

THE APPLICATION OF DIFFERENTIAL TRANSFORMATION METHOD TO SOLVE NONLINEAR DIFFERENTIAL EQUATION GOVERNING JEFFERY-HAMEL FLOW WITH HIGH MAGNETIC FIELD

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Abstract: *The article solves the Jeffery–Hamel flow with analytically method and the effect of external magnetic field is studied. The traditional Navier–Stokes equation of fluid mechanics and Maxwell’s electromagnetism governing equations reduce to nonlinear ordinary differential equations to model this problem. Differential transformation is applied in order to obtain analytical solution of the governing nonlinear differential equations. The velocity profile of the conductive fluid inside the divergent channel studied for various values of Hartmann number.*

The obtained results are finally compared through the illustrative graphs and tables with the numerical solution (Runge–Kutta method). This comparison is shown differential transformation method is a capable tool to solve this nonlinear problem for different α , Hartmann number and Reynolds numbers.

Keywords: *Magneto hydrodynamic, Jeffery–Hamel flow, Differential transformation, Nonlinear ordinary differential equation.*

Nomenclature

B_0	Magnetic field ($wb \cdot m^{-2}$)
C_i	Constant function
F	Transformation of f
$f(\eta)$	Dimensionless velocity
Ha	Hartmann number
P	Pressure term
Re	Reynolds number
r, θ	Cylindrical coordinates
U_{\max}	Maximum value of velocity
u, v	Velocity components along x, y axes, respectively

Greek symbols

ν	Kinematic viscosity
α	Angle of the channel
θ	Any angle
η	Dimensionless angle
$\chi(\tau)$	A function

1. INTRODUCTION

The flow of fluid through a divergent channel has been called Jeffery-Hamel flow after introducing this problem by Jeffery (1915) and Hamel (1916), respectively. On the other hand, the term of Magneto hydrodynamic (MHD) was first introduced by Alfvén in 1970 (L. Bansal 1994). The theory of Magneto hydrodynamics is inducing current in a moving conductive fluid in the presence of magnetic field; Such induced current result force on ions of the conductive fluid. The theoretical study of magnetohydrodynamic (MHD) channel has been a subject of great interest due to its extensive applications in designing cooling systems with liquid metals, MHD generators, accelerators, pumps and flow meters. (J. E. Cha, Y. C. Ahn, Moo-Hwan Kim 2002; M. Tendler 1983; J. Mossino 1979; R. Nijsing, W. Eifler 1980).

In fluid mechanics most of problems are non-linear. It is very important to develop efficient methods to solve them. Up to now, it has been very difficult to obtain analytical approximations of non-linear partial differential equations, even though there are high-performance computers and computation software.

The small disturbance stability of hydromagnetic steady flow between two parallel plates has been investigated by Makinde and Motsa (2001) and Makinde (2003) for plane Poiseuille flow, Kakutani (1964) for plane Couette flow and Makinde and Motsa (2002) for generalized plane Couette flow. Their results show that magnetic field has stabilizing effects on the flow.

Considerable efforts have been done to study the MHD theory for technological application of fluid pumping system in which electrical energy forces the working conductive fluid. damping and controlling of electrically conducting fluid can be achieved by means of an electromagnetic body force (Lorentz force) produced by the interaction of an applied magnetic field and an electric current that usually is externally supplied. Harada *et al.*, (2002). Studied the fundamental characteristics of linear Faraday MHD theoretically and numerically. In 2005, Anwari (2005) continued the Harada *et al.*, (2002) work numerically and theoretically, for various loading configurations. Kim *et al.*, (1997), Ben Salah (1999), and Jang *et al.*, (2000) emphasized on the idea that in such problems, the moving ions drag the bulk fluid with themselves and such MHD system induces continues pumping of conductive fluid without any moving part. Lemoff *et al.*, (2000) and Homsey *et al.*, (2005) worked and developed the same idea mentioned above.

In recent years some researchers used new methods to solve these kinds of problems (D. D. Ganji 2006; D. D. Ganji *et al.*, 2008; D. D. Ganji *et al.*, 2008; M. Gorji, D. D. Ganji, S. Soleimani 2007).

Integral transform methods such as the Laplace and the Fourier transform methods are widely used in engineering problems. These methods transform differential equations into

algebraic equations which are easier to deal with. However, integral transform methods are more complex and difficult when applying to nonlinear problems. The Differential Transformation Method was first applied in the engineering domain by Zhou (1986). The differential transform method is based on Taylor expansion. It constructs an analytical solution in the form of polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions.

The Taylor series method is computationally taken long time for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations.

Differential transform has the inherent ability to deal with nonlinear problems, and consequently Chiou (1996) applied the Taylor transform to solve non-linear vibration problems. Furthermore, the method may be employed for the solution of both ordinary and partial differential equations. Jang *et al.*, (2001) applied the two-dimensional differential transform method to the solution of partial differential equations. Finally, Hassan (2002) adopted the Differential Transformation Method to solve some problems. This method was successfully applied to various application problems (S. Momani, V. Suat Ertürk 2008; R. Abazari, A. Borhanifar 2010; A. A. Joneidi, D. D. Ganji, M. Babaelahi 2009). Recently this kind of problem has been analyzed by some researchers using different methods (M. Esmailpour, D. D. Ganji 2010; Z. Z. Ganji, D. D. Ganji, M. Esmailpour 2009; S. S. Ganji, A. Barari, L. B. Ibsen, G. Domairry 2010; A. Sadighi, D. D. Ganji 2008; K. Hooman *et al.*, 2007).

In this paper, we have applied DTM to find the approximate solutions of nonlinear differential equations governing the MHD Jeffery–Hamel flow, and a comparison between the results and the numerical solution has been provided. The numerical results of this problem are done using Maple 12.

2. GOVERNING EQUATIONS

Consider a system of cylindrical polar coordinates (r, θ, z) with steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at channel walls lie in planes, and intersect in the axis of z . Assuming purely radial motion which means that there is no change in the flow parameter along the z direction. The flow depends on r, θ and further assume that there is no magnetic field in the z -direction. The reduced form of continuity, Navier-Stokes and Maxwell's equations are (W. I. Axford 1961):

$$\frac{\rho \partial}{r \partial r} (ru(r, \theta)) = 0 \tag{1}$$

$$u(r, \theta) \frac{\partial u(r, \theta)}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[\frac{\partial^2 u(r, \theta)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u(r, \theta)}{\partial \theta^2} - \frac{u(r, \theta)}{r^2} \right] - \frac{\sigma B_0^2}{\rho r^2} u(r, \theta) \tag{2}$$

$$\frac{1}{\rho r} \frac{\partial P}{\partial \theta} - \frac{2\nu}{r^2} \frac{\partial u(r, \theta)}{\partial \theta} = 0 \tag{3}$$

Where B_0 is the electromagnetic induction, σ the conductivity of the fluid, $u(r)$ is the velocity along radial direction, P is the fluid pressure, ν the coefficient of kinematic viscosity

and ρ the fluid density. Considering $u_\theta = 0$ for purely radial flow, one can define the velocity parameter as:

$$f(\theta) = ru(r). \quad (4)$$

Introducing the $\eta = \frac{\theta}{\alpha}$ as the dimensionless degree, the dimensionless form of the velocity parameter can be obtained by dividing that to its maximum value as :

$$f(\eta) = \frac{f(\theta)}{f_{\max}}. \quad (5)$$

Substituting Eq. (5) into Eqs. (2) and (3), and eliminating P , one can obtain the ordinary differential equation for the normalized function profile as (T. Kakutani 1964)

$$f'''(\eta) + 2\alpha \text{Re} f(\eta) f'(\eta) + (4 - Ha)\alpha^2 f'(\eta) = 0. \quad (6)$$

With the following reduced form of boundary conditions

$$f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0. \quad (7)$$

Introducing the Reynolds number and the Hartmann number based on the electromagnetic parameter as following, respectively:

$$\text{Re} = \frac{f_{\max} \alpha}{\nu} = \frac{U_{\max} r \alpha}{\nu} \begin{pmatrix} \text{divergent - channel : } \alpha > 0, f_{\max} > 0 \\ \text{convergent - channel : } \alpha < 0, f_{\max} < 0 \end{pmatrix} \quad (8)$$

$$Ha = \sqrt{\frac{\sigma B_0^2}{\rho \nu}}. \quad (9)$$

3. FUNDAMENTALS OF DIFFERENTIAL TRANSFORMATION METHOD

We suppose $\chi(\tau)$ to be analytic function in a domain D and $\tau = \tau_i$ represent any point in D . The function $\chi(\tau)$ is then represented by one power series whose center is located at τ . The Taylor series expansion function of $\chi(\tau)$ is of the form (C.K. Chen, SH. Ho 1999; F. Ayaz 2004):

$$\chi(\tau) = \sum_{k=0}^{\infty} \frac{(\tau - \tau_i)^k}{k!} \left[\frac{d^k \chi(\tau)}{dt^k} \right]_{\tau=\tau_i} \quad \forall \tau \in D. \quad (10)$$

The particular case of Eq. (10) when $\tau = \tau_i$ is referred to as the Maclaurin series of $\chi(\tau)$ and is expressed as:

$$\chi(\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \left[\frac{d^k \chi(\tau)}{dt^k} \right]_{\tau=0} \quad \forall \tau \in D. \quad (11)$$

As explained in the differential transformation of the function $\chi(\tau)$ is defined as follows:

$$\chi(k) = \sum_{k=0}^{\infty} \frac{H^k}{k!} \left[\frac{d^k \chi(\tau)}{dt^k} \right]_{\tau=0}. \quad (12)$$

Where $\chi(\tau)$ is the original function and is the transformed function. The differential spectrum of is confined within the interval, where H is a constant. The differential inverse transform of is defined as follows:

$$\chi(\tau) = \sum_{k=0}^{\infty} \left(\frac{\tau}{H}\right)^k \chi(k). \tag{13}$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The values of function $\chi(\tau)$ at values of argument k are referred to as discrete, i.e. $\chi(0)$ is known as the zero discrete, $\chi(1)$ as the first discrete, etc. The more discrete available, the more precise it is possible to restore the unknown function. The function $\chi(\tau)$ consists of the T -function $\chi(k)$ and its value is given by the sum of the T -function with $(\tau/H)^k$ as its coefficient. In real applications, at the right choice of constant H , the larger values of argument k the discrete of spectrum reduce rapidly. Mathematical operations performed by differential transform method are listed in Table 1.

Table 1
Some of the Basic Operations of Differential Transformation Method

<i>Original function</i>	<i>Transformed function</i>
$\chi(\tau) = \alpha f(\tau) \pm \beta g(\tau)$	$\chi(k) = \alpha F(k) \pm \beta G(k)$
$\chi(\tau) = f'(\tau)$	$\chi(k) = (k + 1) F(k + 1)$
$\chi(\tau) = f''(\tau)$	$\chi(k) = (k + 1)(k + 2) F(k + 2)$
$\chi(\tau) = f'''(\tau)$	$\chi(k) = (k + 1)(k + 2)(k + 3) F(k + 2)$
$\chi(\tau) = f(\tau) g(\tau)$	$\chi(k) = \sum_{l=0}^k F(l) G(k - l)$
$\chi(\tau) = \sin(\omega t + \alpha)$	$\chi(k) = \frac{\omega^k}{k!} \sin\left(\frac{\pi k}{2} + \alpha\right)$
$\chi(\tau) = \cos(\omega t + \alpha)$	$\chi(k) = \frac{\omega^k}{k!} \cos\left(\frac{\pi k}{2} + \alpha\right)$
$\chi(\tau) = e^{\lambda t}$	$\chi(k) = \frac{\lambda^k}{k!}$
$\chi(\tau) = (1 + t)^m$	$\chi(k) = \frac{m(m-1)\dots(m-k+1)}{k!}$
$\chi(\tau) = t^m$	$\chi(k) = \delta(k - m) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$

4. SOLUTION WITH DIFFERENTIAL TRANSFORMATION METHOD

Now we apply Differential Transformation Method into Eq. (6) Taking the differential transform of Eq. (6) with respect to and considering $H = 1$ gives:

$$(k + 1)(k + 2)(k + 3) F[k + 3] + 2\alpha \operatorname{Re} \sum_{r=0}^k ((k - r + 1) F(r) F[k - r + 1]) + (4 - H)\alpha^2(k + 1) F[k + 1] = 0 \tag{14}$$

$$F(0) = 1, \quad F(1) = 0, \quad F(2) = \beta \quad (15)$$

Where $F(k)$ is the differential transformation of $F(\chi)$ and β is a constant which can be obtained through boundary condition, Eq. (7)

$$f(1) = 0, \quad \sum_{k=0}^N F(k) = 0. \quad (16)$$

This problem can be solved for different values of Ha ;

$$F(0) = 1$$

$$F(1) = 0$$

$$F(2) = \beta$$

$$F(3) = 0$$

$$F(4) = \frac{-1}{6} \alpha \operatorname{Re} \beta - \frac{1}{3} \alpha^2 \beta + \frac{1}{12} \alpha^2 \beta Ha$$

$$F(5) = 0$$

$$F(6) = \frac{1}{90} \alpha^2 \operatorname{Re}^2 \beta + \frac{2}{45} \alpha^3 \operatorname{Re} \beta Ha - \frac{1}{30} \alpha^3 \operatorname{Re} \beta^2 + \frac{2}{45} \alpha^4 \beta - \frac{1}{45} \alpha^4 \beta Ha + \frac{1}{360} \alpha^4 \beta Ha^2$$

$$F(7) = 0$$

$$F(8) = -\frac{1}{2520} \alpha^3 \beta \operatorname{Re}^3 - \frac{1}{420} \alpha^4 \beta \operatorname{Re}^2 + \frac{1}{1680} \alpha^4 \beta \operatorname{Re}^2 Ha + \frac{1}{140} \alpha^2 \beta^2 \operatorname{Re}^2 - \frac{1}{210} \alpha^5 \beta \operatorname{Re} \\ + \frac{1}{420} \alpha^5 \beta \operatorname{Re} Ha - \frac{1}{3600} \alpha^5 \beta \operatorname{Re} Ha^2 + \frac{1}{70} \alpha^3 \beta^2 \operatorname{Re} - \frac{1}{280} \alpha^3 \beta^2 \operatorname{Re} Ha \\ - \frac{1}{315} \alpha^6 \beta + \frac{1}{420} \alpha^6 \beta Ha - \frac{1}{1680} \alpha^6 \beta Ha^2 + \frac{1}{20160} \alpha^6 \beta Ha^3$$

$$F(9) = 0. \quad (17)$$

The above process is continuous. Substituting Eq. (17) into the main equation based on DTM. it can be obtained that the closed form of the solutions is:

$$F(\eta) = 1 + \beta \eta^2 + \left(\frac{-1}{6} \alpha \operatorname{Re} \beta - \frac{1}{3} \alpha^2 \beta + \frac{1}{12} \alpha^2 \beta Ha \right) \eta^4 \\ + \left(\frac{1}{90} \alpha^2 \operatorname{Re}^2 \beta + \frac{2}{45} \alpha^3 \operatorname{Re} \beta Ha - \frac{1}{30} \alpha^3 \operatorname{Re} \beta^2 \right. \\ \left. + \frac{2}{45} \alpha^4 \beta - \frac{1}{45} \alpha^4 \beta Ha + \frac{1}{360} \alpha^4 \beta Ha^2 \right) \eta^6 + \dots \quad (18)$$

To obtain the value of β . we substitute the boundary condition from Eq. (7) into Eq. (18) in point $\chi = 1$. So. we have:

$$\begin{aligned}
 F(1) = & 1 + \beta \eta^2 + \left(\frac{-1}{6} \alpha \operatorname{Re} \beta - \frac{1}{3} \alpha^2 \beta + \frac{1}{12} \alpha^2 \beta Ha \right) \eta^4 \\
 & + \left(\frac{1}{90} \alpha^2 \operatorname{Re}^2 \beta + \frac{2}{45} \alpha^3 \operatorname{Re} \beta Ha - \frac{1}{30} \alpha^3 \operatorname{Re} \beta^2 \right. \\
 & \left. + \frac{2}{45} \alpha^4 \beta - \frac{1}{45} \alpha^4 \beta Ha + \frac{1}{360} \alpha^4 \beta Ha^2 \right) \eta^6 + \dots = 0. \quad (19)
 \end{aligned}$$

Solving Eq. (19) gives the value of β . This value is too long that are not shown in this paper we can find the expressions of $F(\eta)$.

5. RESULTS AND DISCUSSION

The objective of the present study was to apply Differential Transformation Method to obtain an explicit analytic solution of the MHD Jeffery–Hamel problem (Fig 1). The magnetic field acts as a control parameter such as the flow Reynolds number and the angle of the walls, in MHD Jeffery–Hamel problems. There is an additional non-dimensional parameter that determines the solutions, namely the Hartmann number.

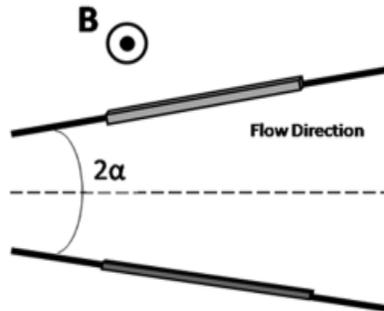


Figure 1: Geometry of the MHD Jeffery-Hamel Flow

Table 2 shows the value of constant β for different α , Ha , Re numbers at the divergent channel.

Table 2
Value of β at Various Re , Ha , α

Re	α	Ha	$\beta = f''(0)$
50	7.5°	250	-1.308020083
50	7.5°	500	-1.093709496
50	7.5°	1000	-0.658867751
50	5°	250	-1.328682532
50	5°	500	-1.203891148
50	5°	1000	-0.955049766
25	7.5°	250	-1.067023522
25	7.5°	500	-0.793987963
25	7.5°	1000	-0.448221153

For comparison, a few limited cases of the DTM solutions are compared with the numerical results.

The comparison between the numerical results and DTM solution for velocity when $Re = 25$ and $\alpha = 5^\circ$ shows in Table 3.

Table 3
Comparison Between the Numerical Results and DTM Solution for Velocity when $Re = 25$ and $\alpha = 5^\circ$

η	$Ha = 500$			$Ha = 750$		
	<i>NM</i>	<i>DTM</i>	<i>Error</i>	<i>NM</i>	<i>DTM</i>	<i>Error</i>
0	1	1	0	1	1	0
0.1	0.99022	0.99023	0.00001	0.991573	0.991488	0.00008
0.2	0.960933	0.960973	0.000039	0.966177	0.965837	0.00034
0.3	0.912273	0.912363	0.000089	0.92345	0.922682	0.000768
0.4	0.844383	0.84454	0.000157	0.862719	0.861344	0.001375
0.5	0.757286	0.757521	0.000235	0.782888	0.780732	0.002156
0.6	0.650719	0.651026	0.000307	0.682271	0.679198	0.003073
0.7	0.523909	0.524246	0.000336	0.558343	0.554355	0.003988
0.8	0.37529	0.375567	0.000277	0.40738	0.402851	0.004529
0.9	0.202125	0.202239	0.000115	0.223928	0.220105	0.003824
1	0	0	0	0	0	0

Also for $Ha = 1000$ and $\alpha = 5^\circ$ this comparison are shown in Table 4. The error bar shows an acceptable agreement between the results observed, which confirms the validity of the DTM.

Table 4
Comparison Between the Numerical Results and DTM Solution for Velocity when $Ha = 1000$ and $\alpha = 5^\circ$

η	$Re = 25$			$Re = 50$		
	<i>NM</i>	<i>DTM</i>	<i>Error</i>	<i>NM</i>	<i>DTM</i>	<i>Error</i>
0	1	1	0	1	1	0
0.1	0.992712	0.992576	0.000136	0.990424	0.990458	0.0000343
0.2	0.970609	0.970061	0.000548	0.961798	0.961935	0.000137
0.3	0.932958	0.931709	0.001249	0.914379	0.914684	0.000306
0.4	0.878458	0.876201	0.002258	0.848433	0.848967	0.000535
0.5	0.805116	0.801534	0.003582	0.764006	0.764807	0.0008
0.6	0.710041	0.704875	0.005167	0.660605	0.661643	0.001038
0.7	0.589131	0.582351	0.00678	0.536772	0.537895	0.001124
0.8	0.436589	0.428818	0.007771	0.389525	0.390421	0.000896
0.9	0.244172	0.237567	0.006605	0.213557	0.213881	0.000323
1	0	0	0	0	0	0

In these tables error is introduced as follow:

$$\text{Error} = |f(\eta)_{NM} - f(\eta)_{DTM}|.$$

Figure 2 shows the magnetic field effect on the velocity profiles for divergent channels. There are good agreements between the numerical solution obtained by the fourth-order Runge-Kutta method and differential transformation method.

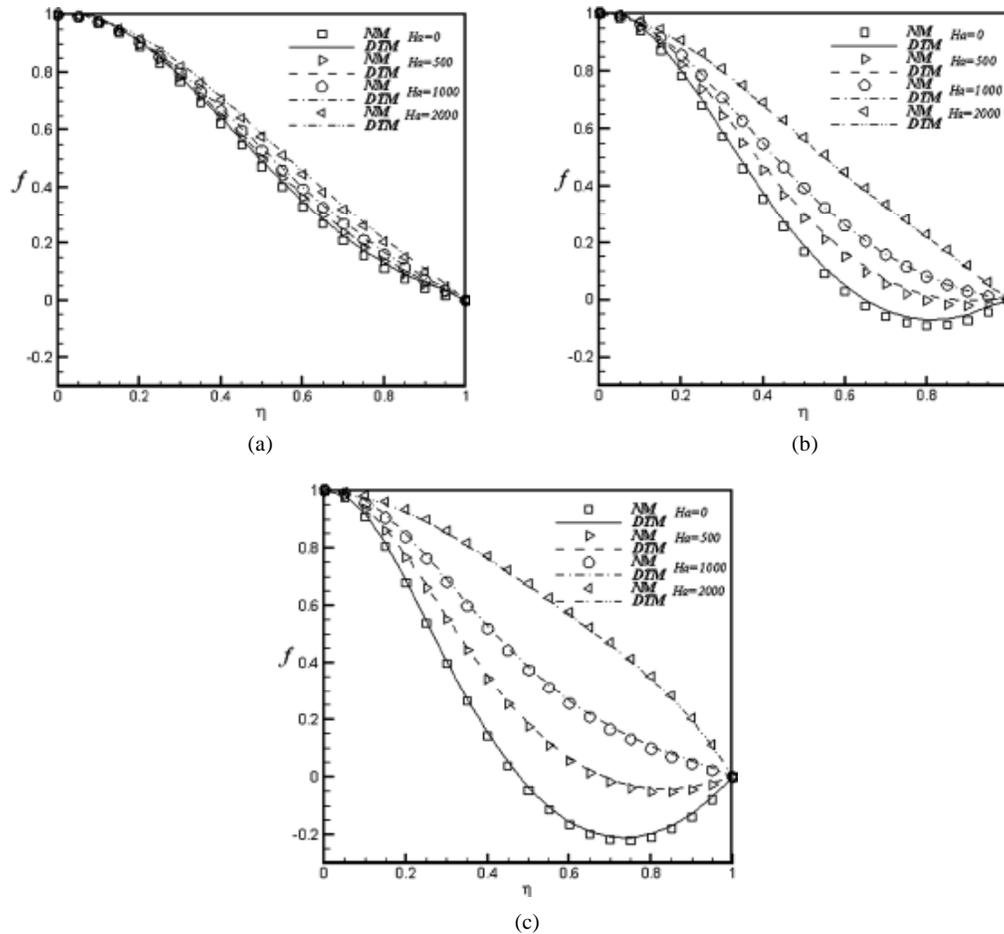


Figure 2: The DTM Solution for Velocity in Divergent Channel for (a) $\alpha = 2.5^\circ$, $Re = 50$, (b) $\alpha = 5^\circ$, $Re = 50$, (c) $\alpha = 7.5^\circ$, $Re = 50$

The results show moderate increases in the velocity with increasing Hartmann numbers at small angle ($\alpha = 25^\circ$) and difference between velocity profiles are more noticeable at greater angles. For specified opening angle, after a critical Reynolds number, we observe that separation and backflow is started. Backflow is excluded in converging channels but is may occur for large Reynolds numbers in diverging channels, see also (Z. Z. Ganji, D. D. Ganji, M. Esmailpour 2009).

As shows in Fig. 2 at $\alpha = 25^\circ$ no separation occurs for all Hartmann numbers but when α increases 5° the separation observes and with more increasing of α stranger backflow occurs in near region of wall channel.

As shows in Fig. 2 increasing Hartmann number will lead to backflow reduction. In greater angles high Hartmann number needed to reduction of backflow.

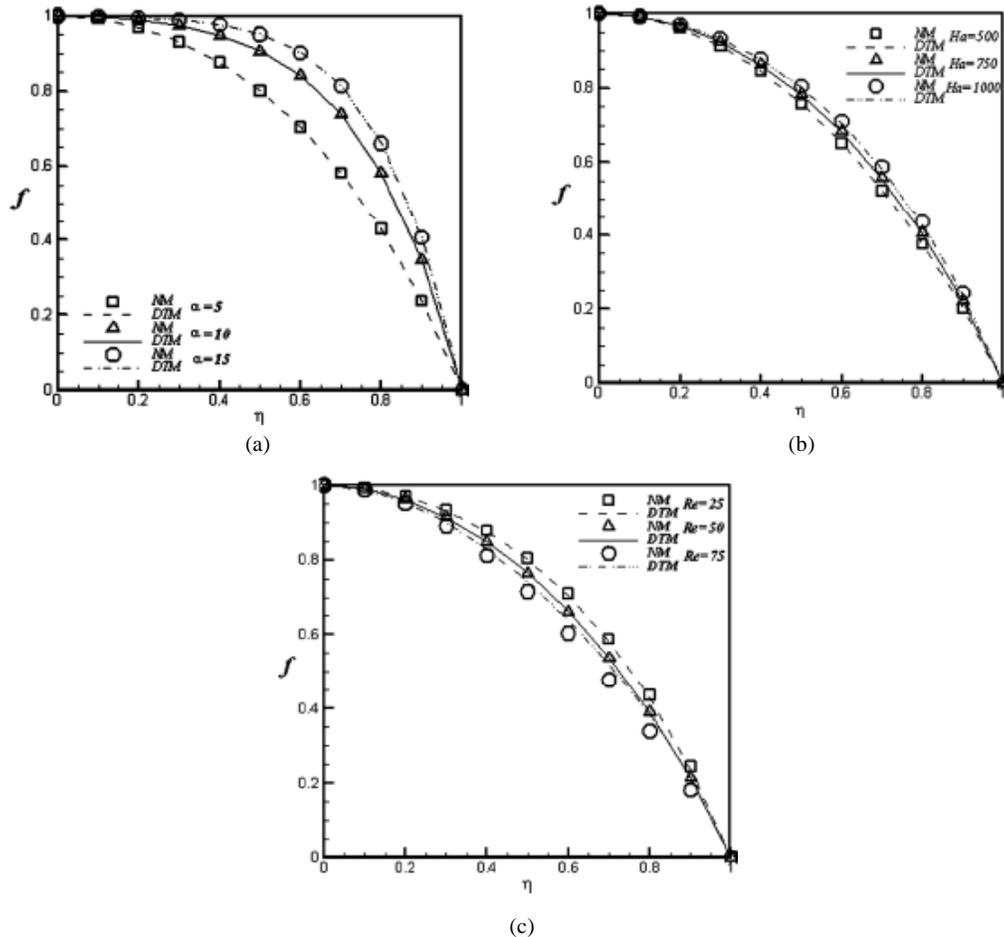


Figure 3: The DTM Solution for Velocity in Divergent Channel for (a) $H = 1000, Re = 25$, (b) $\alpha = 5^\circ, Re = 25$, (c) $H = 1000, \alpha = 5^\circ$

By considering the parameters on the velocity profile, it can be seen in Fig. 3(a) that increasing the angle has more visible effects on this profile. Under magnetic field the Lorentz force affects in opposite of the momentum's direction that stabilizes the velocity profile. (Fig. 3(b)) By increasing α the velocity becomes closer to the maximum value in more expanded region. Increasing Reynolds numbers lead to adverse pressure gradient which causes velocity reduction at near the walls. (Fig. 3(c))

CONCLUSION

In this paper, magneto hydrodynamic Jeffery-Hamel flow has been solved via a sort of analytical method, Differential Transformation Method (DTM). Also this problem is solved by a numerical

method (the Runge-Kutta method of order 4) and the following conclusions have been obtained: a. Differential Transformation Method is a powerful approach for solving MHD Jeffery-Hamel flow in high magnetic field Also it can be observed that there is a good agreement between the present and numerical result. b. Increasing the Reynolds numbers leads to adverse pressure gradient causing velocity reduction at near the walls. c. Increasing Hartmann number will lead to backflow reduction. In greater angles high Hartmann number needed to reduction of backflow.

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