is not true (see Example 4.3).

Example 4.3 Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3\}$ and $\tau_2 = \{0_X, 1_X, A_1\}$, where A_1, A_2 and A_3 are fuzzy sets in X defined as follows:

$$A_1(a) = 1, \quad A_1(b) = 0, \quad A_1(c) = 0;$$

$$A_2(a) = 0, \quad A_2(b) = 0, \quad A_2(c) = 1;$$

$$A_3(a) = 1, \quad A_3(b) = 0, \quad A_3(c) = 1.$$

Let $f : (X, \tau_1) \to (X, \tau_2)$ be the function defined by $f(a) = a, f(b) = b, f(c) = a$. Then f is *gfs*-continuous but not fuzzy *gsc*-irresolute; for a fuzzy set A_3 is *gfs*-closed in (X, τ_2), but $f^{-1}(A_3) = A_3$ is not *gfs*-closed in (X, τ_1).

The following Examples 4.4 and 4.5 show that fuzzy *gc*-irresolute function and fuzzy *gsc*-irresolute function are, in general, independent.

Example 4.4 Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1, A_2 . A_3\}$ and $\tau_2 = \{0_X, 1_X, A_1\}$ where A_1, A_2 and A_3 are fuzzy sets in X defined as follows:

$$A_1(a) = 1, \quad A_1(b) = 0.2, \quad A_1(c) = 0.2;$$

$$A_2(a) = 0.5, \quad A_2(b) = 0.3, \quad A_2(c) = 0.5;$$

$$A_3(a) = 0.5, \quad A_3(b) = 0.7, \quad A_3(c) = 0.5.$$

Let $f : (X, \tau_1) \to (X, \tau_2)$ be the identity function. Then f is fuzzy *gc*-irresolute but not fuzzy *gsc*-irresolute; for a fuzzy set A_1 is *gfs*-closed in (X, τ_2), but $f^{-1}(A_3) = A_3$ is not *gfs*-closed in (X, τ_1). Hence f is fuzzy *gc*-irresolute but not fuzzy *gsc*-irresolute.

Example 4.5 In Example 3.2, f is fuzzy *gsc*-irresolute but not fuzzy *gc*-irresolute.

The following are the properties of fuzzy *gsc*-irresolute functions.

Theorem 4.6 *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function.*

(i) *The following statements are equivalent;*

(a) *f is fuzzy gsc-irresolute.*

(b) *The inverse image of every gfs-open set in Y is gfs-open in X.*

(ii) *If* $f : (X, \tau) \to (Y, \sigma)$ *is fuzzy gsc-irresolute, then* $f(gs\text{-Cl}(A)) \le gs\text{-Cl}(f(A))$ *for all* A in X.

Proof It is similar to that of Theorems 3.4 and 3.5.

Theorem 4.7 *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be functions.*

(i) *If f and g are fuzzy gsc-irresolute, then the composition* $g \circ f$ *is fuzzy gsc-irresolute.*

(ii) *If f is fuzzy gsc-irresolute and g is gfs-continuous, then the composition* $g \circ f$ *is gfscontinuous.*

5. STRONGLY *GFS*-CONTINUOUS AND PERFECTLY *GFS*-CONTINUOUS FUNCTIONS

Definition 5.1 [4] A function $f : X \to Y$ is called perfectly fuzzy continuous if the inverse image of every fuzzy open set in Y is both fuzzy open and fuzzy closed in X.

Definition 5.2 [4]A function $f : X \to Y$ is called strongly *gf*-continuous if the inverse image of every *gf*-open set in Y is fuzzy open in X.

Definition 5.3 [4] A function $f : X \to Y$ is called perfectly *gf*-continuous if the inverse image of every *gf*-open set in Y is both fuzzy open and fuzzy closed in X.

Definition 5.4 A function $f : X \to Y$ is called strongly *gfs*-continuous if the inverse image of every *gfs*-open set in Y is fuzzy open in X.

Definition 5.5 A function $f : X \to Y$ is called perfectly *gfs*-continuous if the inverse image of every *gfs*-open set in Y is both fuzzy open and fuzzy closed in X.

Theorem 5.6 *Strong gfs-continuity* $\Rightarrow$ *strong gf-continuity* $\Rightarrow$ *fuzzy continuity.*

The converses of Theorem 5.6 are not true as Example 5.7 in [4] and the following example show.

Example 5.7 Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_X, 1_X, A_5, A_6\}$ where A_1, A_2, A_3, A_4, A_5 and A_6 are fuzzy sets in X defined by

$0 \le A_1(a) \le 1,\ 0 \le A_1(b) \le 0.4;\ 0 \le A_2(a) \le 0.4,\ 0 \le A_2(b) \le 1;\ A_3(a) = A_3(b) = 0.2;$

$$A_4(a) = 0.7,\ A_4(b) = 0;\ A_5(a) = A_5(b) = 0.7;\ A_6(a) = A_6(b) = 0.2.$$

Let $f : (X, \tau_1) \to (X, \tau_2)$ be the identity function. Then f is strongly *gf*-continuous but not strongly *gfs*-continuous.

Theorem 5.8 *A function $f : X \to Y$ is strongly gfs-continuous if and only if the inverse image of every gfs-closed set in Y is fuzzy closed in X.*

Theorem 5.9 *Let $f : X \to Y$, $g : Y \to Z$ be functions. If f is strongly gfs-continuous and g is gfs-continuous, then $g \circ f$ is fuzzy continuous.*

Theorem 5.10 *Perfect gfs-continuity ⇒ perfect gf-continuity, and perfect gfs-continuity ⇒ strong gfs-continuity.*

The converses of Theorem 5.10 are not true.

Example 5.11 Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A_1, A_2\}$ and $\tau_2 = \{0_X, 1_X, A_3, A_4, A_3 \wedge A_4, A_3 \vee A_4\}$ where A_1, A_2, A_3, and A_4 are fuzzy sets in X defined by

$$0 \le A_1(a) \le 1,\ 0 \le A_1(b) \le 0.3;\ 0 \le A_2(a) \le 1,\ 0.7 \le A_2(b) \le 1;$$

$$A_3(a) = 1,\ A_3(b) = 0.2;\ A_4(a) = 0.3,\ A_4(b) = 0.7.$$

Let $f : (X, \tau_1) \to (X, \tau_2)$ be the identity function. Then f is perfectly *gf*-continuous but not perfectly *gfs*-continuous.(For fuzzy set B in X given by $B(a) = 0.5$, $B(b) = 0.4$, it is *gfs*-open in (X, τ_2), but $f^{-1}(B)$ is neither fuzzy open nor fuzzy closed set in (X, τ_1).)

Example 5.12 Let $X = \{a, b\}$ and $\tau_1 = \{0_X, 1_X, A_1\}$ where A_1 is a fuzzy set in X defined by $0 \le A_1(a) \le 1/2$, $0 \le A_1(b) \le 1$. Let $f : (X, \tau_1) \to (X, \tau_1)$ be the identity function. Then f is strongly *gfs*-continuous but not perfectly *gfs*-continuous. (For fuzzy set B in X given by $B(a) = 0.1$, $B(b) = 0.1$, it is *gfs*-open in (X, τ_2), but $f^{-1}(B)$ is fuzzy open but not fuzzy closed in (X, τ_1)).

Theorem 5.13 *A function $f : X \to Y$ is perfectly gfs-continuous if and only if the inverse image of gfs-closed set in Y is both fuzzy open and fuzzy closed in X.*

Regarding the results above-mentioned so far, we have the table of implications as shown in Table.

Table

⇒	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
a	1	1	1	0	0	0	0	0	0	0
b	0	1	1	0	0	0	0	0	0	0
c	0	0	1	0	0	0	0	0	0	0
d	0	1	1	1	0	0	0	0	0	0
e	0	0	1	0	1	0	0	0	0	0
f	1	1	1	0	0	1	0	0	0	0
g	1	1	1	1	0	0	1	0	0	0
h	1	1	1	1	0	1	1	1	0	0
i	1	1	1	1	1	0	1	0	1	0
j	1	1	1	1	1	1	1	1	1	1

In above table, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, and *j* denote fuzzy continuity, *gf*-continuity, *gfs*-continuity, fuzzy *gc*-irresolute, fuzzy *gsc*-irresolute, perfect fuzzy continuity, strong *gf*-continuity, perfect *gf*-continuity, strong *gfs*-continuity and perfect *gfs*-continuity, respectively. Also 1 denotes 'implies' and 0 denotes 'does not imply'.

6. G*FS*-CONNECTEDNESS AND THEIR PROPERTIES

Definition 6.1 [4] A fts *X* is said to be *fg*-connected if the only fuzzy sets which are both *gf*-open and *gf*-closed are 0_X and 1_X.

Definition 6.2 A fts *X* is said to be generalized fuzzy semi connected (in short, *gfs*-connected) if the only fuzzy sets which are both *gfs*-open and *gfs*-closed are $0X$ and $1X$.

Theorem 6.3 *Every gfs-connected space is fg-connected and every fg-connected space is fuzzy connected* [6].

Proof. It is proved in [4, Theorem 7.2] that every *fg*-connected space is fuzzy connected. So we proved that every *gfs*-connected space is *fg*-connected. Let *X* be a *gfs*-connected space and suppose that *X* is not *fg*-connected. Then there exists a proper fuzzy set A ($A \neq 0_X$, $A = 1_X$) such that A is both *gf*-open and *gf*-closed. Since *gf*-open set is *gfs*-open, X is not *gfs*-connected-a contradiction.

However, the converses are not true as Example 7.3 in [4] and the following example show.

Example 6.4 Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A_1, A_2\}$ where A_1 and A_2 are fuzzy sets in X defined by $0 \leq A_1(a) \leq 1$, $A_1(b) = 0$; $0.8 \leq A_2(a) \leq 1$, $A_2(b) = 0.3$. Then (X, τ) is *fg*-connected but not *gfs*-connected; For any fuzzy set B in X, B is *gfs*-open and *gfs*-closed in (X, τ). Hence (X, τ) is not *gfs*-connected.

Theorem 6.5 *If $f : X \to Y$ is gfs-continuous surjection and X is gfs-connected, then Y is fuzzy connected.*

Theorem 6.6 *If $f : X \to Y$ is fuzzy gsc-irresolute surjection and X is gfs-connected, then Y is gfs-connected.*

Theorem 6.7 *If $f : X \to Y$ is strongly gfs-continuous surjection and X is fuzzy connected, then Y is gfs-connected.*

Theorem 6.8 *A fts X is gfs-connected if and only if it has no non-zero proper gfs-open sets A and B such that* $A + B = 1$.

Corollary 6.9 *A fts X is gfs-connected if and only if it has no non-zero proper gfs-open sets A and B such that* $A + B = 1$, $\mathrm{Cl}(A) + B = A + \mathrm{Cl}(B) = 1$.

7. *GFS*-EXTREMALLY DISCONNECTEDNESS AND *GFS*-COMPACTNESS

Definition 7.1 A fts X is said to be generalized fuzzy semi extremally disconnected (in short, *gfs*-extremally disconnected) if *gs*-Cl(A) is *gfs*-open, whenever A is *gfs*-open.

Theorem 7.2 *Let X be a gfs-extremally disconnected space. Then the following statements are hold*

(i) *For each gfs-closed set A, gs*-Int(A) *is gfs-closed.*

(ii) *For each gfs-open set A, gs*-Cl(A) + *gs*-Cl $(1 - gs\text{-Cl}(A)) = 1$.

(iii) *For each pair of gfs-open set A, B with gs*-Cl(A) + $B = 1$, *gs*-Cl(A) + *gs*-Cl(B) = 1.

Proof (i) Let A be any *gfs*-closed set. Then $1 - A$ is *gfs*-open and so by the hypothesis, $gs\text{-Cl}(1 - A) = 1 - gs\text{-Int}(A)$ is *gfs*-open, which implies that $gs\text{-Int}(A)$ is *gfs*-closed.

(ii) Let A is a *gfs*-open set. Since $1 - gs\text{-Cl}(A) = gs\text{-Int}(1 - A)$, we have

$$gs\text{-Cl}(A) + gs\text{-Cl}(1 - gs\text{-Cl}(A)) = gs - \text{Cl}(A) + gs\text{-Cl}(gs\text{-Int}(1 - A)).$$

Since A is *gfs*-open, $1 - A$ is *gfs*-closed and so by (i) $gs\text{-Int}(1 - A)$ is *gfs*-closed, i.e. $gs\text{-Cl}(gs\text{-Int}(1 - A)) = gs\text{-Int}(1 - A)$. Thus, we get

$$gs\text{-Cl}(A) + gs\text{-Cl}(1-gs\text{-Cl}(A)) = gs\text{-Cl}(A) + gs\text{-Int}(1-A) = gs\text{-Cl}(A) + 1-gs\text{-Cl}(A) = 1.$$

(iii) Let A and B be any *gfs*-open sets such that $gs\text{-Cl}(A) + B = 1$. Then by (ii) we have

$$gs\text{-Cl}(A) + gs\text{-Cl}(1 \,.\, gs\text{-Cl}(A)) = 1 = gs\text{-Cl}(A) + B.$$

This implies that $B = gs\text{-Cl}(1 - gs\text{-Cl}(A))$. But from hypothesis $B = 1 - gs\text{-Cl}(A)$ and thus $gs\text{-Cl}(B) = gs\text{-Cl}(1-gs\text{-Cl}(A))$. Hence $B = gs\text{-Cl}(B)$, and consequently $gs\text{-Cl}(A)+gs\text{-Cl}(B) = 1$.

Definition 7.3 A collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of *gfs*-open sets in X is called *gfs*-open cover of a fuzzy set B in X if $B \le \vee_{\lambda\in\Lambda} A_\lambda$.

Definition 7.4 A fts X is called *gfs*-compact if every *gfs*-open cover of X has a finite subcover.

Definition 7.5 A fuzzy set B in X is said to be *gfs*-compact relative to X (which we shall call a *gfs*-compact set) if for every collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of *gfs*-open sets of X such that $B \le \vee_{\lambda\in\Lambda} A_\lambda$, there exists a finite subset Λ_0 of Λ such that $B \le \vee_{\lambda\in\Lambda 0} A_\lambda$.

Theorem 7.6 *Let X be a gfs-compact fts and A be a gfs-closed set in X. Then A is a gfs*-compact set.

Theorem 7.7 (i) *If* $f : X \to Y$ *is gfs-continuous and X is gfs-compact, then* $f(X)$ *is a fuzzy compact set.*

(ii) *If* $f : X \to Y$ *is fuzzy gsc-irresolute and A is gfs- compact set of X, then* $f(A)$ *is a gfs*-compact set in Y .

(iii) *If* $f : X \to Y$ *is strongly gfs-continuous and X is fuzzy compact, then* $f(X)$ *is a gfs*-compact set in Y .

REFERENCES

[1] K.K. Azad, *On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82** (1981) 14–32.

[2] G. Balasubramanian, *On extensions of fuzzy topologies*, Kybernetika **28** (1992), 239–244.

[3] G. Balasubramanian, *Fuzzy disconnectedness and its stronger forms*, Indian J. Pure Appl. Math. **23** (1993), 27–30.

[4] G. Balasubramanian and P. Sundaram, *On some generalizations of fuzzy continuous functions*, Fuzzy Sets and Systems **86** (1997), 93–100.

[5] C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. 24 (1968), 182–190.

[6] U.V. Fatteh and D.S. Bassan, *Fuzzy connectedness and its stronger forms*, J. Math.Anal. Appl. **111** (1985), 449–464.

[7] N. Levine, *Generalized closed sets in topology*, Rend. Cir. Mat. Palermo 19 (1970), 89–96.

[8] H. Maki, K. Balachandran and R. Devi, *Remark on semi-generalized closed sets and generalized semi-closed sets*, Kyungpook. Math. J. 36 (1996), 155–163.

[9] J.H. Park and J.K. Park, *On Regular generalized fuzzy closed sets and generalizations of fuzzy continuous functions*, Indian J. Pure Appl. Math. **34** (2003), 1013–1024.

[10] P. Sundaram, *Studies on generalizations of continuous maps in topological spaces*, Ph.D. Thesis (Bharathiar University, July 1991).

Jin Han Park & Jin Keun Park
Division of Mathematical Sciences
Pukyong National University
599-1 Daeyeon 3-Dong, Nam-Gu
Pusan 608-737, South Korea