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FUZZY TOPOLOGY BASED ON THE CONSENSUS SPACE

ABSTRACT: By the use of the theory of a consensus space presented by Hisakichi Suzuki, concept of C-fuzzy topology is acquired. It is proved that a Ying's fuzzifying topology is a C-fuzzy topology and each C-fuzzy topology is homeomorphic a special of fuzzy topology. These discussions have built up a new theoretical approach for the fuzzifying topology.

Keywords: *Fuzzy topology; consensus space; C-fuzzy topology; homeomorphism.*

Classification (MSC2000): *54A40; 03E72*

1. INTRODUCTION

Since Chang introduced the concept of fuzzy topology[1], Wong, Lowen, Hutton, Pu and Liu, Ying, etc, discussed respectively various aspects of fuzzy topology[4-6,8-9]. In these author's papers, a fuzzy topology is defined as a classical subset of the fuzzy power set of a non-empty classical set or as a fuzzy subset of the power set of a nonempty classical set. In [9], Ying gave a definition of fuzzifying topology by the use of the semantics of fuzzy logic and developed a new approach to study fuzzy topology. In [2-3], Hisakichi Suzuki gave concepts of consensus set and consensus space and discussed the operation of fuzzy sets by the use of the theory of consensus space.

In this paper, based on the theory of the consensus space, a theoretical approach is established by defining C-fuzzy topology over s set X. It is proved that a Ying's fuzzy topology is a C-fuzzy topology and each C-fuzzy topology is homeomorphic to a special of C-fuzzy topology. By these discussions, we try to build up a connection between Ying's fuzzifying topology and consensus space.

2. PRELIMINARY

Let X be a set, $\mathcal{P}$(X) be a power set of set X and $\mathcal{J}$ be a fuzzy subset of $\mathcal{P}(X)$, Ying gave the following definition:

Definition 2.1[9] *If for any A, B, $A_t \in \mathcal{P}(X)$ ($\forall t \in T$),*

(1) $\mathcal{J}(X) = \mathcal{J}(\phi) = 1$;

(2) $\mathcal{J}(A \cap B) \geq \min\{\mathcal{J}(A), \mathcal{J}(B)\}$;

(3) $\mathcal{J}(\bigcup_{t \in T} A_t) \geq \inf_{t \in T} \mathcal{J}(At)$.

Then $\mathcal{J}$ is called a fuzzifying topology over X.

Clearly, $\mathcal{J}$ is a fuzzifying topology over X if and only if $\mathcal{J}_\lambda = \{A \mid A \in \mathcal{P}(X), \mathcal{J}(A) \geq \lambda\}$ is a topology over X for any $\lambda \in [0, 1]$.

Definition 2.2[2,3] Let U be a set and $(\Omega, \mathcal{A}, P)$ be a probability space. Let

$$\mathcal{F}(U \times \Omega) = \{E \mid E \subseteq \times \Omega, E(u) \in \mathcal{A}, \forall u \in U\}$$

Where $E(u) = \{\omega \mid \omega \in \Omega, (u, \omega) \in E\}$.

Then $\mathcal{F}(U \times \Omega)$ is called a consensus space induced by $(\Omega, \mathcal{A})$ and an element of $\mathcal{F}(U \times \Omega)$ is called a consensus set.

Let E be a consensus set and $(\Omega, \mathcal{A}, P)$ be a probability space, then we have a fuzzy subset μ_E of U defined by

$$\mu_E(u) = P(E(u)), \forall u \in U$$

Definition 2.3[6,7] Let L be a complete lattice with maximal element 1 and minimal element 0. Let $\delta \subseteq L$, if

(1) $0, 1 \in \delta$;

(2) $\gamma_1, \gamma_2 \in \delta \Rightarrow \gamma_1 \wedge \gamma_2 \in \delta$;

(3) For any index set T, $\gamma_t \in \delta$ ($t \in T$) $\Rightarrow \bigvee_{t \in T} \gamma_t \in \delta$.

Then is called a lattice-topology over L.

3. C-FUZZY TOPOLOGY BASED ON THE CONSENSUS SPACE

Let X be a set, $L = \mathcal{P}(X)$ be a power set of X and $(\Omega, \mathcal{A}, P)$ be a probability space. Let $\mathcal{F}(L \times \Omega)$ be a consensus space induced by $(\Omega, \mathcal{A})$ and $E \in \mathcal{F}(L \times \Omega)$ be a consensus set and

$$E(\omega) = \{A \in L, (A, \omega) \in E\}$$

$$E(A) = \{\omega \in \Omega \mid (A, \omega) \in E\}$$

Definition 3.1 If $E(\omega)$ is a lattice-topology over L for any $\omega \in \Omega$, then

$$\mu_E(A) = P(E(A)), \forall A \in L.$$

is called a C-fuzzy topology over L.

Theorem 3.1 *A Ying's fuzzifying topology is a C-fuzzy topology.*

Proof. Let X be a set and $\mathcal{J}$ be a Ying fuzzifying topology, then

$$\mathcal{J}_\lambda = \{A \in \mathcal{P}(X) \mid \mathcal{J}(A) \geq \lambda\}, \forall \lambda \in [0, 1]$$

is a topology over X.

Let $= \Omega$ [0, 1], $\mathcal{A}$ be a Borel field on [0, 1] and P be the usual Lebesgue measure. Let $L = \mathcal{P}(X)$ and $E = \{(A,\lambda) \mid \lambda \in [0, 1], A \in \mathcal{J}_\lambda\}$. Then $E \subseteq L \times \Omega$ and

$$E(A) = \{\lambda \in [0, 1] \mid A \in \mathcal{J}_\lambda\} = [0, \mathcal{J}(A)] \in \mathcal{A}, \forall A \in L$$

$$E(\lambda) = \{A \in L \mid (A, \lambda) \in E\} = \{A \mid A \in \mathcal{J}_\lambda\} = \mathcal{J}_\lambda, \forall \lambda \in [0, 1].$$

Then $E(\lambda)$ is a lattice-topology over L for any $\lambda \in [0, 1]$ and $E \in \mathcal{F}(L \times \Omega)$, then

$$\mu_E(A) = P(E(A)) = P([0, \mathcal{J}(A)]) = \mathcal{J}(A),$$

and consequently $\mathcal{J}$ is a C-fuzzy topology over L.

Example 3.1 Let X be a set, $\mathcal{J}$ be a topology over X and $(\Omega, \mathcal{A}, P)$ be a probability space. Let

$$L_1 = \mathcal{F}(X) = \{f \mid f : \Omega \to \mathcal{P}(X) \text{ is a mapping}\}$$

For $\{f_t \mid t \in T\} \subseteq L_1$, we define lattice operations in L_1 as follows

$$(\bigvee_{t\in T} f_t)(\omega)=\bigcup_{t\in T} f_t(w);\ (\bigwedge_{t\in T} f_t)(\omega)=\bigcap_{t\in T} f_t(\omega)$$

Then L_1 is a complete lattice. Let $\mathcal{A}$ satisfy:

$$\{\omega \in \Omega \mid f(\omega) \in \mathcal{J}\} \in \mathcal{A}, \forall f \in L_1$$

and

$$E = \{(f, \omega) \mid f \in L_1, \omega \in \Omega \text{ and } f(\omega) \in \mathcal{J}\}$$

Then

$$E(f) = \{\omega \in \Omega \mid (f, \omega) \in E\} = \{\omega \in \Omega \mid f(\omega) \in \mathcal{J}\} \in \mathcal{A}$$

$$E(\omega) = \{f \mid (f, \omega) \in E\} = \{f \mid f(\omega) \in \mathcal{J}\}.$$

It follows that $E(\omega)$ is a lattice-topology over L_1 and $E \in \mathcal{F}(L_1 \times \Omega)$. Then $\mathcal{J}_E(f) = P(E(f))$ is a C-fuzzy topology over L_1 and is called a C-fuzzy topology generated by functions.

Example 3.2 Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, X be a set and L_1 be a complete lattice in example 3.1. Let $\mathcal{J}$ be a lattice-topology over L_1 and

$$W = \{(f(\omega), \omega) \mid \omega \in \Omega, f \in \mathcal{J}\},$$

then $W(\omega) = \{f(\omega) \mid f \in \mathcal{J}\}$ is a lattice-topology over $L = \mathcal{P}(X)$.

Let $\mathcal{A}$ satisfy:

$$W(A) = \{\omega \in \Omega \mid A \in W(\omega)\} \in \mathcal{A}, \forall A \in L,$$

then W is a consensus set and $\mu_E(A) = P(W(A))$ is a C-fuzzy topology over L.

Example 3.3 Let X be a set and $L = \mathcal{P}(X)$. Let

$$\Omega_X = \{\mathcal{J} \mid \mathcal{J} \text{ is a lattice-topology over } L\},$$

and

$$\Sigma = \{(A, \mathcal{J}) \mid \mathcal{J} \in \Omega_X, A \in \mathcal{J}\},$$

then $\Sigma(\mathcal{J}) = \{\mathcal{A} \mid \mathcal{A} \in \mathcal{J}\} = \mathcal{J}$ is a lattice-topology over L.

Let $\mathcal{A}_X$ be σ-field over Ω_X and

$$\Sigma(A) = \{\mathcal{J} \mid \mathcal{J} \in \Omega_X, A \in \mathcal{J}\} \in \mathcal{A}_X, \forall A \in L.$$

Let $(\Omega_X, \mathcal{A}_X, P_X)$ be a probability space. Then Σ is a consensus set and $\mathcal{J}_E(A) = P_X(\{\mathcal{J} \mid A \in \mathcal{J}\})$ is a C-fuzzy topology over L.

Example 3.4 Let X be a set and L_1 be a complete lattice in example 3.1. Let

$$\Omega_1 = \{\mathcal{J} \mid \mathcal{J} \text{ is a lattice-topology over } L_1\}$$

$$\Sigma_1 = \{(f, \mathcal{J}) \mid \mathcal{J} \in \mathcal{J} \in \Omega_1, f \in \mathcal{J}\}.$$

Let $(\Omega_1, \mathcal{A}_1, P_1)$ be a probability space and

$$\Sigma_1(f) = \{\mathcal{J} \mid f \in \mathcal{J} \in \mathcal{A}_1\}, \forall f \in L_1.$$

Then Σ_1 is a consensus set and $\mathcal{J}_{\Sigma 1}(f) = P_1(\Sigma_1(f))$ is a C-fuzzy topology over L_1 and is called a C-fuzzy topology generated by topologies.

Theorem 3.2: Let L_1 be a complete lattice in Example 3.1, then a C-fuzzy topology generated by functions must be a C-fuzzy topology generated by topologies.

Proof. Let X, $\mathcal{J}$, $(\Omega, \mathcal{A}, \text{P})$ and L_1 be the same as Example 3.1; let Ω_1 and E_1 be the same as Example 3.4.

Let $E \in \mathcal{F}(L_1 \times \Omega)$ be a consensus set and $E(\omega)$ is a lattice-topology for any $\omega \in \Omega$, and $\mathcal{J}_E(f) = P(E(f))$ be a C-fuzzy topology generated by functions.

Let $\mathcal{A}_1$ be a σ-field over Ω_1 and $\Sigma_1(f) = \{\mathcal{J} \mid f \in \mathcal{J}\} \in \mathcal{A}_1$ for any $f \in L_1$.

Let P_1 be a probability over $\mathcal{A}_1$ and $P_1(\mathcal{U}) = P(\{\omega \mid E(\omega) \in \mathcal{U}\})$, $\forall \mathcal{U} \in \mathcal{A}_1$. then $\mathcal{J}_{\Sigma_1}(f) = P_1(\Sigma(f))$ is a C-fuzzy topology generated by topologies and

$$\begin{aligned}\mathcal{J}_{\Sigma 1}(f) &= P(\{\omega \mid E(\omega) \in \Sigma_1(f)\}) = P(\{\omega \mid f \in E(\omega)\}) \\ &= P(\{\omega \mid f(\omega) \in \mathcal{J}\}) = P(E(f)) = \mathcal{J}_E(f)\end{aligned}$$

Therefore, a C-fuzzy topology $\mathcal{J}_E$ generated by functions is a C-fuzzy topology $\mathcal{J}_{\Sigma_1}$ generated by topologies.

Definition 3.2 Let $S_i (i = 1, 2)$ be complete lattices, $(\Omega_i, \mathcal{A}_i, P_i)$ $(i = 1, 2)$ be probability spaces, $E_i (i = 1, 2)$ be consensus sets of $\mathcal{F}(S_i \times \Omega_i)$ and $\mathcal{J}_{E_i} (i = 1, 2)$ be C-fuzzy topologies. If there is a bijection $\phi : S_1 \to S_2$ such that $\mathcal{J}_{E_2}(\phi(\gamma)) = \mathcal{J}_{E_1}(\gamma)$, $\forall \gamma \in S_1$. Then we say that $\mathcal{J}_{E_2}$ and $\mathcal{J}_{E_2}$ are homeomorphic.

Theorem 3.3 Each C-fuzzy topology over X is homeomorphic to a C-fuzzy topology over a complete lattice S generated by functions.

Proof. Let L = $\mathcal{P}(X)$, $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, $E \in \mathcal{F}(L \times \Omega)$ be a consensus space and $E(\omega) = \{A \mid (A, \omega) \in E\}$ be a lattice-topology for any $\omega \in \Omega$. Let $\mathcal{L} = \{f \mid f : \Omega \to L$ is a mapping$\}$ and $\mathcal{E} = \{f \mid f \in \mathcal{L}, f(\omega) \in E(\omega), \forall\omega \in \Omega\}$.

Then $\mathcal{E}$ is a lattice-topology over $\mathcal{L}$. For $A \in L$, let

$$f_A : \Omega \to \mathcal{L},\ \omega \to f_{\mathrm{A}}^{\omega}$$

Where $f_A^{\omega}(\omega') = \begin{cases} \theta & \omega \neq \omega' \\ A & \omega = \omega' \end{cases}$

Let $S = \{f_A \mid A \in \mathcal{L}\}$, then

$$\bigvee_{t\in T} f_{A_t}(\omega) = \bigcup_{t\in T} f_{A_t}(\omega) = \bigvee_{t\in T} f_{A_t}^{\omega} = f^{\omega}_{\bigcup_{t\in T} A_t} = f_{\bigcup_{t\in T} A_t}(\omega)$$

$$\bigwedge_{t\in T} f_{A_t}(\omega) = \bigcap_{t\in T} f_{A_t}(\omega) = \bigwedge_{t\in T} f_{A_t}^{\omega} = f^{\omega}_{\bigcap_{t\in T} A_t} = f_{\bigcap_{t\in T} A_t}(\omega)$$

It follows that S is a complete lattice.

Let ϕ: $L \to S$, $A \to f_A$, then

$$f_A = f_B \Rightarrow f_A(\omega) = f_B(\omega),\ \forall\omega \in \Omega \Rightarrow f_A^{\omega} = f_B^{\omega}, \forall\omega\in\Omega.$$

It follows that $A = f_A^{\omega}(\omega) = f_B^{\omega}(\omega) = \mathrm{B}$ and consequently is a isomorphism.

Let $\Sigma = \{(f_A, \omega) \mid f_A^{\omega} \in \mathcal{E}\} = \{(f_A, \omega) \mid A \in \mathrm{E}(\omega)\}$,

Then

$$E(f_A) = \{\omega \mid f_A^{\omega} \in \mathcal{E}\} = \{\omega \mid A \in E(\omega)\} = E(A) \in \mathcal{A}$$

$$\Sigma(\omega) = \{f_A \mid (f_A, \omega) \in \Sigma\} = \{f_A \mid f_A^{\omega} \in \mathcal{E}\} = \{f_A \mid A \in E(\omega)\}$$

is a lattice-topology over S. Then we have a C-fuzzy topology $\mathcal{J}_{\Sigma}$ over S generated by functions and

$$\mathcal{J}_{\Sigma}(f_A) = P(E(f_A)) = P(E(A)) = \mathcal{J}_E(A), \forall A \in L.$$

Then $\mathcal{J}_E(A) = \mathcal{J}_{\Sigma}(\phi(A)), \forall A \in L$. Therefore, the C-fuzzy topology $\mathcal{J}_E$ is homeomorphic to the C-fuzzy topology $\mathcal{J}_{\Sigma}$.

4. CONCLUSION

We have given a definition of C-fuzzy topology by the use of the theory of consensus space. We have proved that a Ying's fuzzifying topology is a C-fuzzy topology generated by functions. These discussions have built up a the theoretical foundation for fuzzifying topology.

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