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FUZZY IDEALS OF PSEUDO *MV*-ALGEBRAS

ABSTRACT. The notion of fuzzy (implicative) ideals of a pseudo MV*-algebra is introduced, and its characterizations are established. Conditions for a fuzzy set to be a fuzzy ideal are given. Given a fuzzy set* μ, *the least fuzzy ideal containing* μ *is constructed.*

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1. INTRODUCTION

The ideal theory of pseudo *MV*-algebras is studied in [1] and [2]. In particular, the second author gave characterizations of ideals, and introduced the notion of implicative ideals in pseudo *MV*-algebras (see [2]). In this paper, we introduce the notion of fuzzy (implicative) ideals in a pseudo *MV*-algebra. We give characterizations of fuzzy (implicative) ideals, and provide conditions for a fuzzy set to be a fuzzy ideal. Given a fuzzy set μ, we make the least fuzzy ideal containing μ.

2. PRELIMINARIES

A *pseudo MV-algebra* is an algebra $(M;\oplus, ^{-}, ^{\sim}, 0,1)$ of type (2, 1, 1, 0, 0) such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^{-} \oplus y^{-})^{\sim} :$$

(a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(a2) $x \oplus 0 = 0 \oplus x = x$,

(a3) $x \oplus 1 = 1 \oplus x = 1$,

(a4) $1^{\sim} = 0$, $1^{-} = 0$,

(a5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(a6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$,

(a7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,

(a8) $(x^-)^\sim = x$.

If we define $x \leq y$ if and only if $x^- \oplus y = 1$, then $\leq$ is a partial order such that M is a bounded distributive lattice with the join $x \vee y$ and the meet $x \wedge y$ given by

$$x \vee y = x \oplus x^\sim \odot y = x \odot y^- \oplus y,$$

$$x \wedge y = x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y.$$

Let M be a pseudo MV-algebra M and $x, y, z \in M$. Then the following properties are valid (see [1]).

(b1) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

(b2) $(x \vee y)^- = x^- \wedge y^-$.

(b3) $x \leq y \Rightarrow z \odot x \leq z \odot y, x \odot z \leq y \odot z$.

(b4) $z \oplus (x \wedge y) = (z \oplus x) \wedge (z \oplus y)$.

(b5) $z \odot (x \oplus y) \leq z \odot x \oplus y$.

(b6) $(x^\sim)^- = x$.

(b7) $x \odot 1 = 1 \odot x = x$.

(b8) $x \oplus x^\sim = 1, x^- \oplus x = 1$.

(b9) $x \odot x^- = 0, x^\sim \odot x = 0$.

(b10) $x \odot (y \odot z) = (x \odot y) \odot z$.

A subset I of a pseudo MV-algebra M is called an *ideal* of M (see [2]) if it satisfies:

(c1) $0 \in I$,

(c2) If $x, y \in I$, then $x \oplus y \in I$,

(c3) If $x \in I$, $y \in M$ and $y \leq x$, then $y \in I$.

For every subset $W \subseteq M$, we denote by $\langle W \rangle$ the ideal of M generated by W, that is, $\langle W \rangle$ is the smallest ideal containing W. By [1, Lemma 2.5],

$$\langle W \rangle = \{x \in M \mid x \leq y_1 \oplus \ldots \oplus y_k \text{ for some } y_1, \ldots, y_k \in W\}.$$

3. FUZZY IDEALS

We give the definition of a fuzzy ideal in a pseudo *MV*-algebra.

Definition 3.1. A fuzzy set μ in a pseudo MV-algebra M is called a *fuzzy ideal* of M if it satisfies:

(d1) $(\forall x, y \in M)\ (\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\})$,

(d2) $(\forall x, y \in M)\ (y \leq x \Rightarrow \mu(y) \geq \mu(x))$.

It is easily seen that (d2) forces

(d3) $(\forall x \in M)\ (\mu(0) \geq \mu(x))$.

Example 3.2. Let I be an ideal of a pseudo MV-algebra M and let μ_I be a fuzzy set in M defined by

$$\mu_I(x) := \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Let $x, y \in M$. If $x, y \in I$, then $x \oplus y \in I$ and so

$$\mu_I(x \oplus y) = \alpha = \min\{\mu_I(x), \mu_I(y)\}.$$

If $x \notin I$ or $y \notin I$, then $\mu_I(x) = \beta$ or $\mu_I(y) = \beta$. Thus

$$\mu_I(x \oplus y) \geq \beta = \min\{\mu_I(x), \mu_I(y)\}.$$

Let $x, y \in M$ be such that $y \leq x$. If $y \in I$, then $\mu_I(y) = \alpha \geq \mu_I(x)$. Assume that $y \notin I$. Then $x \notin I$, and thus $\mu_I(y) = \beta = \mu_I(x)$. Therefore μ_I is a fuzzy ideal of M.

Proposition 3.3. *Let μ be a fuzzy ideal of a pseudo MV-algebra M. Then*

(i) $(\forall x, y \in M)\ (\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})$.

(ii) $(\forall x, y \in M)\ (\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\})$.

(iii) $(\forall x, y \in M)\ (\mu(x \wedge y) = \min\{\mu(x), \mu(y)\})$.

(iv) $(\forall x, y \in M)\ (\mu(x \oplus y) = \min\{\mu(x), \mu(y)\})$.

Proof. Since $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$ for all $x, y \in M$, it follows from (d1) and (d2) that

$$\mu(x \odot y) \geq \mu(x \wedge y) \geq \mu(x \vee y) \geq \mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}.$$

Since $x \oplus y \geq x \vee y \geq x, y$ for all $x, y \in M$, we have $\mu(x \oplus y) \leq \mu(x), \mu(y)$ and $\mu(x \vee y) \leq \mu(x), \mu(y)$ by (d2). This completes the proof.

Theorem 3.4. *Let* μ *be a fuzzy set in a pseudo MV-algebra M. Then* μ *is a fuzzy ideal of M if and only if it satisfies* (d1) *and*

$$\text{(d4)}\ (\forall x, y \in M)\ (\mu(x \vee y) \geq \mu(x)).$$

Proof. Let μ be a fuzzy ideal of M and let $x, y \in M$. Since $x \wedge y \leq x$, it follows from (d2) that $\mu(x \wedge y) \geq \mu(x)$. Suppose that μ satisfies (d1) and (d4). Let $x, y \in M$ be such that $y \leq x$. Then $x \wedge y = y$ and so $\mu(y) = \mu(x \wedge y) \geq \mu(x)$ by (d4). Hence μ is a fuzzy ideal of M.

Proposition 3.5. *Every fuzzy ideal* μ *of a pseudo MV-algebra M satisfies the following inequality*

$$(\forall x, y \in M)\ (\mu(y) \geq \min\{\mu(x), \mu(x^{\sim} \odot y)\}). \qquad (1)$$

Proof. Let μ be a fuzzy ideal of a pseudo *MV*-algebra M. Since $y \leq x \vee y = x \oplus x^{\sim} \odot y$ for all $x, y \in M$, it follows from (d1) and (d2) that

$$\mu(y) \geq \mu(x \oplus x^{\sim} \odot y) \geq \min\{\mu(x), \mu(x^{\sim} \odot y)\}.$$

This completes the proof.

Proposition 3.6. *Let* μ *be a fuzzy set in a pseudo MV-algebra M that satisfies* (d3) *and* (1). *Then* μ *satisfies the condition* (d2) *and*

$$(\forall x, y \in M)\ (\mu(y) \geq \min\{\mu(x), \mu(y \odot x^{-})\}). \qquad (2)$$

Proof. Assume that μ satisfies (d3) and (1). Let $x, y \in M$ be such that $y \leq x$. Using (b3) and (b9), we have $x^{\sim} \odot y \leq x^{\sim} \odot x = 0$ and so $x^{\sim} \odot y = 0$. It follows from (d3) and (1) that

$$\mu(y) \geq \min\{\mu(x), \mu(x^{\sim} \odot y)\} = \min\{\mu(x), \mu(0)\} = \mu(x)$$

so that (d2) is valid. Note that

$$(y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x) \leq (y \odot x^{-})^{\sim} \odot (y \odot x^{-}) \oplus x = 0 \oplus x = x$$

so from (d2) that $\mu(x) \leq \mu((y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x))$. Now since

$$x^{\sim} \odot y \leq x \oplus x^{\sim} \odot y = y \odot x^{-} \oplus x,$$

it follows from (d2) that $\mu(x^{\sim} \odot y) \geq \mu(y \odot x^{-} \oplus x)$ so that

$$\mu(y) \geq \min\{\mu(x), \mu(x^{\sim} \odot y)\} \geq \min\{\mu(x), \mu(y \odot x^{-} \oplus x)\}$$

$$\geq \min\{\mu(x), \min\{\mu(y \odot x^{-}), \mu((y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x))\}\}$$
$$\geq \min\{\mu(x), \min\{\mu(y \odot x^{-}), \mu(x)\}\}$$
$$= \min\{\mu(x), \mu(y \odot x^{-})\}.$$

This completes the proof.

Proposition 3.7. *If a fuzzy set* μ *in a pseudo MV-algebra M satisfies conditions* (d3) *and* (2)*, then* μ *is a fuzzy ideal of M.*

Proof. Let $x, y \in M$ be such that $y \leq x$. Then $y \odot x^{-} \leq x \odot x^{-} = 0$ by (b3) and (b9), and thus $y \odot x^{-} = 0$. Using (d3) and (2), we have

$$\mu(y) \geq \min\{\mu(x), \mu(y \odot x^{-})\} = \min\{\mu(x), \mu(0)\} = \mu(x).$$

Thus (d2) is valid. Note that

$$(x \oplus y) \odot y^{-} = (x \oplus (y^{-})^{\sim}) \odot y^{-} = x \wedge y^{-} \leq x$$

for all $x, y \in M$ so from (2) and (d2) that

$$\mu(x \oplus y) \geq \min\{\mu(y), \mu((x \oplus y) \odot y^{-})\} \geq \min\{\mu(y), \mu(x)\}.$$

Hence (d1) is valid, and μ is a fuzzy ideal of M.

Combining Propositions 3.5, 3.6 and 3.7, we have the following characterization of a fuzzy ideal in a pseudo *MV*-algebra.

Theorem 3.8. *For a fuzzy set* μ *in a pseudo MV-algebra M, the following are equivalent:*

(i) μ *is a fuzzy ideal of M.*

(ii) μ *satisfies the conditions* (d3) *and* (1).

(iii) μ *satisfies the conditions* (d3) *and* (2).

Proposition 3.9. *Let* μ *be a fuzzy set in a pseudo MV-algebra M. If* μ *satisfies conditions* (d3) *and*

$$(\forall x, y, z \in M)\ (\mu(x \odot y) \geq \min\{\mu(x \odot y \odot z), \mu(z^{\sim} \odot y)\}), \tag{3}$$

then μ *is a fuzzy ideal of M. Moreover,* μ *satisfies*:

(i) $(\forall x, y \in M)\ (\mu(x \odot y) = \mu(x \odot y \odot y))$,

(ii) $(\forall x \in M)\ (\forall n \in \mathbb{N})\ (\mu(x) = \mu(x^{n}))$, *where* $x^{n} = x^{n-1} \odot x = x \odot x^{n-1}$ *and* $x^{0} = 1$.

Proof. Taking $x = y$, $y = 1$ and $z = x^{-}$ in (3) and using (a8) and (b7), we have $\mu(y) = \mu(y\odot 1) \geq \min\{\mu(y\odot 1\odot x^{-}), \mu((x^{-})^{\sim}\odot 1)\} = \min\{\mu(y\odot x^{-}), \mu(x)\}$.

It follows from Theorem 3.8 that μ is a fuzzy ideal of M. Now taking $z = y$ in (3) and using (b9) and (d3), we get

$$\begin{aligned}\mu(x\odot y) &\geq \min\{\mu(x\odot y\odot y), \mu(y^{\sim}\odot y)\}\\ &= \min\{\mu(x\odot y\odot y), \mu(0)\}\\ &= \mu(x\odot y\odot y).\end{aligned}$$

On the other hand, since $x\odot y\odot y \leq x\odot y$, we see that $\mu(x\odot y\odot y) \geq \mu(x\odot y)$. Then (i) holds.

The proof of (ii) is by induction on n. If $n = 1$, then (ii) is obviously true. If we put $x = 1$ and $y = x$ in (i), then

$$\mu(x) = \mu(1\odot x) = \mu(1\odot x\odot x) = \mu(x^2).$$

Now assume that (ii) is valid for every positive integer $k > 2$. Then

$$\mu(x^{k+1}) = \mu(x^{k-1}\odot x\odot x) = \mu(x^{k-1}\odot x) = \mu(x^k) = \mu(x).$$

Therefore (ii) is true.

Lemma 3.10. *For any fuzzy set* μ *in a pseudo MV-algebra* M, *the condition* (3) *is equivalent to the following condition:*

$$(\forall x, y, z\in M)\ (\mu(x\odot y) \geq \min\{\mu(x\odot y\odot z^{-}), \mu(z\odot y)\}. \tag{4}$$

Proof. (3) $\Rightarrow$ (4): Let $x, y, z\in M$. By (3),

$$\mu(x\odot y) \geq \min\{\mu(x\odot y\odot z^{-}), \mu((z^{-})^{\sim}\odot y)\}.$$

Since $(z^{-})^{\sim} = z$, we have (4).

(4) $\Rightarrow$ (3): Applying (4) we see that

$$\mu(x\odot y) \geq \min\{\mu(x\odot y\odot (z^{\sim})^{-}), \mu(z^{\sim}\odot y)\}.$$

From this we obtain (3), because $(z^{\sim})^{-} = z$ by (b6).

In [2] we introduced the notion of implicative ideals in pseudo *MV*- algebras. An ideal I of a pseudo *MV*-algebra M is said to be *implicative* if it satisfies the following implication:

$$(\forall x, y, z\in M)\ (x\odot y\odot z\in I,\ z^{\sim}\odot y\in I \Rightarrow x\odot y\in I).$$

Definition 3.11. Let μ be a fuzzy ideal of a pseudo MV-algebra M. We say that μ is *fuzzy implicative* if it satisfies the condition (3) (or (4)).

Proposition 3.12. *Let I be an ideal of a pseudo MV-algebra M. Then I is implicative if and only if the fuzzy set* μ_I *which is described in Example 3.2 is a fuzzy implicative ideal of M.*

Proof. Straightforward.

Lemma 3.13. *Let* μ *be a fuzzy ideal of a pseudo MV-algebra M. Then*

$$(\forall x, y \in M)\ (\mu(x \odot y) \geq \min\{\mu(x \odot y \odot y), \mu(y \wedge y^{\sim})\}).$$

Proof. Applying (b7) and (b8) we have

$$x \odot y = (x \odot y) \odot 1 = (x \odot y) \odot (y \oplus y^{\sim}),$$

and so $x \odot y \leq (x \odot y) \odot y \oplus y^{\sim}$ by (b5). Using (b4) we obtain

$$\begin{aligned} x \odot y &\leq y \wedge (x \odot y \odot y \oplus y^{\sim}) \\ &\leq (x \odot y \odot y \oplus y) \wedge (x \odot y \odot y \oplus y^{\sim}) \\ &= x \odot y \odot y \oplus (y \wedge y^{\sim}). \end{aligned}$$

It follows from (d2) and (d1) that

$$\mu(x \odot y) \geq \mu(x \odot y \odot y \oplus (y \wedge y^{\sim}) \geq \min\{\mu(x \odot y \odot y), \mu(y \wedge y^{\sim})\}.$$

This completes the proof.

Theorem 3.14. *Let* μ *be a fuzzy ideal of a pseudo MV-algebra M. Then* the following statements are equivalent:

(i) μ *is fuzzy implicative.*

(ii) $(\forall x, y \in M)\ (\mu(x \odot y) = \mu(x \odot y \odot y))$.

(iii) $(\forall x \in M)\ (x^2 = 0 \Rightarrow \mu(x) = \mu(0))$.

(iv) $(\forall x \in M)\ (\mu(x \wedge x^{-}) = \mu(0))$.

(v) $(\forall x \in M)\ (\mu(x \wedge x^{\sim}) = \mu(0))$.

Proof. (i) ⇒ (ii): This is by Proposition 3.9.

(ii) ⇒ (iii): Taking $x = 1$ and $y = x$ in (ii), we get

$$\mu(x) = \mu(1 \odot x) = \mu(1 \odot x \odot x) = \mu(x^2).$$

Then (iii) is obviously true.

(iii) $\Rightarrow$ (iv): Using (b3) and (b9) we have

$$(x \wedge x^{-})^2 = (x \wedge x^{-}) \odot (x \wedge x^{-}) \leq x \odot x^{-} = 0.$$

Consequently, $(x \wedge x^{-})^2 = 0$. Hence $\mu(x \wedge x^{-}) = \mu(0)$.

(iv) $\Rightarrow$ (v): Since $x \wedge x^{\sim} = x^{\sim} \wedge x = x^{\sim} \wedge (x^{\sim})^{-}$, it follows from (iv) that $\mu(x \wedge x^{\sim}) = \mu(0)$.

(v) $\Rightarrow$ (i): By Lemma 3.13, $\mu(x \odot y) \geq \min\{\mu(x \odot y \odot y), \mu(y \wedge y^{\sim})\}$. Therefore $\mu(x \odot y) \geq \min\{\mu(x \odot y \odot y), \mu(0)\} = \mu(x \odot y \odot y)$. Applying (b3) and (b5) we get

$$x \odot y \odot y \leq x \odot y \odot (z \vee y) = x \odot y \odot (z \oplus z^{\sim} \odot y) \leq x \odot y \odot z \oplus z^{\sim} \odot y.$$

Since μ is a fuzzy ideal, we have

$$\mu(x \odot y \odot y) \geq \mu(x \odot y \odot z \oplus z^{\sim} \odot y) \geq \min\{\mu(x \odot y \odot z), \mu(z^{\sim} \odot y)\}.$$

Thus μ satisfies the condition (3), i.e., μ is fuzzy implicative.

Using the level subset of a fuzzy set, we give a characterization of a fuzzy ideal.

Theorem 3.15. *Let* μ *be a fuzzy set in a pseudo MV-algebra M. Then* μ is a fuzzy ideal of M if and only if its nonempty level subset

$$U(\mu; \alpha) := \{x \in M \mid \mu(x) \geq \alpha\}$$

is an ideal of M for all $\alpha \in [0, 1]$.

Proof. Assume that μ is a fuzzy ideal of M and let $\alpha \in [0, 1]$ be such that $U(\mu; \alpha) \neq \theta$. Obviously $0 \in U(\mu; \alpha)$. Let $x, y \in M$ be such that $x, y \in U(\mu; \alpha)$. Then $\mu(x) \geq \alpha$. and $\mu(y) \geq \alpha$. It follows from (d1) that

$$\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\} \geq \alpha$$

so that $x \oplus y \in U(\mu; \alpha)$. Let $x, y \in M$ be such that $x \in U(\mu; \alpha)$ and $y \leq x$. Then $\mu(y) \geq \mu(x) \geq \alpha$ by (d2), and so $y \in U(\mu; \alpha)$. Therefore $U(\mu; \alpha)$ is an ideal of M.

Conversely suppose that $U(\mu; \alpha)$ is a nonempty ideal of M for all $\alpha \in [0, 1]$. If (d1) is not valid, then there exist $a, b \in M$ such that $\mu(a \oplus b) < \min\{\mu(a), \mu(b)\}$. Taking

$$\beta = \frac{1}{2}(\mu(a \oplus b) + \min\{\mu(a), \mu(\mathrm{b})\}),$$

we get $\mu(a\oplus b) < \beta < \min\{\mu(a), \mu(b)\}$. Therefore $a, b \in U(\mu; \beta)$ but $a\oplus b \notin U(\mu;\beta)$. This is a contradiction, and (d1) is valid. Finally let $x, y \in M$ be such that $y \leq x$. Assume that $\mu(y) < \mu(x)$ and let $\gamma = \frac{1}{2}(\mu(y) + \mu(x))$. Then $\mu(y) < \gamma < \mu(x)$ and thus $x \in U(\mu; \gamma)$ and $y \notin U(\mu; \gamma)$. This is impossible, and μ is a fuzzy ideal of M.

Corollary 3.16. *If* μ *is a fuzzy ideal of a pseudo MV-algebra* M, *then the set*

$$M_a := \{x \in M \mid \mu(x) \geq \mu(a)\}$$

is an ideal of M for every a ∈ M.

Proof. Straightforward.

Theorem 3.17. *Let* μ *be a fuzzy set in a pseudo MV-algebra* M. *Then* μ *satisfies the condition* (3) *if and only if for all* $x, y, z \in M$ *and* $\alpha \in [0, 1]$, *whenever* $x \odot y \odot z \in U(\mu; \alpha)$ *and* $z^{\sim} \odot y \in U(\mu; \alpha)$ *then* $x \odot y \in U(\mu; \alpha)$.

Proof. Let $x, y, z \in M$ and $\alpha \in [0, 1]$ be such that $x \odot y \odot z \in U(\mu; \alpha)$ and $z^{\sim} \odot y \in U(\mu; \alpha)$. Then $\mu(x \odot y \odot z) \geq \alpha$ and $\mu(z^{\sim} \odot y) \geq \alpha$. It follows from (3) that

$$\mu(x \odot y) \geq \min\{\mu(x \odot y \odot z), \mu(z^{\sim} \odot y)\} \geq \alpha$$

so that $x \odot y \in U(\mu; \alpha)$. Conversely, if the condition (3) is not valid, then

$$\mu(a \odot b) < \min\{\mu(a \odot b \odot c), \mu(c^{\sim} \odot b)\}$$

for some $a, b, c \in M$. Let

$$\beta := \frac{1}{2}(\mu(a \odot b) + \min\{\mu(a \odot b \odot c), \mu(c^{\sim} \odot b)\}).$$

Then $\mu(a \odot b) < \beta < \min\{\mu(a \odot b \odot c), \mu(c^{\sim} \odot b)\}$, and so $a \odot b \odot c \in U(\mu; \beta)$ and $c^{\sim} \odot b \in U(\mu; \beta)$ but $a \odot b \neq U(\mu; \beta)$. This is a contradiction.

Theorems 3.15 and 3.17 together yield the following corollary.

Corollary 3.18. *Let* μ *be a fuzzy set in a pseudo MV-algebra* M. *Then* μ *is a fuzzy implicative ideal of* M *if and only if for each* $\alpha \in [0, 1]$, $U(\mu; \alpha) = \theta$ *or* $U(\mu; \alpha)$ *is an implicative ideal of* M.

Theorem 3.19. *For a fuzzy set* μ *in a pseudo MV-algebra* M, *let* μ^* *be a fuzzy set in* M *defined by*

$$\mu^*(x) := \sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\}$$

for all $x \in$ M. Then μ^* is the least fuzzy ideal of M that contains μ.

Proof. For any $\beta \in \mathrm{Im}(\mu^*)$, let $\beta_n = \beta - \frac{1}{n}$ for $n \in \mathbb{N}$. Let $x \in U(\mu^*; \beta)$. Then $\mu^*(x) \geq \beta$, which implies that

$$\sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\} \geq \beta > \beta - \frac{1}{n} = \beta_n, \forall_n \in \mathbb{N}.$$

Hence for any $n \in \mathbb{N}$ there exists $\gamma_n \in \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\}$ such that $\gamma_n > \beta_n$. Thus $x \in \langle U(\mu; \gamma_n)\rangle$ for all $n \in \mathbb{N}$. Consequently, $x \in \bigcap_{n\in\mathbb{N}} \langle U(\mu; \gamma_n)\rangle$. On the other hand, if $x \in \bigcap_{n\in\mathbb{N}} \langle U(\mu; \gamma_n)\rangle$, then $\gamma_n \in \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\}$ for any $n \in \mathbb{N}$. Therefore

$$\beta - \frac{1}{n} = \beta_n < \gamma_n \leq \sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\} = \mu^*(x), \forall_n \in \mathbb{N}.$$

Since n is arbitrary, it follows that $\beta \leq \mu^*(x)$ so that $x \in U(\mu^*; \beta)$. Hence $U(\mu^*;\beta) = \bigcap_{n\in\mathbb{N}} \langle U(\mu; \gamma_n)\rangle$, which is an ideal of M. Therefore we conclude that μ^* is a fuzzy ideal of M by Theorem 3.15. We now prove that μ^* contains μ. For any $x \in M$, let $\beta \in \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$. Then $x \in U(\mu; \beta)$ and thus $x \in \langle U(\mu; \beta)\rangle$. Therefore $\beta \in \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\}$, which implies that

$$\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\} \subseteq \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\}.$$

It follows that

$$\begin{aligned}\mu(x) &\leq \sup \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\} \\ &\leq \sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha)\rangle\} \\ &= \mu^*(x)\end{aligned}$$

which shows that μ^* contains μ. Finally, let ν be a fuzzy ideal of M containing μ. Let

$x \in M$ and let $\mu^*(x) = \beta$. Then $x \in U(\mu^*; \beta) = \bigcap_{n \in \mathbb{N}} \langle U(\mu; \gamma_n) \rangle$, and so $x \in \langle U(\mu; \gamma_n) \rangle$ for all $n \in \mathbb{N}$. Consequently,

$$x \leq y_1 \oplus \ldots \oplus y_k \text{ for some } y_1, \ldots, y_k \in U(\mu, \gamma_n)$$

(see the last paragraph of Section 2). It is easy to check that

$$\mu(x) \geq \min \{\mu(y_1), \ldots, \mu(y_k)\} \geq \gamma_n.$$

Then $\nu(x) \geq \mu(x) \geq \gamma_n > \beta_n = \beta - \frac{1}{n}$ for every $n \in \mathbb{N}$, so that $\nu(x) \geq \beta = \mu^*(x)$ since n is arbitrary. This shows that $\mu^* \subseteq \nu$, and the proof is complete.

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