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MAX-MIN APPROACH OF AN OSCILLATION OF A MASS ATTACHED TO AN ELASTIC WIRE

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Abstract: *In this paper, the analytical solution of an oscillation of a mass attached to an elastic wire is obtained using the new developed analytic method entitled the Max-Min Approach (*MMA*). This method can solve equations without tangible restriction of sensitivity to the degree of nonlinear term, furthermore, this method is a quite approximate as a direct result of the reduction in the size of calculations. The analytical results are compared with the exact solution. The comparison reveals that a good agreement exists between the numerical solution and* MMA *solution. On the other hand, it is shown that in comparison with other approximate methods such as Energy Balance Method (*EBM*), Variational Iteration Method (*VIM*), and Parameter Expansion Method (*PEM*) this method is very effective and because of approaching (the solutions that were obtained by this method) to results of other methods and exact solutions, this method solves nonlinear engineering difficulties honestly.*

Keywords: *Max-Min approach method (*MMA*), An oscillation, Mass attached, Stretched elastic wire, Numerical solution.*

1. INTRODUCTION

In this paper, the analytical solution of a mass attached to an elastic wire is explored and the *MMA* is compared with exact results.

As a result of the complexity, strong nonlinearity, and the existence of symmetric as a boundary conditions, most of the analytical and numerical methods face different problems in solving this equation.

Max-Min approach method (*MMA*) which has been recently developed by *HE* (2008) is one of the most successful and efficient methods in solving nonlinear equations.

1.1 Mathematical Formulation

The self consistent non-linear oscillation equation of mass attached to the center of elastic wire is a kind of conservative nonlinear oscillatory system having an irrational elastic item.

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$$u'' + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0, \qquad 0 < \lambda \le 1\,. \tag{1}$$

Initial conditions which should be satisfied are as follows:

$$U(0) = A, \qquad U'(0) = 0\,. \tag{2}$$

This system oscillates between symmetric bounds [$-A$, A], and it's angular. Frequency and corresponding periodic solution are dependent on the amplitude A.

Various kinds of analytical methods including Max-Min Approach (MMA) [1-3], Parameter Expansion Method (*PEM*) [4-13], Variational Iteration Method (*VIM*) [14-24], Energy Balance Method (*EBM*) [25-30], were used to handle strongly nonlinear systems. Max-Min Approach method proposed by Ji-Huan He [31] is proved to be a very effective and convenient way for handling nonlinear issues [32]. The main objective of this paper is to apply a method to the discussed nonlinear oscillator and to compare the results of new method with other methods and numerical solutions.

1.2. Basic Idea of Max-Min Approach

The *Max-Min Approach* method began with an ancient Chinese inequality called He chengtian's Inequality (*He, 2008*), and was proposed by He (2008).

Chengtian's inequality was mainly used to find the fractional day of the month in the following form:

$$29\,\frac{26}{49}\,\text{days} > 1\,\text{month} > 29\,\frac{9}{17}\,. \tag{3}$$

By using the weighting factors (15 and 1), He chengtian obtained the following fractional day:

$$\text{The fractional day} = \frac{26\times 15 + 9\times 1}{49\times 15 + 17\times 1} = \frac{399}{752}\,. \tag{4}$$

Therefore:

$$1\,\text{month} = 29\,\frac{3999}{752}\,\text{days} \tag{5}$$

He chengtian actually used the following inequality:

If

$$\frac{a}{b} < x < \frac{d}{c}\,. \tag{6}$$

Where a, b, c, and d are real numbers, then

$$\frac{a}{b} < \frac{ma+nd}{mb+nc} < \frac{d}{c}. \quad (7)$$

And x is approximated by

$$x = \frac{ma+nd}{mb+nc} \quad (8)$$

Where m and n are weighting factors

1.3. Application of Max-Min Method

To solve Eq. (1) by means of *MMA*, Eq. (1) is re-written in the following form:

$$u'' + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0, \qquad 0 < \lambda \le 1. \quad (9)$$

The trial function is also used to determine the angular frequency *i.e.*

$$u(t) = A\cos(\omega t). \quad (10)$$

Where ω is the frequency and should be determined. It is clear that the square of frequency ω^2 is never less than that in the solution, and never exceeds the square of frequency obtained from solution of Eq. (10), therefore:

$$1 < \omega^2 < 1 - \frac{\lambda}{\sqrt{1+A^2}}, \quad (11)$$

$$1 < \omega^2 < \frac{\sqrt{1+A^2}}{\sqrt{1+A^2}} - \frac{\lambda}{\sqrt{1+A^2}}, \quad (12)$$

$$1 < \omega^2 < \frac{\sqrt{1+A^2} - \lambda}{\sqrt{1+A^2}}. \quad (13)$$

According to He chengtian's interpolation and from Eq. (8), we have

$$\omega^2 = \frac{m + n\left(\sqrt{1+A^2} - \lambda^2\right)}{m + n\left(\sqrt{1+A^2}\right)}. \quad (14)$$

Where m and n are weighting factors, $k = n/(m+n)$; Thus the frequency can be approximated as follows:

$$\omega = \sqrt{1 - \frac{K\lambda}{\sqrt{1+A^2}}}. \quad (15)$$

In approximate solution explains;

$$u(t) = A\cos\left[\left(1 - \frac{K\lambda}{\sqrt{1+A^2}}\right)^{\frac{1}{2}} t\right]. \tag{16}$$

In view of the approximate solution (Eq. (16)), Eq. (1) is re-written in the following form:

$$u'' - \left(\frac{\lambda}{\sqrt{1+u^2}} + K\right) u = (1-K)\, u\,. \tag{17}$$

If Eq. (16) is the exact solution, then the right hand side of Eq. (17) is vanishing completely. Since the approach is only an approximation to the exact solution, it is set as follows:

$$B = \int_0^{\frac{T}{4}} [u - Ku] \cos(\omega t)\, dt = 0\,. \tag{18}$$

By substituting Eq. (16) into Eq. (18), where $T = \frac{2\pi}{\omega}$, it is obtained as follows:

$$B_1 = \int_0^{\frac{T}{4}} [A\cos(\omega t)] \cos(\omega t)\, dt = \int_0^{\frac{T}{2}} A\cos^2(\varphi)\, d\varphi = A\left[\frac{x}{2} + \frac{\sin(2\varphi)}{2}\right]_0^{\frac{x}{2}}, \tag{19}$$

$$B_2 = \int_0^{\frac{T}{4}} [-kA\cos(\omega t)] \cos(\omega t)\, dt = \int_0^{\frac{T}{2}} -kA\cos^2(\varphi)\, d\varphi = -\left[\frac{Kx}{2} + \frac{\sin(2\varphi)}{2}\right]_0^{\frac{x}{2}}. \tag{20}$$

Because of two terms in the above integral equation, they are solved separately, and it is supposed:

$$B = B_1 + B_2, \tag{21}$$

$$\Rightarrow \qquad B = \begin{cases} B_1 = A\dfrac{3\pi}{4} \\ B_2 = -KA\dfrac{3\pi}{4} \end{cases}, \tag{22}$$

$$B = A\frac{3\pi}{4} - KA\frac{3\pi}{4} \Rightarrow B = A\frac{3\pi}{4}(1-K) = 0\,. \tag{23}$$

Therefore:

$$K = 1. \tag{24}$$

The obtained frequency is:

$$1 < \omega^2 < 1 - \frac{\lambda}{\sqrt{1+A^2}}\,. \tag{25}$$

In addition, the approximate period is:

$$T = \frac{2\pi}{\sqrt{1 - \dfrac{\lambda}{\sqrt{1+A^2}}}} = \frac{2\pi\sqrt{1+A^2}}{\sqrt{1+A^2 - \lambda\sqrt{1+A^2}}}\,. \tag{26}$$

To show the remarkable accuracy of the calculated results, the approximate period obtained by the He's Energy Balance Method, Variational Iteration Method (*N.jamshidi, D.D.ganji 2009*), and the parameter expansion method (*Xu. 2007*) are compared with exact solution in Table1 for various values of λ. In Figs. (1), (2), (3), (4), and (5), the values of approximate and numerical solution are plotted to make a graphical comparison.

Table 1
Comparison Between Different Methods to Obtained Frequency

A	$T_{MMA} = T_{EBM}$	T_{VIM}	T_{PEM}	T_{exact}
$\lambda = 0.1$, $A = 0.1$	6.621737	6.621237	6.621688	6.621688
1	6.535726	6.517854	6.537455	6.537507
10	6.320056	6.314768	6.322926	6.322938
100	6.282868	6.286328	6.287188	6.287188
$\lambda = 0.5$, $A = 0.1$	8.869817	8.863794	8.869254	8.869257
1	7.972828	7.814722	7.988547	7.992133
10	6.474308	6.445572	6.489765	6.490208
100	6.301664	6.298952	6.303275	6.606280
$\lambda = 0.75$, $A = 0.1$	12.49906	12.47385	12.49670	12.49673
1	9.564070	9.168186	9.604912	9.625404
10	6.576648	6.531632	6.601000	6.602092
100	6.310970	6.306880	6.313393	6.313040
$\lambda = 0.95$, $A = 0.1$	27.18491	26.86123	27.15432	27.15678
1	11.87273	10.96676	11.97333	12.07527
10	6.662118	6.603018	6.694228	6.696116
100	6.318440	6.313246	6.321522	6.321539

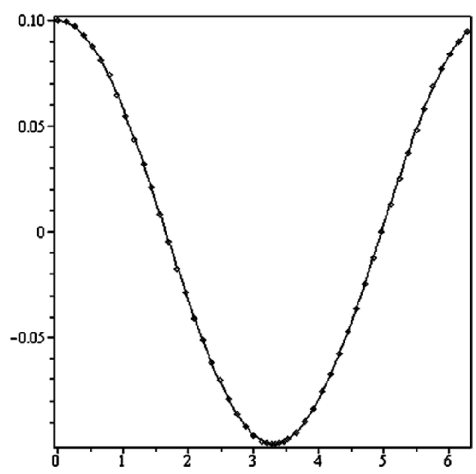

Figure 1: Comparison of the Approximate Solution with Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Numerical Solution ($\lambda = 0.1$, $A = 0.1$)

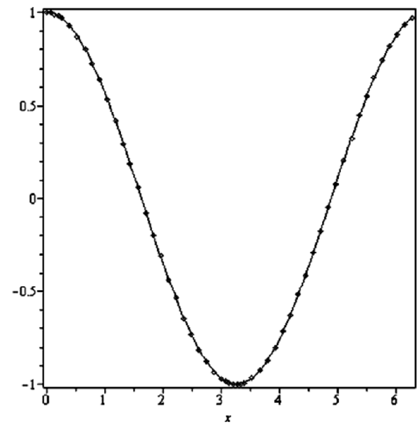

Figure 2: Comparison of the Approximate Solution with Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Numerical Solution ($\lambda = 0.1$, $A = 0.1$)

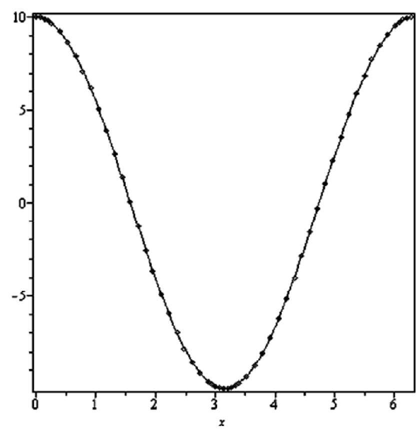

Figure 3: Comparison of the Approximate Solution with Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Numerical Solution ($\lambda = 0.1$, $A = 10$)

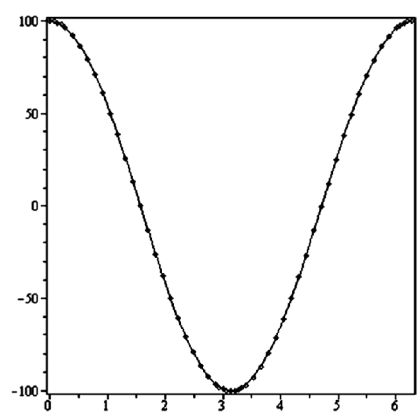

Figure 4: Comparison of the Approximate Solution with Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Numerical Solution ($\lambda = 0.1$, $A = 0.1$)

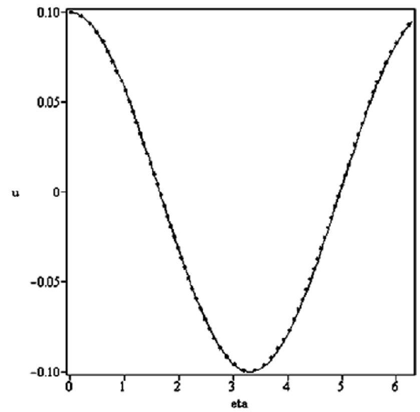

Figure 5: Comparison of the MMA with the Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Exact Solution ($\lambda = 0.1$, $A = 0.1$)

Example 1: The motion equation of the pendulum is considered with harmonic stringer point in Fig. 6.

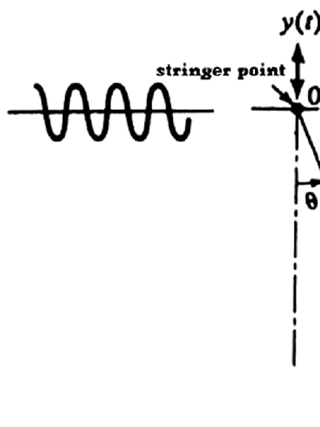

Figure 6: Pendulum with Harmonic Stringer Point

The motion equation of the pendulum with harmonic stringer point is as follows:

$$\theta'' + \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l}\right)\sin(\theta) = 0 \qquad \theta(0) = A, \qquad \theta'(0) = 0\,, \tag{27}$$

$$\sin(\theta) = \theta - \frac{1}{6}\theta^2 + \frac{1}{120}\theta^5 - \frac{1}{5040}\theta^7\,. \tag{28}$$

According to the initial conditions, a trial-function in the form of the following equation was chosen to satisfy the boundary conditions.

$$\theta(t) = A\cos\omega t \;\;\Rightarrow\;\; \theta''(t) = -\,\omega^2\theta(t)\,. \tag{29}$$

We can re-write Eq. (27) in the following form:

$$-\omega^2\theta + \left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right)\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5\right) = 0\,. \tag{30}$$

Therefore:

$$\omega^2 = \left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right)\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{144}\theta^5\right) \Rightarrow \omega^2 = \left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right)\left(1 - \frac{\theta^2}{12}\right)^2. \tag{31}$$

If we choose as a trial function, so we can write the maximum and minimum values like this:

$$0 < \omega^2 < \left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right). \tag{32}$$

According to Chengtian [13] interpolation, x can be found by:

$$\omega^2 = \frac{m(0) + n\left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right)}{m+n}\,. \tag{33}$$

Where m and n are weighting factors, $k = n/(n+m)$. Therefore the frequency can be approximated as follows:

$$\omega^2 K = \left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right). \tag{34}$$

Moreover the frequency can be approximated by:

$$\omega = \sqrt{K\left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right)}\,. \tag{35}$$

Its approximate solution reads:

$$\theta(t) = A\cos\left(\sqrt{K\left(\frac{g}{l} - \frac{\omega_0^2 y\cos(\omega_0 t)}{l}\right)}\,t\right). \tag{36}$$

Eq. (22) is the exact solution. Since the approach is only an approximation to the exact solution, it is set as follows:

$$B = \int_0^{\frac{T}{4}}\left[\theta - \frac{1}{6}A\theta^3 + \frac{1}{144}A\theta^5\right]\cos(\omega t)\,dt\,. \tag{37}$$

By substituting Eq. (36) into Eq. (37), where $T = 2\pi/\omega$, it is obtained as follows:

$$B_1 = \int_0^{\frac{T}{4}} \left[\left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right) (1-K).A\cos(\omega t) \right] \cos(\omega t)\, dt \tag{38}$$

$$\Rightarrow \quad B_1 = \frac{\left[\left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right) (1-K).A \right] \pi}{4} \tag{39}$$

$$B_2 = \int_0^{\frac{T}{4}} \left[\left(-\frac{1}{6} \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right) \right) . (A\cos(\omega t))^3 \right] \cos(\omega t)\, dt \tag{40}$$

$$\Rightarrow \quad B_2 = \frac{\left[-\frac{1}{6} \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right) A^3 \right] \pi}{32} \tag{41}$$

$$B_3 = \int_0^{\frac{T}{4}} \left[\left(\frac{1}{144} \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right) \right) . (A\cos(\omega t))^5 \right] \cos(\omega t)\, dt \tag{42}$$

$$\Rightarrow \quad B_3 = \frac{\left[\frac{5}{144} \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right) A^5 \right] \pi}{32} \tag{43}$$

Because of two terms in the above integral equation, they are solved separately, and it is supposed that:

$$B = B_1 + B_2 + B_3. \tag{44}$$

By setting $B_1 + B_2 + B_3 = 0$, for avoiding the secular term, the value of k yields, therefore:

$$K = 1 - \frac{A^2}{8} + \frac{5A^4}{1152}. \tag{45}$$

From Eqs. (42) and (32), we have:

$$\omega = \sqrt{K \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right)}. \tag{46}$$

Substituting Eq. (45) into Eq. (46) gives:

$$\Rightarrow \omega = \sqrt{\left(1 - \frac{1}{8} A^2 + \frac{5}{1152} A^4 \right) \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right)}. \tag{47}$$

In addition, the approximate period is:

$$T = \frac{2\pi}{\sqrt{\left(1 - \frac{1}{8} A^2 + \frac{5}{1152} A^4 \right) \left(\frac{g}{l} - \frac{\omega_0^2 y \cos(\omega_0 t)}{l} \right)}}. \tag{48}$$

In Figs. (7), (8), and (9), the values of approximate and numerical solution for different value of y and A are plotted to make a graphical comparison.

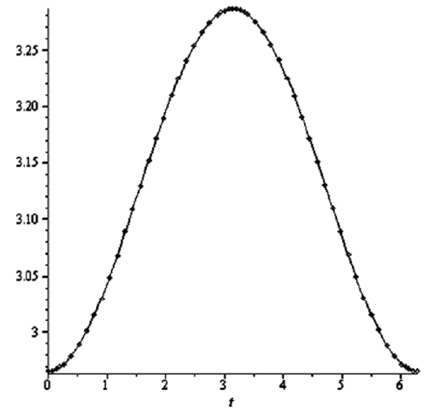

Figure 7: Comparison of the Approximate Solution with Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Numerical Solution ($\lambda = 1$, $A = 0.1$)

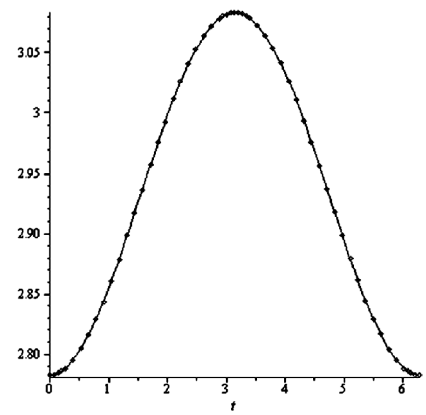

Figure 8: Comparison of the Approximate Solution with Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Numerical Solution ($\lambda = 1$, $A = 1$)

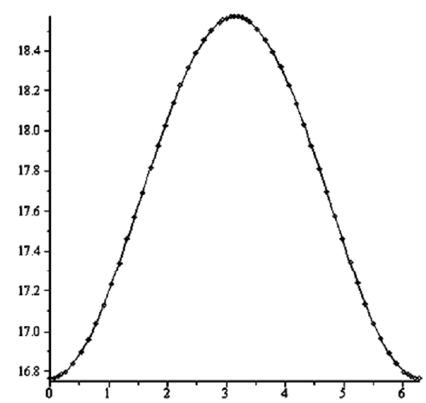

Figure 9: Comparison of the Approximate Solution with Numerical Solution; Dashed Line: Approximated Solution, Solid Line: Numerical Solution (($\lambda = 1$, $A = 10$))

CONCLUSION

He overcame the Max-Min Method to some strong nonlinear equations such as a mass attached to a stretched elastic wire. On the other hand, accuracy of the method was investigated by a comparison which was made between Max-Min Method (*MMA*), Energy Balance Method (*EBM*), Variational Iteration Method (*VIM*), Parameter Expansion Method (PEM) and the exact solution. This method was profitable to obtain analytical solution for all oscillators and vibration problems, such as in the field of civil structures, fluid mechanics, electromagnetic and waves, etc. After some practical and theoretical experiments in graphs and tables, this method was compared to some other methods such as Energy Balance Method (*EBM*), Variational Iteration Method (*VIM*), Parameter Expansion Method (*PEM*), and thereafter it was concluded that the answer was as exact as the others. It is necessary to mention that, this method (*MMA*) made a possible and comfortable choice which cleared the approximate periodic solutions without any expanding of relevant term. One of these limitations occurs as the governing equation contained a damping term, it was not qualified in its present form. To sum up, it can be understood that the results of the current test are completely equivalent to the exact values.

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