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REDEFINED FUZZY B-ALGEBRAS

ABSTRACT: *Using the* belongs to *relation* $(\in)$ *and* quasi-coincidence with *relation* (q) *between fuzzy points and fuzzy sets, the concept of* (α, β)*-fuzzy B-algebras where* α *and* β *are any two of* $\{\in, q, \in \vee q, \in \wedge\}$ *with* $\alpha \neq \in \wedge q$ *is introduced, and related properties are investigated. We give a condition for an* $(\in, \in \vee q)$*-fuzzy B-algebra to be an* $(\in, \in)$*-fuzzy B-algebra. We provide characterizations of an* $(\in, \in \vee q)$*-fuzzy B-algebra. We show that a proper* $(\in, \in)$*-fuzzy B-algebra* $\mathscr{A}$ *of* X *with additional conditions can be expressed as the union of two proper non-equivalent* $(\in, \in)$*-fuzzy B-algebras of* X. *We also prove that if* $\mathscr{A}$ *is a proper* $(\in, \in \vee q)$*-fuzzy B-algebra of a B-algebra* X *such that*

$$\# \{ \mathscr{A}(\mathrm{x}) \mid \mathscr{A}(\mathrm{x}) < 0.5\} \geq 2;$$

then there exist two proper non-equivalent $(\in, \in \vee q)$*-fuzzy B-algebras of* X *such that* $\mathscr{A}$ *can be expressed as the union of them.*

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1. INTRODUCTION

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([7, 8]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5, 6] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Recently, the present authors ([9]) introduced a new notion, called a BH-algebra, which is a generalization of BCH/ BCI/BCK-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. The second author together with J. Neggers [13] introduced and investigated a class of algebras, i.e., the class of B-algebras, which is related to several classes of algebras of interest such as BCH/BCI/BCK-algebras and which seems to have rather nice properties without being excessively complicated otherwise. J. R. Cho and H. S. Kim [4] discussed further relations between B-algebras and other classes of algebras, such as quasigroups. It is well known that every group determines a B-algebra, called a *group-derived* B-algebra. It is natural to have a question of interest to determine whether or not all B-algebras are so group-derived. It is proved that this is not the case in general, and thus that this class of algebras contains the class of groups indirectly via the group-derived principle (See [1]). In this paper, using the *belongs to* relation ($\in$) and *quasi-coincidence with* relation (q) between fuzzy points and fuzzy sets, we introduce the concept of (α, β)-fuzzy B-algebras where α and β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and investigate related properties. We give a condition for an ($\in$, $\in \vee$f)-fuzzy B-algebra to be an ($\in$, $\in$)-fuzzy B-algebra. We provide characterizations of an ($\in$, $\in \vee$q)-fuzzy B-algebra. We show that a proper ($\in$, $\in$)-fuzzy B-algebra $\mathscr{A}$ of X with additional conditions can be expressed as the union of two proper non-equivalent ($\in$, $\in$)-fuzzy B-algebras of X. We also prove that if $\mathscr{A}$ is a proper ($\in$, $\in \vee$ q)-fuzzy B-algebra of a B-algebra X such that $\#\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} \geq 2$, then there exist two proper non-equivalent ($\in$, $\in \vee$ q)-fuzzy B-algebras of X such that $\mathscr{A}$ can be expressed as the union of them.

2. PRELIMINARIES

A *B-algebra* is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(i) $(\forall x \in X)\ (x * x = 0)$,

(ii) $(\forall x \in X)\ (x * 0 = x)$,

(iii)$(\forall x, y, x \in X)\ ((x * y) * z = x * (z * (0 * y)))$.

A non-empty subset N of a B-algebra X is called a *B-subalgebra* of X if $x * y \in N$ for any $x, y \in N$. A non-empty subset N of a B-algebra X is said to be *normal* if $(x * a) * (y * b) \in N$ whenever $x * y \in N$ and $a * b \in N$. Note that any normal subset N of a B-algebra X is a B-algebra of X, but the converse need not be true (see [10]). A non-empty subset N of a B-algebra X is called a *normal B-subalgebra* of X if it is both a B-algebra and normal.

Lemma 2.1. [13] *If X is a B-algebra, then* $x * y = x * (0 * (0 * y))$ *for all* $x, y \in X$.

Example 2.2. [13] Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation " $*$ " on X by

$$x * y := \frac{n(x-y)}{n+y}.$$

Then $(X; *, 0)$ is a B-algebra.

Example 2.3. [13] Let $\mathbb{Z}$ be the group of integers under usual addition and let $\alpha \notin \mathbb{Z}$. We adjoin the special element α to $\mathbb{Z}$. Let $X := \mathbb{Z} \cup \{\alpha\}$. Define $\alpha+0 = \alpha$, $\alpha+n = n - 1$ where $n \neq 0$ in $\mathbb{Z}$ and $\alpha + \alpha$ is an arbitrary element in X. Define a mapping $\phi : X \to X$ by $\phi(\alpha) = 1$, $\phi(n) = -n$ where $n \in \mathbb{Z}$. If we define a binary operation "$*$" on X by $x * y := x+\phi(y)$, then $(X; *, 0)$ is a non-group derived B-algebra.

A fuzzy set $\mathscr{A}$ in a set X of the form

$$\mathscr{A}(y) := \begin{cases} t \in (0,1] & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set $\mathscr{A}$ in a set X, Pu and Liu [15] gave meaning to the symbol $x_t \alpha \mathscr{A}$, where $\alpha \in \{\in, \mathrm{q}, \in \vee \mathrm{q}, \in \wedge \mathrm{q}\}$.

To say that $x_t \in \mathscr{A}$ (resp. $x_t \mathrm{q} \mathscr{A}$) means that $\mathscr{A}(x) \geq t$ (resp. $\mathscr{A}(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set $\mathscr{A}$.

To say that $x_t \in \vee \mathrm{q}\ \mathscr{A}$ (resp. $x_t \in \wedge \mathrm{q}\ \mathscr{A}$) means that $x_t \in \mathscr{A}$ or $x_t \mathrm{q} \mathscr{A}$ (resp. $x_t \in \mathscr{A}$ and $x_t \mathrm{q} \mathscr{A}$).

3. REDEFINED FUZZY B-ALGEBRAS

In what follows, let X denote a B-algebra unless otherwise specified, and let α and β denote any one of $\in$, q, $\in \vee$ q, or $\in \wedge$ q unless otherwise specified. To say that $x_t \overline{\alpha} \mathscr{A}$ means that $x_t \alpha \mathscr{A}$ does not hold.

Definition 3.1. [10] A fuzzy set $\mathscr{A}$ in X is called a *fuzzy B-algebra* if it satisfies the inequality

$$(\forall x, y \in X)\ (\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}). \tag{1}$$

Proposition 3.2. *For any fuzzy set* $\mathscr{A}$ *in* X, *the condition* (1) *is equivalent to the following condition*

$$(\forall x, y \in X)\ (\forall t_1, t_2 \in (0, 1])\ (x_{t_1}, y_{t_2} \in \mathscr{A} \Rightarrow (x * y)_{\min\{t_1, t_2\}} \in \mathscr{A}). \tag{2}$$

Proof. Assume that the condition (1) is valid. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}, y_{t_2} \in \mathscr{A}$. Then $\mathscr{A}(x) \geq t_1$ and $\mathscr{A}(y) \geq t_2$, which imply from (1) that

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\} \geq \min\{t_1, t_2\}.$$

Hence $(x * y)_{\min\{t_1, t_2\}} \in \mathscr{A}$.

Conversely suppose that the condition (2) is valid. Note that $x_{\mathscr{A}(x)} \in \mathscr{A}$ and $y_{\mathscr{A}(y)} \in \mathscr{A}$ for all $x, y \in X$. Thus $(x * y)_{\min\{\mathscr{A}(x), \mathscr{A}(y)\}} \in \mathscr{A}$ by (2), and so $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}$.

Definition 3.3. A fuzzy set $\mathscr{A}$ in X is said to be an (α, β)-*fuzzy B-algebra* of X, where $\alpha \neq \in \wedge$ q, if it satisfies the following conditions:

$$(\forall x, y \in X)\ (\forall t_1, t_2 \in (0, 1])\ (x_{t_1} \alpha \mathscr{A},\ y_{t_2} \alpha \mathscr{A} \Rightarrow (x * y)_{\min\{t_1, t_2\}} \beta \mathscr{A}). \tag{3}$$

Let $\mathscr{A}$ be a fuzzy set in X such that $\mathscr{A}(x) \leq 0.5$ for all $x \in X$. Let $x \in X$ and $t \in (0, 1]$ be such that $x_t \in \wedge\, q \mathscr{A}$. Then $\mathscr{A}(x) \geq t$ and $\mathscr{A}(x) + t > 1$. It follows that

$$1 < \mathscr{A}(x) + t \leq \mathscr{A}(x) + \mathscr{A}(x) = 2\mathscr{A}(x)$$

so that $\mathscr{A}(x) > 0.5$. This means that $\{x_t \mid x_t \in \wedge\, q \mathscr{A}\} = \phi$. Therefore the case $\alpha = \in \wedge$ q in Definition 3.3 will be omitted.

Example 3.4. Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then $(X; *, 0)$ is a B-algebra ([1]). Let $\mathscr{A}$ be a fuzzy set in X defined by $\mathscr{A}(0) = 0.6$, $\mathscr{A}(b) = 0.7$, and $\mathscr{A}(a) = \mathscr{A}(c) = 0.3$. Then $\mathscr{A}$ is an $(\in, \in \vee$ q)-fuzzy B-algebra of X. But

(1) $\mathscr{A}$ is not an $(\in, \in)$-fuzzy B-algebra of X since $b_{0.63} \in \mathscr{A}$ and $b_{0.68} \in \mathscr{A}$, but $(b * b)_{\min\{0.63, 0.68\}} = 0_{0.63} \overline{\in} \mathscr{A}$.

(2) $\mathscr{A}$ is not a (q, $\in \vee$ q)-fuzzy B-algebra of X since $b_{0.43}$ q $\mathscr{A}$ and $a_{0.79}$ q $\mathscr{A}$, but $(b * a)_{\min\{0.43,0.79\}} = a_{0.43} \overline{\in \vee q} \mathscr{A}$ because $\mathscr{A}(a) = 0.3 \ngeq 0.43$ and $\mathscr{A}(a) + 0.43 = 0.3 + 0.43 = 0.73 \not> 1$.

(3) $\mathscr{A}$ is not an ($\in \vee$ q, $\in \vee$ q)-fuzzy B-algebra of X since $b_{0.5} \in \vee$ q $\mathscr{A}$ and $c_{0.8} \in \vee$ q $\mathscr{A}$, but $(b * c)_{\min\{0.5,0.8\}} = c_{0.5} \overline{\in \vee q} \mathscr{A}$ because $\mathscr{A}(c) = 0.3 \ngeq 0.5$ and $\mathscr{A}(c) + 0.5 = 0.3 + 0.5 = 0.8 \not> 1$.

(4) $\mathscr{A}$ is not an ($\in \vee$ q, q)-fuzzy B-algebra of X since $b_{0.66} \in \vee$ q $\mathscr{A}$ and $a_{0.78} \in \vee$ q $\mathscr{A}$, but $(a * b)_{\min\{0.78,0.66\}} = c_{0.66}\, \overline{q}\, \mathscr{A}$ because $\mathscr{A}(c) + 0.66 = 0.3 + 0.66 = 0.96 \not> 1$.

Theorem 3.5. *Every* ($\in \vee$ q, $\in \vee$ q)-*fuzzy B-algebra is an* ($\in$, $\in \vee$ q)-*fuzzy B-algebra.*

Proof. Let $\mathscr{A}$ be an ($\in \vee$ q, $\in \vee$ q)-fuzzy B-algebra of X. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mathscr{A}$ and $y_{t_2} \in \mathscr{A}$. Then $x_{t_1} \in \vee$ q $\mathscr{A}$ and $y_{t_2} \in \vee$ q $\mathscr{A}$, which imply that $(x * y)_{\min\{t_1, t_2\}} \in \vee$ q $\mathscr{A}$. Hence $\mathscr{A}$ is an ($\in$, $\in \vee$ q)-fuzzy B-algebra of X.

Theorem 3.6. *Every* ($\in$, $\in$)-*fuzzy B-algebra is an* ($\in$, $\in \vee$ q)-*fuzzy B-algebra.*

Proof. Straightforward.

Example 3.4 shows that the converse of Theorems 3.5 and 3.6 need not be true.

Proposition 3.7. *If* $\mathscr{A}$ *is a non-zero* (α, β)-*fuzzy B-algebra of X, then* $\mathscr{A}(0) > 0$.

Proof. Assume that $\mathscr{A}(0) = 0$. Since $\mathscr{A}$ is non-zero, there exists $x \in X$ such that $\mathscr{A}(x) = t > 0$. If $\alpha = \in$ or $\alpha = \in \vee$ q, then $x_t \alpha \mathscr{A}$, but $(x * x)_{\min\{t,t\}} = 0_t \overline{\beta} \mathscr{A}$. This is a contradiction. If α = q, then $x_1 \alpha \mathscr{A}$ because $\mathscr{A}(x)+1 = t+1 > 1$. But $(x * x)_{\min\{1,1\}} = 0_1 \overline{\beta} \mathscr{A}$, which is a contradiction. Hence $\mathscr{A}(0) > 0$.

For a fuzzy set $\mathscr{A}$ in X, we denote $X_0 := \{x \in X \mid \mathscr{A}(x) > 0\}$.

Theorem 3.8. *If* $\mathscr{A}$ *is a nonzero* ($\in$, $\in$)-*fuzzy B-algebra of X, then the set* X_0 *is a B-subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mathscr{A}(x) > 0$ and $\mathscr{A}(y) > 0$. Suppose that $\mathscr{A}(x * y) = 0$. Note that $x_{\mathscr{A}(x)} \in \mathscr{A}$ and $y_{\mathscr{A}(y)} \in \mathscr{A}$, but $(x * y)_{\min\{\mathscr{A}(x), \mathscr{A}(y)\}} \overline{\in} \mathscr{A}$ because $\mathscr{A}$

$(x * y) = 0 < \min\{\mathscr{A}(x), \mathscr{A}(y)\}$. This is a contradiction, and thus $\mathscr{A}(x * y) > 0$, which shows that $x * y \in X_0$. Consequently X_0 is a B-algebra of X.

Theorem 3.9. *If* $\mathscr{A}$ *is a nonzero* $(\in, \mathrm{q})$*-fuzzy B-algebra of X, then the set* X_0 *is a B-subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mathscr{A}(x) > 0$ and $\mathscr{A}(y) > 0$. If $\mathscr{A}(x * y) = 0$, then

$$\mathscr{A}(x * y) + \min\{\mathscr{A}(x), \mathscr{A}(y)\} = \min\{\mathscr{A}(x), \mathscr{A}(y)\} \leq 1.$$

Hence $(x * y)_{\min\{\mathscr{A}(x), \mathscr{A}(y)\}} \overline{q}\, \mathscr{A}$, which is a contradiction since $x_{\mathscr{A}(x)} \in \mathscr{A}$ and $y_{\mathscr{A}(y)} \in \mathscr{A}$. Thus $\mathscr{A}(x * y) > 0$, and so $x * y \in X_0$. Therefore X_0 is a B-algebra of X.

Theorem 3.10. *If* $\mathscr{A}$ *is a nonzero* $(\mathrm{q}, \in)$*-fuzzy B-algebra of X, then the set* X_0 *is a B-subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mathscr{A}(x) > 0$ and $\mathscr{A}(y) > 0$. Thus $\mathscr{A}(x) + 1 > 1$ and $\mathscr{A}(y) + 1 > 1$, which imply that $x_1\, \mathrm{q}\, \mathscr{A}$ and $y_1\, \mathrm{q}\, \mathscr{A}$. If $\mathscr{A}(x * y) = 0$, then $\mathscr{A}(x * y) < 1 = \min\{1, 1\}$. Therefore $(x * y)_{\min\{1,1\}} \overline{\in} \mathscr{A}$, which is a contradiction. It follows that $\mathscr{A}(x * y) > 0$ so that $x*y \in X_0$. This completes the proof.

Theorem 3.11. *If* $\mathscr{A}$ *is a nonzero* (q, q)*-fuzzy B-algebra of X, then the set* X_0 *is a B-subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\mathscr{A}(x) > 0$ and $\mathscr{A}(y) > 0$. Thus $\mathscr{A}(x) + 1 > 1$ and $\mathscr{A}(y) + 1 > 1$, and therefore $x_1\, \mathrm{q}\, \mathscr{A}$ and $y_1\, \mathrm{q}\, \mathscr{A}$. If $\mathscr{A}(x * y) = 0$, then $\mathscr{A}(x*y) + \min\{1, 1\} = 0 + 1 = 1$, and so $(x*y)_{\min\{1,1\}} \overline{q}\, \mathscr{A}$. This is impossible, and hence $\mathscr{A}(x * y) > 0$, i.e., $x * y \in X_0$. This completes the proof.

Corollary 3.12. *If A is one of the following*

(i) *a nonzero* $(\in, \in \wedge q)$*-fuzzy B-algebra of X,*

(ii) *a nonzero* $(\in, \in \vee q)$*-fuzzy B-algebra of X,*

(iii) *a nonzero* $(\in \vee \mathrm{q}, \mathrm{q})$*-fuzzy B-algebra of X,*

(iv) *a nonzero* $(\in \vee \mathrm{q}, \in)$*-fuzzy B-algebra of X,*

(v) *a nonzero* $(\in \vee \mathrm{q}, \in \wedge \mathrm{q})$*-fuzzy B-algebra of X,*

(vi) *a nonzero* $(\mathrm{q}, \in \wedge \mathrm{q})$*-fuzzy B-algebra of X,*

(vii) *a nonzero* $(\mathrm{q}, \in \vee \mathrm{q})$*-fuzzy B-algebra of X,*

then the set X_0 *is a B-algebra of X.*

Proof. The proof is similar to the proof of Theorems 3.8, 3.9, 3.10, and/or 3.11.

Theorem 3.13. *Every nonzero* (q, q)-*fuzzy B-algebra of X is constant on* X_0.

Proof. Let $\mathscr{A}$ be a nonzero (q, q)-fuzzy B-algebra of X. Assume that $\mathscr{A}$ is not constant on X_0. Then there exists $y \in X_0$ such that $t_y = \mathscr{A}(y) \neq \mathscr{A}(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. Suppose $t_y < t_0$ and choose $t_1, t_2 \in (0, 1]$ such that $1 - t_0 < t_1 < 1 - t_y < t_2$. Then $\mathscr{A}(0) + t_1 = t_0 + t_1 > 1$ and $\mathscr{A}(y) + t_2 = t_y + t_2 > 1$, and so $0_{t_1} q \mathscr{A}$ and $y_{t_2} q \mathscr{A}$. Since

$$\mathscr{A}(y * 0) + \min\{t_1, t_2\} = \mathscr{A}(y) + t_1 = t_y + t_1 < 1,$$

we have $(y * 0)_{\min t_1 t_2} \overline{q} \mathscr{A}$ which is a contradiction. Next assume that $t_y > t_0$. Then $\mathscr{A}(y) + (1 - t_0) = t_y + 1 - t_0 > 1$ and so $y_{1-t_0} q \mathscr{A}$. Since

$$\mathscr{A}(y * y) + (1 - t_0) = \mathscr{A}(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

we get $(y * y)_{\min\{1-t_0, 1-t_0\}} \overline{q} \mathscr{A}$. This is impossible. Therefore $\mathscr{A}$ is constant on X_0.

Theorem 3.14. *Let $\mathscr{A}$ be a non-zero* (α, β)-*fuzzy B-algebra of X where* (α, β) *is one of the following:*

- $(\in, q)$, • $(\in, \in \wedge q)$,
- $(q, \in)$, • $(q, \in \wedge q)$,
- $(\in \vee q, q)$, • $(\in \vee q, \in \wedge q)$,
- $(\in \vee q, \in)$,

Then $\mathscr{A} = \chi_{X_0}$, *the characteristic function of* X_0.

Proof. Assume that there exists $x \in X_0$ such that $\mathscr{A}(x) < 1$. For $\alpha = \in$, choose $t \in (0, 1]$ such that $t < \min\{1 - \mathscr{A}(x), \mathscr{A}(x), \mathscr{A}(0)\}$. Then $x_t \alpha \mathscr{A}$ and $0_t \alpha \mathscr{A}$, but $(x * 0)_{\min\{t,t\}} = x_t \overline{\beta} \mathscr{A}$ where $\beta = q$ or $\beta = \in \wedge q$. This is a contradiction. Now let $\alpha = q$. Then $x_1 \alpha \mathscr{A}$ and $0_1 \alpha \mathscr{A}$, but $(x * 0)_{\min\{1,1\}} = x_1 \overline{\beta} \mathscr{A}$ for $\beta = \in$ or $\beta = \in \wedge q$, a contradiction. Finally let $\alpha = \in \vee q$ and choose $t \in (0, 1]$ such that $x_t \in \mathscr{A}$ but $x_t \overline{q} \mathscr{A}$. Then $x_t \alpha \mathscr{A}$ and $0_1 \alpha \mathscr{A}$, but $(x * 0)_{\min\{t,1\}} = x_t \overline{\beta} \mathscr{A}$ for $\beta = q$ or $\beta = \in \wedge q$. This is impossible. Note that $x_1 \alpha \mathscr{A}$ and $0_1 \alpha \mathscr{A}$ but $(x * 0)_{\min\{1,1\}} = x_1 \overline{\in} \mathscr{A}$, a contradiction. Therefore $\mathscr{A} = \chi_{X_0}$.

Theorem 3.15. *Let S be a B-subalgebra of X and let* $\mathscr{A}$ *be a fuzzy set in X such that*

(i) $\mathscr{A}(x) = 0$ *for all* $x \in X \setminus S$,

(ii) $\mathscr{A}(x) \geq 0.5$ *for all* $x \in S$.

Then $\mathscr{A}$ *is a* $(\mathrm{q}, \in \vee\, \mathrm{q})$*-fuzzy B-algebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} q \mathscr{A}$ and $y_{t_2} q \mathscr{A}$, that is, $\mathscr{A}(x) + t_1 > 1$ and $\mathscr{A}(y) + t_2 > 1$. Then $x * y \in S$ because if not then $x \in X \setminus S$ or $y \in X \setminus S$. Thus $\mathscr{A}(x) = 0$ or $\mathscr{A}(y) = 0$, and so $t_1 > 1$ or $t_2 > 1$. This is a contradiction. If $\min\{t_1, t_2\} > 0.5$, then $\mathscr{A}(x * y) + \min\{t_1, t_2\} > 1$ and thus $(x * y)_{\min\{t_1, t_2\}} q \mathscr{A}$. If $\min\{t_1, t_2\} \leq 0.5$, then $\mathscr{A}(x * y) \geq 0.5 \geq \min\{t_1, t_2\}$ and so $(x * y)_{\min\{t_1, t_2\}} \in \mathscr{A}$. Therefore $(x * y)_{\min\{t_1, t_2\}} \in \vee q \mathscr{A}$. This completes the proof.

Theorem 3.16. *Let* $\mathscr{A}$ *be a* $(\mathrm{q}, \in \vee\, \mathrm{q})$*-fuzzy B-algebra of X such that* $\mathscr{A}$ *is not constant on* X_0. *Then there exists* $x \in X$ *such that* $\mathscr{A}(x) \geq 0.5$. *Moreover,* $\mathscr{A}(x) \geq 0.5$ *for all* $x \in X_0$.

Proof. Assume that $\mathscr{A}(x) < 0.5$ for all $x \in X$. Since $\mathscr{A}$ is not constant on X_0, there exists $x \in X_0$ such that $t_x = \mathscr{A}(x) \neq \mathscr{A}(0) = t_0$. Then either $t_0 < t_x$ or $t_0 > t_x$. For the first case, choose $\delta > 0.5$ such that $t_0 + \delta < 1 < t_x + \delta$. It follows that $x_\delta \mathrm{q} \mathscr{A}$, $\mathscr{A}(x * x) = \mathscr{A}(0) = y_0 < \delta = \min\{\delta, \delta\}$ and $\mathscr{A}(x * x) + \min\{\delta, \delta\} = \mathscr{A}(0) + \delta = t_0 + \delta < 1$ so that $(x * x)_{\min\{\delta,\delta\}} \overline{\in \vee q} \mathscr{A}$. This is a contradiction. Now if $t_0 > t_x$, we can choose $\delta > 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. Then $0_\delta \mathrm{q} \mathscr{A}$ and $x_1 \mathrm{q} \mathscr{A}$, but $(x * 0)_{\min\{1,\delta\}} = x_\delta \overline{\in \vee q} \mathscr{A}$ since $\mathscr{A}(x) < 0.5 < \delta$ and $\mathscr{A}(x) + \delta = t_x + \delta < 1$. This leads a contradiction. Therefore $\mathscr{A}(x) \geq 0.5$ for some $x \in X$. We now show that $\mathscr{A}(0) \geq 0.5$. Assume that $\mathscr{A}(0) = t_0 < 0.5$. Since there exists $x \in X$ such that $\mathscr{A}(x) = t_x \geq 0.5$, it follows that $t_0 < t_x$. Choose $t_1 > t_0$ such that $t_0 + t_1 < 1 < t_x + t_1$. Then $\mathscr{A}(x) + t_1 = t_x + t_1 > 1$, and so $x_{t_1} q \mathscr{A}$. Now we get

$$\mathscr{A}(x * x) + \min\{t_1, t_1\} = \mathscr{A}(0) + t_1 = t_0 + t_1 < 1,$$
$$\mathscr{A}(x * x) = \mathscr{A}(0) = t_0 < t_1 = \min\{t_1, t_1\}.$$

Hence $(x*x)_{\min\{t_1,t_1\}}\overline{\in\vee q}\mathscr{A}$, a contradiction. Therefore $\mathscr{A}(0) \geq 0.5$. Finally suppose that $t_x = \mathscr{A}(x) < 0.5$ for some $x \in X_0$. Take $t > 0$ such that $t_x + t < 0.5$. Then $\mathscr{A}(x) + 1 = t_x + 1 > 1$ and $\mathscr{A}(0) + (0.5 + t) > 1$, which imply that $x_1\, q\mathscr{A}$ and $0_{0.5+t}\, q\mathscr{A}$. But $(x*0)_{\min\{1,0.5+t\}} = x_{0.5+t}\ \overline{\in\vee q}\mathscr{A}$ since $\mathscr{A}(x * 0) = \mathscr{A}(x) < 0.5 + t < \min\{1, 0.5 + t\}$ and

$$\mathscr{A}(x * 0) + \min\{1, 0.5 + t\} = \mathscr{A}(x) + 0.5 + t = t_x + 0.5 + t < 0.5 + 0.5 = 1.$$

This is a contradiction. Hence A (x) ³ 0.5 for all x Î X_0. This completes the proof.

Theorem 3.17. *A fuzzy set* $\mathscr{A}$ *in X is an* $(\in, \in \vee\, \mathrm{q})$-*fuzzy B-algebra of X* if and only if it satisfies:

$$(\forall x, y \in X)\ (\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}). \tag{4}$$

Proof. Suppose that $\mathscr{A}$ is an $(\in, \in \vee\, \mathrm{q})$-fuzzy B-algebra of X and let $x, y \in X$. If $\min\{\mathscr{A}(x), \mathscr{A}(y)\} < 0.5$, then $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}$. For, assume that $\mathscr{A}(x*y) < \min\{\mathscr{A}(x), \mathscr{A}(y)\}$ and choose t such that $\mathscr{A}(x*y) < t < \min\{\mathscr{A}(x), \mathscr{A}(y)\}$. Then $x_t \in \mathscr{A}$ and $y_t \in \mathscr{A}$ but $(x * y)_{\min\{t,t\}} = (x*y)_t\overline{\in\vee q}\mathscr{A}$, a contradiction. Hence $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}$ whenever $\min\{\mathscr{A}(x), \mathscr{A}(y)\} < 0.5$. Now suppose that $\min\{\mathscr{A}(x), \mathscr{A}(y)\} \geq 0.5$. Then $x_{0.5} \in \mathscr{A}$ and $y_{0.5} \in \mathscr{A}$, which imply that

$$(x * y)_{\min\{0.5,0.5\}} = (x * y)_{0.5} \in \vee\, \mathrm{q}\ \mathscr{A}.$$

Thus $\mathscr{A}(x * y) \geq 0.5$. Otherwise, $\mathscr{A}(x * y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently, $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$ for all $x, y \in X$. Conversely assume that (4) is valid. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mathscr{A}$ and $y_{t_2} \in \mathscr{A}$. Then $\mathscr{A}(x) \geq t_1$ and $\mathscr{A}(y) \geq t_2$. If $\mathscr{A}(x * y) < \min\{t_1, t_2\}$, then $\min\{\mathscr{A}(x), \mathscr{A}(y)\} \geq 0.5$. Otherwise, we have

$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\} \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\} \geq \min\{t_1, t_2\}$, a contradiction. It follows that

$\mathscr{A}(x * y) + \min\{t_1, t_2\} > 2\mathscr{A}(x * y) \geq 2\min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\} = 1$ so that $(x*y)_{\min\{t_1,t_2\}q\mathscr{A}}$. Therefore $\mathscr{A}$ is an $(\in, \in \vee\, \mathrm{q})$-fuzzy B-algebra of X.

Proposition 3.18. *Let $\mathscr{A}$ be an* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebra of X. Then*

(i) $(\forall x \in X)\ (\mathscr{A}(0) \geq \min\{\mathscr{A}(x), 0.5\})$,

(ii) $(\forall x \in X)\ (\mathscr{A}(0 * x) \geq \min\{\mathscr{A}(x), 0.5\})$,

(iii) $(\forall x, y \in X)\ (\mathscr{A}(x * (0 * y)) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\})$,

(iv) $(\forall x, y \in X)\ (\forall n \in \mathbb{N})\ (\mathscr{A}(x^n * x) \geq \min\{\mathscr{A}(x), 0.5\}$ whenever n is odd),

(v) $(\forall x, y \in X)\ (\forall n \in \mathbb{N})\ (\mathscr{A}(x^n * x) = \min\{\mathscr{A}(x), 0.5\}$ whenever n is even),

where $x^n * y = \underbrace{x * (\cdots * (x * (x * y)) \cdots)}_{n}$ *for all* $x, y \in X$.

Proof. Since $x * x = 0$ for all $x \in X$, it follows from Theorem 3.17 that

$$\mathscr{A}(0) = \mathscr{A}(x * x) \geq \min\{\mathscr{A}(x), \mathscr{A}(x), 0.5\} = \min\{\mathscr{A}(x), 0.5\}$$

for all $x \in X$. Thus (i) is valid. For any $x, y \in X$, we have

$$\mathscr{A}(0 * x) \geq \min\{\mathscr{A}(0), \mathscr{A}(x), 0.5\} = \min\{\mathscr{A}(x), 0.5\}$$

by Theorem 3.17 and (i) which shows that (ii) is valid, and

$$\mathscr{A}(x * (0 * y)) \geq \min\{\mathscr{A}(x), \mathscr{A}(0 * y), 0.5\} \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$$

by Theorem 3.17 and (ii). Therefore (iii) holds. Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k. Observe that $\mathscr{A}(x * x) = \mathscr{A}(0) \geq \min\{\mathscr{A}(x), 0.5\}$. Suppose that $\mathscr{A}(x^{2k-1} * x) \geq \min\{\mathscr{A}(x), 0.5\}$ for a positive integer k. Then

$$\begin{aligned} \mathscr{A}(x^{2(k+1)-1} * x) &= \mathscr{A}(x^{2k+1} * x) = \mathscr{A}(x^{2k-1} * (x * (x * x))) \\ &= \mathscr{A}(x^{2k-1} * x) \geq \min\{\mathscr{A}(x), 0.5\} \end{aligned}$$

which proves (iv). Similarly we obtain (v).

Theorem 3.19. *A fuzzy set $\mathscr{A}$ in X is an* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebra of X if and only if the set*

$$U(\mathscr{A}; t) := \{x \in X \mid \mathscr{A}(x) \geq t\}$$

is a B-subalgebra of X for all $t \in (0, 0.5]$.

Proof. Assume that $\mathscr{A}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy B-algebra of X. Let $x, y \in U(\mathscr{A}; t)$ for $t \in (0, 0.5]$. Then $\mathscr{A}(x) \geq t$ and $\mathscr{A}(y) \geq t$. It follows from Theorem 3.17 that

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\} \geq \min\{t, 0.5\} = t$$

so that $x * y \in U(\mathscr{A}; t)$. Therefore $U(\mathscr{A}; t)$ is a subalgerba of X. Conversely, let $\mathscr{A}$ be a fuzzy set in X such that the set

$$U(\mathscr{A}; t) := \{x \in X \mid \mathscr{A}(x) \geq t\}$$

is a B-subalgebra of X for all $t \in (0, 0.5]$. If there exist $x, y \in X$ such that $\mathscr{A}(x * y) < \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$, then we can take $t \in (0, 1)$ such that $A(x * y) < t < \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$. Thus $x, y \in U(\mathscr{A}; t)$ and $t < 0.5$, and so $x * y \in U(\mathscr{A}; t)$, i.e., $\mathscr{A}(x * y) \geq t$. This is a contradiction. Therefore

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$$

for all $x, y \in X$. Using Theorem 3.17, we conclude that $\mathscr{A}$ is an $(\in, \in \vee \text{q})$-fuzzy B-algebra of X.

We give conditions for a fuzzy set to be an $(\in, \in \vee \text{q})$-fuzzy B-algebra.

Theorem 3.20. *If a fuzzy set $\mathscr{A}$ in X satisfies conditions* (ii) *and* (iii) *in Proposition 3.18, then $\mathscr{A}$ is an* $(\in, \in \vee \text{q})$*-fuzzy B-algebra of X.*

Proof. Assume that $\mathscr{A}$ satisfies conditions (ii) and (iii) in Proposition 3.18 and let $x, y \in X$. Then

$$\begin{aligned} \mathscr{A}(x * y) &= \mathscr{A}(x * (0 * (0 * y))) \quad \text{by Lemma 2.1} \\ &\geq \min\{\mathscr{A}(x), \mathscr{A}(0 * y), 0.5\} \quad \text{by Proposition 3.18(ii)} \\ &\geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}. \quad \text{by Proposition 3.18(iii)} \end{aligned}$$

Using Theorem 3.17, we conclude that $\mathscr{A}$ is an $(\in, \in \vee \text{q})$-fuzzy B-algebra of X.

Theorem 3.21. *Let S be a B-subalgebra of a B-algebra X. For any $t \in (0, 0.5]$, there exists an* $(\in, \in \vee \text{q})$*-fuzzy B-algebra $\mathscr{A}$ of X such that $U(\mathscr{A}; t) = S$.*

Proof. Let $\mathscr{A}$ be a fuzzy set in X defined by

$$\mathscr{A}(x) = \begin{cases} t & \text{if } x \in S, \\ 0 & \text{otherwise}, \end{cases}$$

for all $x \in X$ where $t \in (0, 0.5]$. Obviously, $U(\mathscr{A}; t) = S$. Assume that $\mathscr{A}(x * y) < \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$ for some $x, y \in X$. Since $\#\text{Im}(\mathscr{A}) = 2$, it follows that $\mathscr{A}(x * y) = 0$ and $\min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\} = t$, and so $\mathscr{A}(x) = t = \mathscr{A}(y)$, so that $x, y \in S$ but $x * y \notin S$. This is a contradiction, and so $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$. Using Theorem 3.17, we know that $\mathscr{A}$ is an $(\in, \in\vee\text{q})$-fuzzy B-algebra of X.

Theorem 3.22. *For any subset S of X, the characteristic function χ_S of S is an* $(\in, \in\vee q)$*-fuzzy B-algebra of X if and only if S is a B-subalgebra of X.*

Proof. Assume that χ_S is an $(\in, \in\vee q)$-fuzzy B-algebra of X. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, and so $x_1 \in \chi_S$ and $y_1 \in \chi_S$. It follows that $(x * y)_1 = (x * y)_{\min\{1,1\}} \in\vee q\ \chi_S$ which yields $\chi_S(x * y) > 0$. Hence $x * y \in S$, and thus S is a B-subalgebra of X. Conversely if S is a B-subalgebra of X, then χ_S is an $(\in, \in)$-fuzzy B-algebra of X. It follows from Theorem 3.6 that χ_S is an $(\in, \in\vee q)$-fuzzy B-algebra of X.

Theorem 3.23. *Let* $\{\mathscr{A}_i \mid i \in \Lambda\}$ *be a family of* $(\in, \in\vee q)$*-fuzzy B-algebras of X. Then* $\mathscr{A} := \bigcap_{i\in\Lambda} \mathscr{A}_i$ *is an* $(\in, \in\vee q)$*-fuzzy B-algebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mathscr{A}$ and $y_{t_2} \in \mathscr{A}$. Assume that $(x * y)_{\min\{t_1,t_2\}} \overline{\in\vee q} \mathscr{A}$. Then $\mathscr{A}(x * y) < \min\{t_1, t_2\}$ and $\mathscr{A}(x * y) + \min\{t_1, t_2\} \leq 1$, which imply that

$$\mathscr{A}(x * y) < 0.5 \tag{5}$$

Let $\Omega_1 := \{i \in \Lambda \mid (x * y)_{\min\{t_1, t_2\}} \in \mathscr{A}_i\}$ and

$$\Omega_2 := \{i \in \Lambda \mid (x * y)_{\min\{t_1, t_2\}}\ q \mathscr{A}_i\} \cap \{j \in \Lambda \mid (x * y)_{\min\{t_1,t_2\}} \overline{\in} \mathscr{A}_j\}.$$

Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \phi$. If $\Omega_2 = \phi$, then $(x * y)_{\min\{t_1, t_2\}} \in \mathscr{A}_i$ for all $i \in \Lambda$, that is, $\mathscr{A}_i(x * y) \geq \min\{t_1, t_2\}$ for all $i \in \Lambda$, which yields $\mathscr{A}(x * y) \geq \min\{t_1, t_2\}$. This is a contradiction. Hence $\Omega_2 \neq \phi$, and so for every $i \in \Omega_2$ we have $\mathscr{A}_i(x*y) < \min\{t_1, t_2\}$ and $\mathscr{A}_i(x*y)+\min\{t_1, t_2\} > 1$. It follows that $\min\{t_1, t_2\} > 0.5$. Now $x_{t_1} \in \mathscr{A}$ implies $\mathscr{A}(x) \geq t_1$ and thus $\mathscr{A}_i(x) \geq \mathscr{A}(x) \geq t_1 \geq \min\{t_1, t_2\} > 0.5$ for all $i \in \Lambda$. Similarly we get $\mathscr{A}_i(y) > 0.5$ for all $i \in \Lambda$. Next suppose that $t := \mathscr{A}_i(x * y) < 0.5$. Taking $t < r < 0.5$, we get $x_r \in \mathscr{A}_i$ and $y_r \in \mathscr{A}_i$, but $(x * y)_{\min\{r, r\}} = (x * y)_r \overline{\in\vee q} \mathscr{A}_i$. This contradicts that $\mathscr{A}_i$ is an $(\in, \in\vee q)$-fuzzy B-algebra of X. Hence $\mathscr{A}_i(x * y) \geq 0.5$ for all $i \in \Lambda$, and so $\mathscr{A}(x * y) \geq 0.5$ which contradicts (5). Therefore $(x * y)_{\min\{t_1, t_2\}} \in\vee q\ \mathscr{A}$ and consequently $\mathscr{A}$ is an $(\in, \in\vee q)$-fuzzy B-algebra of X.

Theorem 3.24. *Let* $f : X \to Y$ *be a homomorphism of B-algebras and let* $\mathscr{A}$ *and* $\mathscr{B}$ *be* $(\in, \in\vee q)$*-fuzzy B-algebras of X and Y, respectively. Then*

(i) $f^{-1}(\mathscr{B})$ *is an* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebra of X.*

(ii) *If* $\mathscr{A}$ *satisfies the sup property, i.e., for any subset T of X there exists* $x_0 \in T$ *such that*

$$\mathscr{A}(x_0) = \bigvee\{\mathscr{A}(x) \mid x \in T\},$$

then $f(\mathscr{A})$ *is an* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebra of Y when f is onto.*

Proof. (i) Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in f^{-1}(\mathscr{B})$ and $y_{t_2} \in f^{-1}(\mathscr{B})$. Then $(f(x))_{t_1} \in \mathscr{B}$ and $(f(y))_{t_2} \in \mathscr{B}$. Since $\mathscr{B}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy B-algebra of Y, it follows that

$$(f(x * y))_{\min\{t_1, t_2\}} = (f(x) * f(y))_{\min\{t_1, t_2\}} \in \vee\, \mathrm{q}\mathscr{B}$$

so that $(x*y)_{\min\{t_1, t_2\}} \in \vee\, \mathrm{q} f^{-1}(\mathscr{B})$. Therefore $f^{-1}(\mathscr{B})$ is an $(\in, \in \vee \mathrm{q})$-fuzzy B-algebra of X.

(ii) Let $a, b \in Y$ and $t_1, t_2 \in (0, 1]$ be such that $a_{t_1} \in f(\mathscr{A})$ and $b_{t_2} \in f(\mathscr{A})$. Then $(f(\mathscr{A}))(a) \geq t_1$ and $(f(\mathscr{A}))(b) \geq t_2$. Since $\mathscr{A}$ has the sup property, there exists $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$\mathscr{A}(x) = \bigvee\{\mathscr{A}(z) \mid z \in f^{-1}(a)\}$$

and

$$\mathscr{A}(y) = \bigvee\{\mathscr{A}(w) \mid w \in f^{-1}(b)\}.$$

Then $x_{t_1} \in \mathscr{A}$ and $x_{t_2} \in \mathscr{A}$. Since $\mathscr{A}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy B-algebra of X, we have $(x * y)_{\min\{t_1, t_2\}} \in \vee q\mathscr{A}$. Now $x * y \in f^{-1}(a * b)$ and so $(f(\mathscr{A}))(a * b) \geq \mathscr{A}(x * y)$. Thus

$$(f(\mathscr{A}))(a * b) \geq \min\{t_1, t_2\} \text{ or } (f(\mathscr{A}))(a * b) + \min\{t_1, t_2\} > 1$$

which means that $(a*b)_{\min\{t_1, t_2\}} \in \vee q\, f(\mathscr{A})$. Consequently, $f(\mathscr{A})$ is an $(\in, \in \vee \mathrm{q})$-fuzzy B-algebra of Y.

A fuzzy set $\mathscr{A}$ in X is said to be *proper* if $\mathrm{Im}(\mathscr{A})$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 3.25. *Let X be a B-algebra. Then a proper* $(\in, \in)$*-fuzzy B-algebra* $\mathscr{A}$ *of X such that* $\#\mathrm{Im}(\mathscr{A}) \geq 3$ *can be expressed as the union of two proper non-equivalent* $(\in, \in)$*-fuzzy B-algebras of X.*

Proof. Let $\mathscr{A}$ be a proper $(\in, \in)$-fuzzy B-algebra of X with $\mathrm{Im}(\mathscr{A}) = \{t_0, t_1, \ldots, t_n\}$, where $t_0 > t_1 > \ldots > t_n$ and $n \geq 2$. Then

$$U(\mathscr{A}; t_0) \subseteq U(\mathscr{A}; t_1) \subseteq \ldots \subseteq U(\mathscr{A}; t_n) = X$$

is the chain of $\in$-level B-subalgebras of $\mathscr{A}$. Define fuzzy sets $\mathscr{B}$ and $\mathscr{C}$ in X by

$$\mathscr{B}(x) = \begin{cases} r_1 & \text{if } x \in U(\mathscr{A}; t_1), \\ t_2 & \text{if } x \in U(\mathscr{A}; t_2) \setminus U(\mathscr{A}; t_1), \\ \cdots & \\ t_n & \text{if } x \in U(\mathscr{A}; t_n) \setminus U(\mathscr{A}; t_{n-1}), \end{cases}$$

and

$$\mathscr{C}(x) = \begin{cases} t_0 & \text{if } x \in U(\mathscr{A}; t_0), \\ t_1 & \text{if } x \in U(\mathscr{A}; t_1) \setminus U(\mathscr{A}; t_0), \\ r_2 & \text{if } x \in U(\mathscr{A}; t_3) \setminus U(\mathscr{A}; t_1), \\ t_4 & \text{if } x \in U(\mathscr{A}; t_4) \setminus U(\mathscr{A}; t_3), \\ \cdots & \\ t_n & \text{if } x \in U(\mathscr{A}; t_n) \setminus U(\mathscr{A}; t_{n-1}), \end{cases}$$

respectively, where $t_2 < r_1 < t_1$ and $t_4 < r_2 < t_2$. Then $\mathscr{B}$ and $\mathscr{C}$ are $(\in, \in)$-fuzzy B algebras of X with

$$U(\mathscr{A}; t_1) \subseteq U(\mathscr{A}; t_2) \subseteq \ldots \subseteq U(\mathscr{A}; t_n) = X$$

and

$$U(\mathscr{A}; t_0) \subseteq U(\mathscr{A}; t_1) \subseteq U(\mathscr{A}; t_3) \subseteq \ldots \subseteq U(\mathscr{A}; t_n) = X$$

as respective chains of $\in$-level B-subalgebras, and $\mathscr{B}, \mathscr{C} \leq \mathscr{A}$. Thus $\mathscr{B}$ and $\mathscr{C}$ are non-equivalent, and obviously $\mathscr{B} \cup \mathscr{C} = \mathscr{A}$. This completes the proof.

Note that every $(\in, \in)$-fuzzy B-algebra is an $(\in, \in \vee\, q)$-fuzzy B-algebra, but the converse is not true in general. Now we give a condition for an $(\in, \in \vee\, q)$-fuzzy B-algebra to be an $(\in, \in)$-fuzzy B-algebra.

Theorem 3.26. *Let* $\mathscr{A}$ *be an* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebra of X such that* $\mathscr{A}(x) <$ 0.5 *for all* $x \in X$. *Then* $\mathscr{A}$ *is an* $(\in, \in)$*-fuzzy B-algebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mathscr{A}$ and $y_{t_2} \in \mathscr{A}$. Then $\mathscr{A}(x) \geq t_1$ and $\mathscr{A}(y) \geq t_2$. It follows from Theorem 3.17 that

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\} = \min\{\mathscr{A}(x), \mathscr{A}(y)\} \geq \min\{t_1, t_2\}$$

so that $(x * y)_{\min\{t_1, t_2\}} \in \mathscr{A}$. Hence $\mathscr{A}$ is an $(\in, \in)$-fuzzy B-algebra of X.

For any fuzzy set $\mathscr{A}$ in X and $t \in (0, 1]$, we denote

$$\mathscr{A}_t = \{x \in X \mid x_t\, q\, \mathscr{A}\} \text{ and } [A]_t = \{x \in X \mid x_t \in \vee q\, \mathscr{A}\}.$$

Obviously, $[\mathscr{A}]_t = U(\mathscr{A}; t) \cup \mathscr{A}_t$.

Theorem 3.27. *A fuzzy set* $\mathscr{A}$ *in X is an* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebra of X if and only if* $[\mathscr{A}]_t$ *is a B-subalgebra of X for all* $t \in (0, 1]$.

We call $[\mathscr{A}]_t$ an $(\in \vee \mathrm{q})$-*level B-subalgebra* of $\mathscr{A}$.

Proof. Let $\mathscr{A}$ be an $(\in, \in \vee \mathrm{q})$-fuzzy B-algebra of X and let $x, y \in [\mathscr{A}]_t$ for $t \in (0, 1]$. Then $x_t \in \vee \mathrm{q}\, \mathscr{A}$ and $y_t \in \vee \mathrm{q}\, \mathscr{A}$, that is, $\mathscr{A}(x) \geq t$ or $\mathscr{A}(x) + t > 1$, and $\mathscr{A}(y) \geq t$ or $\mathscr{A}(y) + t > 1$. Since $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$ by Theorem 3.17, we have $\mathscr{A}(x * y) \geq \min\{t, 0.5\}$. Otherwise, $x_t \overline{\in \vee q} \mathscr{A}$ or $y_t \overline{\in \vee q} \mathscr{A}$, a contradiction. If $t \leq 0.5$, then $\mathscr{A}(x * y) \geq \min\{t, 0.5\} = t$ and so $x * y \in U(\mathscr{A}; t) \subseteq [\mathscr{A}]_t$. If $t > 0.5$, then $\mathscr{A}(x * y) \geq \min\{t, 0.5\} = 0.5$ and thus $\mathscr{A}(x*y)+t > 0.5+0.5 = 1$. Hence $(x * y)_t\, \mathrm{q}\, \mathscr{A}$, and so $x*y \in \mathscr{A}_t \subseteq [\mathscr{A}]_t$. Therefore $[\mathscr{A}]_t$ is a B-subalgebra of X Conversely, let $\mathscr{A}$ be a fuzzy set in X and $t \in (0, 1]$ be such that $[\mathscr{A}]_t$ is a B-subalgebra of X. If possible, let

$$\mathscr{A}(x * y) < t < \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$$

for some $t \in (0, 0.5)$ and $x, y \in X$. Then $x, y \in U(\mathscr{A}; t) \subseteq [\mathscr{A}]_t$, which implies that $x * y \in [\mathscr{A}]_t$. Hence $\mathscr{A}(x * y) \geq t$ or $\mathscr{A}(x * y) + t > 1$, a contradiction. Therefore

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$$

for all $x, y \in X$. Using Theorem 3.17, we conclude that $\mathscr{A}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy B-algebra of X.

Theorem 3.28. *Let* $\mathscr{A}$ *be a proper* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebra of X such that* $\#\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} \geq 2$. *Then there exist two proper non-equivalent* $(\in, \in \vee \mathrm{q})$*-fuzzy B-algebras of X such that* $\mathscr{A}$ *can be expressed as the union of them.*

Proof. Let $\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} = \{t_1, t_2, \ldots, t_r\}$, where $t_1 > t_2 > \ldots > t_r$ and $r \geq 2$. Then the chain of $(\in \vee q)$-level B-subalgebras of $\mathscr{A}$ is

$$[\mathscr{A}]_{0.5} \subseteq [\mathscr{A}]_{t_1} \subseteq [\mathscr{A}]_{t_2} \subseteq \ldots \subseteq [\mathscr{A}]_{t_r} = X.$$

Let $\mathscr{B}$ and $\mathscr{C}$ be fuzzy sets in X defined by

$$\mathscr{B}(x) = \begin{cases} t_1 & if \quad x \in [\mathscr{A}]_{t_1}, \\ t_2 & if \quad x \in [\mathscr{A}]_{t_2} \setminus [\mathscr{A}]_{t_1}, \\ \cdots & \\ t_r & if \quad x \in [\mathscr{A}]_{t_r} \setminus [\mathscr{A}]_{t_{r-1}}, \end{cases}$$

and

$$\mathscr{C}(x) = \begin{cases} \mathscr{A}(x) & if \quad x \in [\mathscr{A}]_{0.5}, \\ k & if \quad x \in [\mathscr{A}]_{t_2} \setminus [\mathscr{A}]_{0.5}, \\ t_3 & if \quad x \in [\mathscr{A}]_{t_3} \setminus [\mathscr{A}]_{t_2}, \\ \cdots & \\ t_r & if \quad x \in [\mathscr{A}]_{t_r} \setminus [\mathscr{A}]_{t_{r-1}}, \end{cases}$$

respectively, where $t_3 < k < t_2$. Then $\mathscr{B}$ and $\mathscr{C}$ are $(\in, \in \vee q)$-fuzzy B-algebras of X; and $\mathscr{B}$, $\mathscr{C} \leq \mathscr{A}$. The chains of $(\in \vee q)$-level B-subalgebras of $\mathscr{B}$ and $\mathscr{C}$ are, respectively, given by

$$[\mathscr{A}]_{t_1} \subseteq [\mathscr{A}]_{t_2} \subseteq \ldots \subseteq [\mathscr{A}]_{t_r}$$

and

$$[\mathscr{A}]_{0.5} \subseteq [\mathscr{A}]_{t_2} \subseteq \ldots \subseteq [\mathscr{A}]_{t_r}.$$

Therefore $\mathscr{B}$ and $\mathscr{C}$ are non-equivalent and clearly $\mathscr{A} = \mathscr{B} \cup \mathscr{C}$. This completes the proof.

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