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FUZZY IDEALS WITH OPERATORS IN *BCC*-ALGEBRAS

ABSTRACT: *The notions of BCC-algebras with operators (brie,* Ω–BCC *algebras) and* Ω*–fuzzy* BCC*-ideals of* Ω-BCC*-algebras are given. Some properties of* Ω*-fuzzy* BCC*-ideals of* Ω-BCC*-algebras are investigated.*

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1. INTRODUCTION

In 1966, Y. Imai and K. Iséki [8] defined a class of algebras of type (2,0) called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra. The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [11]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori [9] introduced a notion of *BCC*-algebras, and W. A. Dudek [1, 2] redefined the notion of *BCC*-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [6], W. A. Dudek and X. H. Zhang introduced a notion of *BCC*-ideals in *BCC*-algebras, and W. A. Dudek and Y. B. Jun established the fuzzification of *BCC*-ideals in *BCC*-algebras (see [3, 4, 5]). In this paper, we introduce the notions of *BCC*-algebras with operators (briefly, Ω-*BCC*-algebras) and Ω-fuzzy *BCC*-ideals of Ω-*BCC*-algebras. We investigate some properties of Ω-fuzzy *BCC*-ideals of Ω-*BCC*-algebras.

2. BASIC CONCEPTS AND RESULTS

We record here some basic concepts and clarify notions which are used in the sequel. For more details, we refer to the textbook BCK-Algebras (Meng and Jun [10]) and references which are given in this article.

Recall that a *BCC-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms:

(C1) $((x * y) * (z * y)) * (x * z) = 0$,

(C2) $0 * x = 0$,

(C3) $x * 0 = x$,

(C4) $x * y = 0$ and $y * x = 0$ imply $x = y$:

for every $x, y, z \in X$. For any *BCC*-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X. In a *BCC*-algebra X, the following holds (see [7]).

- $x * x = 0$,
- $x * y \leq x$,
- $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

for all $x, y \in X$. A nonempty subset S of a *BCC*-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A nonempty subset A of a *BCC*-algebra X is called a *BCK-ideal* of X if it satisfies

- $0 \in A$,
- $(\forall x \in X)\ (\forall y \in A)\ (x * y \in A \Rightarrow x \in A)$.

A nonempty subset A of a *BCC*-algebra X is called a *BCC-ideal* of X if it satisfies

- $0 \in A$,
- $(\forall x, z \in X)\ (\forall y \in A)\ ((x * y) * z \in A \Rightarrow x * z \in A)$.

Note that every *BCC*-ideal of a *BCC*-algebra X is a subalgebra of X. For a fuzzy set μ in X and $t \in [0, 1]$, the set $\mu_t := \{x \in X \mid \mu(x) \geq t\}$ is called a *level set* of μ. A fuzzy set μ in a *BCC*-algebra X is called a *fuzzy subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy set μ in a *BCC*-algebra X is called a *fuzzy BCK-ideal* of X it it satisfies

- $(\forall x \in X)\ (\mu(0) \geq \mu(x))$.
- $(\forall x, y \in X)\ (\mu(x) \geq \min\{\mu(x * y), \mu(y)\})$.

A fuzzy set μ in a *BCC*-algebra X is called a *fuzzy BCC-ideal* of X if it satisfies

- $(\forall x \in X)\ (\mu(0) \geq \mu(x))$.
- $(\forall x, y, z \in X)\ (\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\})$.

3. FUZZY *BCC*-IDEALS WITH OPERATORS

We begin with the definition of a *BCC*-algebra with operators.

Definition 3.1. A *BCC-algebra with operators* is an algebraic system consisting of a *BCC*-algebra X, a nonempty set Ω and a function defined on the product set $\Omega \times X$ and having values in X such that, if ax denotes the element in X determined by the element x of X and the element α of Ω, then $\alpha(x * y) = \alpha x * \alpha y$ holds for any $x, y \in X$ and $\alpha \in \Omega$. We shall usually use the phrase "X is an Ω-*BCC*-algebra" to a *BCC*-algebra with operators. A subalgebra S of an Ω-*BCC*-algebra X is said to be an Ω-*subalgebra* of X if $\alpha x \in S$ for every $\alpha \in \Omega$ and $x \in S$. A *BCC*-ideal A of an Ω-*BCC*-algebra X is said to be an Ω-*BCC-ideal* of X if $\alpha x \in A$ for every $\alpha \in \Omega$ and $x \in A$.

Example 3.2. Let X be a *BCC*-algebra and let Ω be a nonempty set. If we define $\alpha x = 0$ (or, $\alpha x = x$) for all $x \in X$ and $\alpha \in \Omega$, then X is an Ω-*BCC*-algebra.

Proposition 3.3. *Let X be an Ω-BCC-algebra. For each $\alpha \in \Omega$, the set*

$$X_\alpha := \{x \in X \mid \alpha x = x\}$$

is a subalgebra of X. Moreover, if X satisfies the right cancellation law, then X_α is a BCK-ideal of X.

Proof. Let $x, y \in X_\alpha$ for each $\alpha \in \Omega$. Then $\alpha x = x$ and $\alpha y = y$, which imply that $\alpha(x * y) = \alpha x * \alpha y = x * y$, i.e., $x * y \in X_\alpha$. Hence X_α is a subalgebra of X. Assume that X satisfies the right cancellation law and let $x, y \in X$ be such that $x * y \in X_\alpha$ and $y \in X_\alpha$. Then $\alpha x * y = \alpha x * \alpha y = \alpha(x * y) = x * y$, and hence $\alpha x = x$ by the right cancellation law. Thus $x \in X_\alpha$, and X_α is a *BCK*-ideal of X.

Proposition 3.4. *In an Ω-BCC-algebra X, the following hold:*

(i) $(\forall \alpha \in \Omega)\ (\alpha 0 = 0)$.

(ii) $(\forall x, y \in X)\ (\forall \alpha \in \Omega)\ (x \leq y \Rightarrow \alpha x \leq \alpha y)$.

Proof. (i) For any $x \in X$ and $\alpha \in \Omega$ we have $\alpha 0 = \alpha(x * x) = \alpha x * \alpha x = 0$.

(ii) Let $x, y \in X$ be such that $x \leq y$ and let $\alpha \in \Omega$. Then $\alpha x * \alpha y = \alpha(x * y) = \alpha 0 = 0$, and so $\alpha x \leq \alpha y$.

Proposition 3.5. *Let X be an* Ω-*BCC-algebra. For each* $\alpha \in \Omega$, *the set*

$$X(\alpha, 0) := \{x \in X \mid \alpha x = 0\}$$

is a BCC-ideal of X.

Proof. Proposition 3.4 (i) implies $0 \in X(\alpha, 0)$. Let $x, y, z \in X$ and $\alpha \in \Omega$ be such that $(x * y) * z \in X(\alpha, 0)$ and $y \in X(\alpha, 0)$. Then

$$\alpha(x * z) = \alpha x * \alpha z = (\alpha x * 0) * \alpha z = (\alpha x * \alpha y) * \alpha z$$

$$= \alpha(x * y) * \alpha z = \alpha((x * y) * z) = 0,$$

and so $x * z \in X(\alpha, 0)$. Thus $X(\alpha, 0)$ is a *BCC*-ideal of X.

Definition 3.6. Let X be an Ω-*BCC*-algebra. A fuzzy set μ in X is called an Ω-*fuzzy subalgebra* (resp. Ω-*fuzzy BCK-ideal,* Ω-*fuzzy BCC-ideal*) of X if it is a fuzzy subalgebra (resp. *fuzzy BCK-ideal, fuzzy BCC-ideal*) of X such that the following inequality:

$$(\forall x \in X)\ (\forall \alpha \in \Omega)\ (\mu(\alpha x) \geq \mu(x)).$$

Example 3.7. Let μ be a fuzzy set in an Ω-*BCC*-algebra X defined by

$$\mu(x) := \begin{cases} 0.7 & \textit{if } \ \alpha x = 0 \ \textit{ for each } \ \alpha \in \Omega, \\ 0.4 & \textit{otherwise}. \end{cases}$$

It is easy to verify that μ is an Ω-fuzzy *BCC*-ideal of X.

Proposition 3.8. *If* μ *is an* Ω-*fuzzy subalgebra of an* Ω-*BCC-algebra X*, then

(i) $(\forall x, y \in X)\ (\forall \alpha \in \Omega)\ (\mu(\alpha(x * y)) \geq \min\{\mu(\alpha x), \mu(\alpha y)\})$.

(i) $(\forall x \in X)\ (\forall \alpha \in \Omega)\ (\mu(\alpha 0)\ (= \mu(0)) \geq \mu(\alpha x))$.

Proof. Straightforward.

Given Ω-fuzzy *BCC*-ideal, we make a fuzzy *BCC*-ideal.

Theorem 3.9. *Let* μ *be an* Ω-*fuzzy BCC-ideal of an* Ω-*BCC-algebra X. For any* $\alpha \in \Omega$, *let* ν *be a fuzzy set in X given by* $\nu(x) = \mu(\alpha x)$ *for all* $x \in$ X. Then ν is a fuzzy BCC-ideal of X.

Proof. Using Proposition 3.4(i), we have

$$\nu(0) = \mu(\alpha 0) = \mu(0) \geq \mu(\alpha x) = \nu(x)$$

for all $x \in X$ and $\alpha \in \Omega$. For any $x, y, z \in X$ and $\alpha \in \Omega$, we get

$$\nu(x * z) = \mu(\alpha(x * z)) = \mu(\alpha x * \alpha z) \geq \min\{\mu((\alpha x * \alpha y) * \alpha z), \mu(\alpha y)\}$$
$$= \min\{\mu(\alpha((x * y) * z), \mu(\alpha y)\} = \min\{\nu((x * y) * z), \nu(y)\}.$$

Therefore ν is a fuzzy *BCC*-ideal of X.

Theorem 3.10. *Let X be an* Ω*-BCC-algebra. If* μ *is an* Ω*-fuzzy BCC-ideal of X, then for each* $\alpha \in \Omega$ *the set*

$$G_\alpha := \{x \in X \mid \mu(\alpha x) = \mu(0)\}$$

is an Ω*-BCC-ideal of X.*

Proof. Clearly $0 \in G_\alpha$. Let $x, y, z \in X$ be such that $(x * y) * z \in G_\alpha$ and $y \in G_\alpha$. Then

$$\mu(\alpha(x * z)) = \mu(\alpha x * \alpha z) \geq \min\{\mu((\alpha x * \alpha y) * \alpha z), \mu(\alpha y)\}$$
$$= \min\{\mu(\alpha((x * y) * z), \mu(\alpha y)\} = \mu(0),$$

and so $x * z \in G_\alpha$. This completes the proof.

If μ is an Ω-fuzzy *BCC*-ideal of an Ω-*BCC*-algebra X, then μ_t, $t \in [0, 1]$, is either empty or a *BCC*-ideal of X (see [3, Theorem 4.9]). Let $x \in \mu_t$ and $\alpha \in \Omega$. Then $\mu(\alpha x) \geq \mu(x) \geq t$, and so $\alpha x \in \mu_t$. Thus μ_t is an Ω-*BCC*-ideal of X. Now let μ be a fuzzy set in an Ω-*BCC*-algebra X for which every nonempty level set is an Ω-*BCC*-ideal of X. Then μ is a fuzzy *BCC*-ideal of X (see [3, Theorem 4.9]). Assume that there exists $x \in X$ and $\alpha \in \Omega$ such that $\mu(\alpha x) < \mu(x)$. Taking $t = \frac{1}{2}(\mu(\alpha x) + \mu(x))$ implies that $\mu(\alpha x) < t < \mu(x)$, and so $\alpha x \notin \mu_t$ and $x \in \mu_t$. This contradicts to the fact that every nonempty level set is an Ω-*BCC*-ideal of X. This shows that μ is an Ω-fuzzy *BCC*-ideal of *X:* Hence we have the following theorem.

Theorem 3.11. *A fuzzy set* μ *in an* Ω*-BCC-algebra X is an* Ω*-fuzzy BCC-ideal of X if and only if for every t* $\in [0, 1]$ *the level set* μ_t *is either empty or an* Ω*-BCC-ideal of X.*

Theorem 3.12. *Let A be a nonempty subset of an* Ω*-BCC-algebra X and* let ν be a fuzzy set in X defined by

$$\nu(x) := \begin{cases} s & \text{if } x \in A, \\ t & \text{otherwise}, \end{cases}$$

for all $x \in X$ *and* $s > t$ *in* [0, 1]. *Then* ν *is an* Ω*-fuzzy BCC-ideal of X if and only if A is an* Ω*-BCC-ideal of X*.

Proof. Note that

$$\nu_r := \begin{cases} X & \text{if } 0 \le r \le t, \\ A & \text{if } t < r \le s, \\ \phi & \text{if } s < r \le 1. \end{cases}$$

Hence the proof follows from Theorem 3.11.

Proposition 3.13. *Every* Ω*-fuzzy BCK-ideal* μ *of an* Ω*-BCC-algebra X satisfies the following inequality*:

$$(\forall x, y \in X)\ (\forall \alpha \in \Omega)\ (\mu(\alpha(x * y)) \ge \min \{\mu(\alpha x), \mu(\alpha y)\}).$$

Proof. Since $x * y \le x$ for all $x, y \in X$, $\alpha(x * y) \le \alpha x$ by Proposition 3.4 (ii). Since μ is order reversing, it follows that

$$\mu(\alpha(x * y)) \ge \mu(\alpha x) \ge \min \{\mu(\alpha x * \alpha y), \mu(\alpha y)\}$$
$$= \min \{\mu(\alpha(x * y)), \mu(\alpha y)\} \ge \min \{\mu(\alpha x), \mu(\alpha y)\},$$

which completes the proof.

Let $\{A_t \mid t \in T\}$, where $\phi \ne T \subseteq [0, 1]$, be a collection of Ω-*BCC*-ideals of an Ω-*BCC*-algebra X such that

$$(\forall s, t \in T)\ (s > t \Leftrightarrow A_s \subset A_t).$$

Then $\bigcup_{t \le s} A_s$ and $\bigcap_{s < t} A_s$ are Ω-*BCC*-ideals of X. Combining Theorem 3.11 and [5, Proposition 3.8] induce the following theorem.

Theorem 3.14. *Let* $\{A_t \mid t \in T\}$, *where* $\phi \ne T \subseteq [0, 1]$, *be a collection of* Ω*-BCC ideals of an* Ω*-BCC-algebra X such that*

(i) $X = \bigcup_{t \in T} A_t$

(ii) $(\forall s, t \in T)\ (s > t \Leftrightarrow A_s \subset A_t)$.

Then a fuzzy set μ *on* X *defined by* $\mu(x) = \sup\ \{t \in T \mid x \in A_t\}$ *for all* $x \in X$ *is an* Ω*-fuzzy BCC-ideal of* X.

Theorem 3.15. *Let* μ *be a fuzzy set in an* Ω*-BCC-algebra* X *and let* $\mathrm{Im}(\mu) = \{t_0, t_1, \ldots, t_n\}$, *where* $t_0 > t_1 > \ldots > t_n$. *If* $A_0 \subset A_1 \subset \ldots \subset A_n = X$ *are* Ω*-BCC-ideals of* X *such that* $\mu(A_k \setminus A_{k-1}) = t_k$ *for* $k = 0, 1, \ldots, n$, *where* $A_{-1} = \phi$ *then* μ *is an* Ω*-fuzzy BCC-ideal of* X.

Proof. Using [5, Proposition 3.11] we know that μ is a fuzzy *BCC*-ideal of X. Let $x \in X$ and $\alpha \in \Omega$. Then $x \in A_k \setminus A_{k-1}$ for some $k \in \{1, 2, \ldots, n\}$. Since A_k is an Ω-*BCC*-ideal of X, $\alpha x \in A_k$. Hence $\mu(\alpha x) \geq t_k = \mu(x)$. This shows that μ is an Ω-fuzzy *BCC*-ideal of X.

Theorem 3.16. *If every* Ω*-fuzzy BCC-ideal* μ *in an* Ω*-BCC-algebra* X *has the finite image, then every descending chain of* Ω*-BCC-ideals of* X *terminates at finite step.*

Proof. Let μ be a fuzzy set in X defined by

$$\mu(x) := \begin{cases} \dfrac{n}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, n = 0, 1, 2, \ldots, \\ 1 & \text{if } x \in \cap A_n, n = 0, 1, 2, \ldots, \end{cases}$$

where $X = A_0 \supset A_1 \supset A_2 \supset \ldots$ is a strictly descending chain of Ω-*BCC*-ideals of X which does not terminate at finite step. Then μ is a fuzzy *BCC*-ideal of X (see [5, Proposition 3.16]). If $x \in A_n \setminus A_{n+1}$ and $\alpha \in \Omega$, then $\alpha x \in A_n$. Hence $\mu(\alpha x) \geq \dfrac{n}{n+1} = \mu(x)$. Now if $x \in A_n$ and $\alpha \in \Omega$, then $\alpha x \in \cap A_n$. Thus $\mu(\alpha x) = 1 = \mu(x)$. Therefore μ is an Ω-fuzzy *BCC*-ideal of X which has an infinite number of different values. This is impossible, and the result is valid.

Theorem 3.17. *Let* μ *be an* Ω*-fuzzy BCC-ideal of an* Ω*-BCC-algebra* X. *If* $\mathrm{Im}(\mu)$ *is a well-ordered subset of* $[0, 1]$, *then every ascending chain of* Ω*-BCC-ideals of* X *terminates at finite step.*

Proof. Let μ be a fuzzy set in X defined by

$$\mu(x) := \begin{cases} 0 & \text{if } x \notin \bigcup_{n \in \mathbb{N}} A_n, \\ \frac{1}{k} & \text{where } k = \min\{n \in \mathbb{N} \mid x \in A_n\} \end{cases}$$

where $A_1 \subset A_2 \subset A_3 \subset \ldots$ is a strictly ascending chain of Ω-*BCC*-ideals of X which does not terminate at finite step. Then μ is a fuzzy *BCC*- ideal of X (see [5, Proposition 3.17]). Now if $x \notin \bigcup_{n \in \mathbb{N}} A_n$, then obviously $\mu(\alpha x) \geq 0 = \mu(x)$ for all $\alpha \in \Omega$. Assume that $x \in \bigcup_{n \in \mathbb{N}} A_n$ and $\alpha \in \Omega$. Then $x \in A_n \setminus A_{n-1}$ for some $n \in \mathbb{N}$. Thus $\alpha x \in A_n$, and hence $\mu(\alpha x) \geq \frac{1}{n} = \mu(x)$. Therefore μ is an Ω-fuzzy *BCC*-ideal of X which has an infinite number of different values. This is impossible, and the result is valid.

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