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FUZZY QUOTIENT SPACE OF A VECTOR SPACE

ABSTRACT: In this paper we have introduced the concept of cosets of a subspace in a vector space V generated by an element of the space and a translational invariant fuzzy subset μ of the space. We have proved some results analogous to certain basic results of the classical vector space . Finally we have defined the quotient space of V generated by a subspace and μ.

Keywords: *Cosets, Translational invariant fuzzy subset , Quotient space.*

1. INTRODUCTION

The notion of *fuzzy subset* was initiated by Zadeh [3]. Rosenfeld introduced the concept of *fuzzy subgroups* in his classical paper [2] in 1971. Ray [1] introduced the concept of *translational invariant fuzzy subset*. In [1] Ray obtained *quotient group* of a *group* generated by a *subgroup* and a *fuzzy subset*. Ali and Ray [4] obtained *quotient ring* of a *ring* generated by an *ideal* and a *fuzzy subset*. In this paper the result is extended to *vector space*.

2. PRELIMINARIES

Let $*$ be a *binary operation* on a nonempty *set* S and μ be a *fuzzy subset* of S.

Definition 2.1. [1] μ is said to be *left translational invariant* with respect to $*$ if $\mu(x) = \mu(y) \Rightarrow \mu(a * x) = \mu(a*y)\ \forall\ x , y , a \in S$.

Definition 2.2. [1] μ is said to be *right translational invariant* with respect to $*$ if $\mu(x) = \mu(y) \Rightarrow \mu(x*a) = \mu(y*a)\ \forall\ x, y , a \in S$.

Definition 2.3. [1] μ is said to be *translational invariant* with respect to $*$ if μ is both *left* and *right translational invariant* with respect to $*$.

Remark 2.4. If μ is *commutative*, i.e., $\mu(x*y) = \mu(y*x)\ \forall\ x , y \in S$, then μ is *left translational invariant* if and only if μ is *right translational invariant*.

The above notion of *translational invariant fuzzy subset* can be extended to any *set* with more than one *binary operation*.

Example 2.5. Consider the *ring* $Z_6 = \{0, 1, 2, 3, 4, 5\}$, the *ring* of *integers modulo* 6.

Let μ be a *fuzzy subset* of Z_6 defined as follows **:**

$$\mu(0) = \mu(3) = 1$$

$$\mu(1) = \mu(4) = .5$$

$$\mu(2) = \mu(5) = .3$$

It can be easily verified that μ is a *translational invariant fuzzy subset* of Z_6 with respect to *addition* and *multiplication modulo* 6.

Definition 2.6. A *fuzzy subset* μ of a *vector space* $V(F)$ is said to be *translational invariant* if μ is *translational invariant* with respect to both *vector addition* and *scalar multiplication* i.e.,

$\mu(a) = \mu(b) \Rightarrow \mu(a + x) = \mu(b + x)$ and

$\mu(a) = \mu(b) \Rightarrow \mu(\alpha a) = \mu(\alpha b)\ \forall\ a, b, x \in V$ and $\forall\ \alpha \in F$.

Example 2.7. Let F be any *field*. Consider the *vector space* $V = F[x]$, the set of all polynomials in x over F.

Define $\mu : V \rightarrow [0, 1]$ as $\mu(f) = 1/\text{deg.}f$, $\text{deg.}f \neq 0$
$= 1$, otherwise.

Then μ is *translational invariant* with respect to *scalar multiplication* but not with respect to *vector addition*.

However $\nu : V \rightarrow [0, 1]$ define as $\nu(f)$ = constant term of f, is *translational invariant* with respect to both *vector addition* and *scalar multiplication*.

Definition 2.8. Suppose W is a *subspace* of a *vector space* V and μ is a *fuzzy subset* of V. Suppose $a \in V$, and consider the *subset* $C(a, \mu, W)$ of V, given as follows**:**

$C(a, \mu, W) = \{x \in V : \mu(x) = \mu(a + w)$, for some $w \in W\}$.

We call $C(a, \mu, W)$ the *coset* of W in V generated by a and μ.

Proposition 2.9. $a \in C(a, \mu, W)$ and $a + W \subseteq C(a, \mu, W)\ \forall\ a \in V$.

Proof. Since $\mu(a) = \mu(a + 0)\ \forall\ a \in V$ it follows that $a \in C(a, \mu, W)\ \forall\ a \in V$.

Now let $x \in a + W$, then $x = a + w$ for some $w \in W$.

Hence $\mu(x) = \mu(a + w)$, $w \in W$, and so $x \in C(a, \mu, W)$.

Consequently $a + W \subseteq C(a, m, W)\ \forall\ a \in V$.

Example 2.10. Consider the *vector space* $\boldsymbol{R}^2$ over $\boldsymbol{R}$ and fix a *vector* $v = (v_1, v_2)$ in $\boldsymbol{R}^2$.

Let μ be a *fuzzy subset* of $\boldsymbol{R}^2$ satisfying $\mu(a) = \mu(b)$ if and only if $b - a = \text{n}v\ \text{n} \in \boldsymbol{Z}$.

Then μ is a *translational invariant* with respect to both *vector addition* as well *scalar multiplication* in $\boldsymbol{R}^2$.

Consider the *vector subspace* $W = \langle(1, 1)\rangle$ and let $a = (1, 2)$.

Then $a + W$ = set of all *vectors* lying on the line passing through (1, 2) and parallel to (1, 1). whereas $C(a, \mu, W)$ = set of all *vectors* lying on the above line as well as on lines parallel to it and at distances integral multiple of $|v_1 - v_2| \left.\right/ \tfrac{1}{\sqrt{2}}$ from it .

So here we see that $a + W$ is a *proper subset* of $C(a, \mu, W)$.

3. SOME RESULTS

In this section we have proved some results analogous to certain basic results of classical *vector space*.

Theorem 3.1. Suppose W is a *subspace* of the *vector space* V and μ is a *fuzzy subset* of V. Let $a, b \in V$. If $a - b \in W$ then

$C(a, \mu, W) = C(b, \mu, W)$.

Proof. Let $a - b \in W$.

Then $x \in C(a, \mu, W)$

$\Rightarrow \mu(x) = \mu(a + w),\ w \in W$

$\Rightarrow \mu(x) = \mu(b - b + a + w),\ -b + a + w \in W$

$\Rightarrow x \in C(b, \mu, W)$.

Hence $C(a, \mu, W) \subseteq C(b, \mu, W)$

Similarly, $C(b, \mu, W) \subseteq C(a, \mu, W)$

Thus we get $C(a, \mu, W) = C(b, \mu, W)$.

Corollary 3.2. Suppose W is a *subspace* of the *vector space* V and μ is a *fuzzy subset* of V. Let $a \in V$. If $a \in W$, then $C(a, \mu, W) = C(0, \mu, W)$, where 0 is the *zero element* of V.

Proof. In Theorem 3.1 if we take $b = 0$, we shall get the required result.

Henceforth, unless otherwise mentioned, μ is always assumed to be a *translational invariant fuzzy subset* of V and W is assumed to be a *subspace* of V.

Proposition 3.3. Let a, b $\in V$. Then

$C(a, \mu, W) = C(b, \mu, W) \Leftrightarrow b \in C(a, \mu, W)$.

Proof. Let $C(a, \mu, W) = C(b, \mu, W)$.

As $b \in C(b, \mu, W)$, we have $b \in C(a, \mu, W)$.

Now $b \in C(a, \mu, W)$ implies $\mu(b) = \mu(a + w)$, $w \in W$

which gives $\mu(a) = \mu(b - w)$.

Now $x \in C(a, \mu, W)$

$\Rightarrow \mu(x) = \mu(a + w_1)$, $w_1 \in W$

$\Rightarrow \mu(x) = \mu(b - w + w_1)$, $-w + w_1 \in W$

$\Rightarrow x \in C(b, \mu, W)$.

Hence $C(a, \mu, W) \subseteq C(b, \mu, W)$.

Again $x \in C(b, \mu, W)$

$\Rightarrow \mu(x) = \mu(b + w_2)$, $w_2 \in W$

$\Rightarrow \mu(x) = \mu(a + w + w_2)$, $w + w_2 \in W$

$\Rightarrow x \in C(a, \mu, W)$.

Hence $C(b, \mu, W) \subseteq C(a, \mu, W)$.

Consequently $C(a, \mu, W) = C(b, \mu, W)$.

Similarly we can prove **:**

Proposition 3.4. Let $a, b \in V$. Then

$C(a, \mu, W) = C(b, \mu, W) \Leftrightarrow a \in C(b, \mu, W)$.

Theorem 3.5. Let $a, b \in V$. Then either $C(a, \mu, W)$ and $C(b, \mu, W)$ are disjoint or $C(a, \mu, W) = C(b, \mu, W)$.

Proof. Suppose $C(a, \mu, W)$ and $C(b, \mu, W)$ are not disjoint. Then there exists $x \in V$ such that $x \in C(a, \mu, W)$ and $x \in C(b, \mu, W)$.

Now $x \in C(a, \mu, W) \Rightarrow C(x, \mu, W) = C(a, \mu, W)$ and

$x \in C(b, \mu, W) \Rightarrow C(x, \mu, W) = C(b, \mu, W)$.

Hence $C(a, \mu, W) = C(b, \mu, W)$.

This proves the theorem.

Theorem 3.6. Let $a, b \in V$. If $a - b \in C(0, \mu, W)$ or $b - a \in C(0, \mu, W)$, then $C(a, \mu, W) = C(b, \mu, W)$.

Proof. Let $a - b \in C(0, \mu, W)$. Then $\mu(a - b) = \mu(0 + w) = \mu(w), w \in W$.

From which we get $\mu(a) = \mu(b + w)$.

Now $\mu(a) = \mu(b + w) \Rightarrow a \in C(b, \mu, W)$

$\Rightarrow C(a, \mu, W) = C(b, \mu, W)$.

Similar is the case if $b - a \in C(0, \mu, W)$.

Theorem 3.7. Let $a, b \in V$. If $C(a, \mu, W) = C(b, \mu, W)$ then

$a - b \in C(0, \mu, W)$ or $b - a \in C(0, \mu, W)$.

Proof. Let $C(a, \mu, W) = C(b, \mu, W)$, then by Theorem 3 .4 we have $a \in C(b, \mu, W)$ which implies $\mu(a) = \mu(b + w)$, for some $w \in W$.

Therefore $\mu(a - b) = \mu(w) \Rightarrow \mu(a - b) = \mu(0 + w) \Rightarrow a - b \in C(0, \mu, W)$.

Similarly we can show that $b - a \in C(0, \mu, W)$.

Theorem 3.8. Let $a \in V$. Then $C(a, \mu, W) = \cup(x + W), x \in C(a, \mu, W)$.

Proof. It is known that $a + W \subseteq C(a, \mu, W)$. If $a + W = C(a, \mu, W)$ then the theorem is proved. If not, let $b \in C(a, \mu, W) - (a + W)$.

Since $b \in C(a, \mu, W)$, we have $C(b, \mu, W) = C(a, \mu, W)$.

Also since b does not belong to $a + W$, so $b + W$ and $a + W$ are disjoint.

We observe that $b + W \subseteq C(b, \mu, W) = C(a, \mu, W)$.

If $C(a, \mu, W) = (a + W) \cup (b + W)$, we are done.

If not , we shall consider all mutually disjoint *cosets* of W formed by the *elements* of $C(a, \mu, W)$ and ultimately get the desired result.

Theorem 3.9. V is *partitioned* into disjoint *cosets* of W generated by the *elements* of V and the *fuzzy subset* μ.

Proof. For each $a \in V$, we have $a \in C(a, \mu, W)$.

Also for any $b \in V$, if $b \in C(a, \mu, W)$ then $C(a, \mu, W) = C(b, \mu, W)$.

Hence $V = \cup\{C(a, \mu, W), a \in V\}$.

This completes the proof.

4. FUZZY QUOTIENT SPACE GENERATED BY A FUZZY SUBSET

Let V be a *vector space* and W be a *subspace* of V. Suppose μ is a *translational invariant fuzzy subset* of V.

Let $C(V, \mu, W) = \{C(a, \mu, W) : a \in V\}$.

Let $C(a, \mu, W), C(b, \mu, W) \in C(V, \mu, W)$.

Suppose $C(x, \mu, W) = C(a, \mu, W)$ and $C(y, \mu, W) = C(b, \mu, W)$.

Then $x \in C(a, \mu, W)$ and $y \in C(b, \mu, W)$

$\Rightarrow \mu(x) = \mu(a + w_1)$ and $\mu(y) = \mu(b + w_2)$, $w_1, w_2 \in W$

$\Rightarrow \mu(x + y) = \mu(a + w_1 + y) = \mu(a + y + w_1) = \mu(a + b + w_2 + w_1)$

$\Rightarrow x + y \in C(a + b, \mu, W)$, since $w_2 + w_1 \in W$

$\Rightarrow C(x + y, \mu, W) = C(a + b, \mu, W)$.

Thus if $C(x, \mu, W) = C(a, \mu, W)$ and $C(y, \mu, W) = C(b, \mu, W)$,

then $C(x + y, \mu, W) = C(a + b, \mu, W)$.

Again suppose $C(a, \mu, N) = C(b, \mu, N)$,

Then $a \in C(b, \mu, W)$

$\Rightarrow\ \mu(a) = \mu(b + w),\ w \in W$

$\Rightarrow\ \mu(\alpha a) = \mu(\alpha b + \alpha w),\ \alpha \in F$

$\Rightarrow\ \alpha a \in C(\alpha b, \mu, W)$, since $\alpha w \in W$

$\Rightarrow\ C(\alpha a, \mu, W) = C(\alpha b, \mu, W)$.

Thus if $C(a, \mu, W) = C(b, \mu, W)$,

then $C(\alpha a, \mu, W) = C(\alpha b, \mu, W)$.

Hence we can define one *binary operation* called *vector addition* and *a scalar multiplication*, in $C(V, \mu, W)$, the *set* of all *cosets* of W in V generated by μ, as follows :

For any $C(a, \mu, W)$, $C(b, \mu, W) \in C(V, \mu, W)$ and $\alpha \in F$

$C(a, \mu, W) + C(b, \mu, W) = C(a + b, \mu, W)$ and

$\alpha C(a, \mu, W) = C(\alpha a, \mu, W)$.

Theorem 4.1. Let μ be a *translational invariant fuzzy subset* of a *vector space* V and W a *subspace* of V. Then $C(V, \mu, W)$, the *set* of all *cosets* of W in V generated by μ, is a *vector space* with respect to the *vector addition* and *scalar multiplication* defined by

$C(a, \mu, W) + C(b, \mu, W) = C(a + b, \mu, W)$, and

$\alpha C(a, \mu, W) = C(\alpha a, \mu, W)$, where

$C(a, \mu, W)$, $C(b, \mu, W) \in C(V, \mu, W)$ and $\alpha \in F$.

Proof. Let $C(a, \mu, W)$, $C(b, \mu, W)$, $C(d, \mu, W) \in C(V, \mu, W)$.

Then $(C(a, \mu, W) + C(b, \mu, W)) + C(d, \mu, W)$

$= C(a + b, \mu, W) + C(d, \mu, W)$

$= C((a + b) + d, \mu, W)$

$= C(a + (b + d), \mu, W)$

$= C(a, \mu, W) + (C(b, \mu, W) + C(d, \mu, W))$.

Also $C(a, \mu, W) + C(b, \mu, W) = C(a + b, \mu, W)$

$= C(b + a, \mu, W)$

$= C(b, \mu, W) + C(a, \mu, W)$.

$(0, \mu, W)$ is the *zero element* of $C(V, \mu, W)$.

$(- a, \mu, W)$ is the *additive inverse* of $C(a, \mu, W)$.

Therefore W is an *abelian group*.

Further we have

$$\alpha \{C(a, \mu, W) + (C(b, \mu, W)\} = \alpha C((a+b), \mu, W)$$
$$= C(\alpha (a+b), \mu, W)$$
$$= C((\alpha a + \alpha b), \mu, W)$$
$$= C(\alpha a, \mu, W) + C(\alpha b, \mu, W)$$
$$= \alpha C(a, \mu, W) + \alpha C(b, \mu, W).$$

Again ,

$$(\alpha + \beta) C(a, \mu, W) = C((\alpha + \beta) a, \mu, W)$$
$$= C(\alpha a + \beta a, \mu, W)$$
$$= C(\alpha a, \mu, W) + C(\beta a, \mu, W)$$
$$= \alpha C(a, \mu, W) + \beta C(a, \mu, W).$$

Also,

$$\alpha\{\beta C(a, \mu, W)\} = \alpha C(\beta a, \mu, W)$$
$$= C(\alpha(\beta a), \mu, W)$$
$$= C((\alpha\beta)a, \mu, W) = (\alpha\beta) C(a, \mu, W).$$

And,

$1C(a, \mu, W) = C(1a, \mu, W) = C(a, \mu, W)$, where 1 is the *identity* of F.

Hence $C(V, \mu, W)$ is a *vector space* .

Definition 4.2. The *vector space* $C(V, \mu, W)$ is called the *fuzzy quotient space* or *factor space* of V generated by W and μ.

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