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SOLVING NONLINEAR HEAT TRANSFER EQUATION WITH TIME-DEPENDENT THERMAL CONDUCTIVITY BY USING HOMOTOPY PERTURBATION METHOD AND VIM

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Abstract: *We solve for temperature distribution of annular fins with time-dependent thermal conductivity. To this end, Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) are employed to determine temperature distribution of the annular fin when thermal conductivity varies in time. The results of HPM and VIM are compared with numerical results obtained by using direct integration Runge-Kutta method.*

Since thermal conductivity plays an important role on fin efficiency, we tried to solve heat transfer equation with thermal conductivity as a function of temperature. In this research, some new analytical methods called Homotopy perturbation method (HPM), Variational iteration method (VIM) are introduced to be applied to evaluate the temperature distribution of annular fin with temperature-dependent thermal conductivity and to determine the temperature distribution within the fin and also the comparison of the applied methods (together) are shown graphically. The validity of the solutions were veriûed by comparison with numerical results obtained using a Runge–Kutta method.

Keywords: *Heat transfer, Annular fin, Homotopy perturbation method (HPM), Variational iteration method (VIM)*

1. INTRODUCTION

Advanced technological applications require highly efficient cooling systems which improves cooling rates while the cost and weight of the corresponding mechanical system is kept at a reasonable level. As a result, developing new heat transfer technologies has played a key role in advancing engineering components whose performance is directly associated with heat transfer and cooling rate. The early method of increasing external surfaces of industrial components is widely used in various applications. The reader is referred to the extensive review by Kern and Kraus on the extended surface method for further details.

As technology improves, it was realized that devices have to be cooled in a more effective ways and require high-performance heat transfer components with progressively smaller weights,

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volumes, and costs. So, one of the most significant importance is the optimization of the design of fins for high performance, light weight, and compact heat transfer components. Kern and Kraus [1] have presented an extensive review on this topic.

Except for a limited number of nonlinear scientific problems and heat transfer, finding their exact analytical solutions is difficult. Perturbation method is one of the well-known methods to solve the nonlinear equations which were studied by a large number of researchers such as Bellman [2], Cole [3] and O'Malley [4]. The common perturbation methods are restricted, and also because the basis of the common perturbation method was upon the existence of a small parameter, developing the method for different applications is exceedingly difficult. Yu and Chen [5] investigated the optimal ûn length of a convective–radiative straight ûn with rectangular proûle under convective boundary conditions and variable thermal conductivity. Yu and Chen [6] assumed that the linear variation of the thermal conductivity and exponential function with the distance of the heat transfer coefficient and then, solved the nonlinear conducting-convecting-radiating heat transfer equation by the differential transformation method. Bouaziz *et al.*, [7] presented the efficiency of longitudinal ûns with temperature-dependent thermo-physical properties [8, 9].

Hence, among approximate analytical solutions, variational iteration method (VIM) [10-14] and homotopy–perturbation method (HPM) [15-23] are the most effective and convenient ones for both weakly and strongly nonlinear equations. For more information one may follow Refs. [24, 25] to see a concise comparison between VIM, HPM and HAM which strongly reveal that He's method are far effective and accurate.

Therefore, we present an analytical solution of nonlinear problem of heat transfer in annular fins with time-dependent thermal conductivity and examine the results of HPM and VIM methods in contrast with numerical results computed using Runge-Kutta integration scheme. The aim of this paper is to give the analytic solution of the nonlinear equation of the annular fins with time-dependent thermal conductivity and compare the HPM and VIM results with numerical results given [27].

NOMENCLATURE

A_s	Fin surface area	(m^2)
A_c	Cross-sectional area of the fin	(m^2)
h	Coefficient of natural convection	(W/m^2K)
r	Radius	(m)
B_i	Biot number	($\frac{hr_i}{k_\infty}$)
t	Thickness of the annular fin	(m)
r_i	Inner radius of the annular fin	(m)
r_0	Outer radius of the annular fin	(m)
$K(T)$	Thermal conductivity	(W/mK)
k_a	Thermal conductivity in $T = T_a$	(W/mK)

L	Fin length	(m)
T	Temperature	(K)
T_a	Environment temperature	(K)
T_b	Temperature at the base	(K)

Greek Symbols

β	Dimensionless parameter describing variation of the thermal conductivity
λ	The radius ratio, (r_o/r_i)
R	Dimensionless radial coordinate
θ	Dimensionless temperature
κ	Parameter describing the variation of the thermal conductivity
$\delta\theta$	Dimensionless thickness of the fin, t/ri

Subscripts

a	Ambient
s	Surface
b	Base
n	Number of iteration

2. APPLICATION OF VIM AND HPM IN HEAT TRANSFER

In this section, we will apply HPM and VIM to the nonlinear equation of annular fins with temperature-dependent thermal conductivity. The example to be studied is the one-dimensional heat transfer in a cylindrical fin with the length of L, thickness of t, radius of the fin r, interior radius of the fin r_i and the cross section area of A_s and the perimeter of A_c (see Fig. 1). The fin surface transfers heat through convection. Suppose the temperature of the surrounding air is T_a. We assume that base temperature of the fin, T_b, and convection heat transfer coefficient, h, are constant while conduction coefficient, k, can be variable.

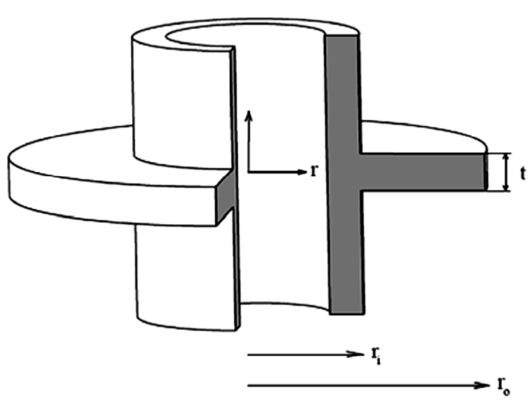

Figure 1: Geometry of a Cylindrical Fin

The energy equation and the boundary conditions for the fin are as follows:

$$t\,\frac{d}{dr}\left[\text{so it's temperature dependent } r\,\frac{dT}{dr}\right] = 2\,hr\,(T - T_{\infty}) \quad (1)$$

T-inf = Ta

$$T = T_b \quad \text{at} \quad r = r_i \quad (2)$$

$$\frac{dT}{dr} = 0 \quad \text{at} \quad r = r_o \quad (3)$$

Assuming k as a linear function of temperature, we have:

$$k(T) = k_a[1 + \beta(T - T_a)] \quad (4)$$

where β represents the rate of effectiveness of temperature variation on thermal conductivity coefficient, $k(T)$.

And:

$$A_c = 2\pi rt, \quad (5)$$

$$\left\{\begin{array}{c} A_c = 2\pi rdr + 2\pi Lt \\ 2\pi Lt \cong 0 \text{ (hence it is negligible)} \end{array}\right\} \Rightarrow A_s = 2\pi rdr \quad (6)$$

Using dimensionless parameters:

$$\theta = \frac{(T - T_{\infty})}{(T_b - T_{\infty})} \quad (7)$$

$$B_i = \frac{hr_i}{k_{\infty}}\,s \quad (8)$$

$$\beta = k\,(T_b - T_{\infty}) \quad (9)$$

$$R = \frac{(r - r_i)}{r_i} \quad (10)$$

$$\lambda = \frac{r_o}{r_i} \quad (11)$$

$$\delta = \frac{t}{r_i} \quad (12)$$

And substituting Eqs. (4-12) in Eq. (1) we have:

$$\frac{d^2\theta}{dR^2} + \beta\left(\frac{d\theta}{dR}\right) + \beta\theta\,\frac{d^2\theta}{dR^2} + \frac{\beta}{(1+R)}\,\theta\,\frac{d\theta}{dR} + \frac{1}{(1+R)}\,\frac{d\theta}{dR} - \frac{2Bi}{\delta}\,\theta = 0,\ 0 < R < \lambda - 1 \quad (13)$$

$$\theta = 1 \quad \text{at} \quad R = 0, \tag{14}$$

$$\frac{d\theta}{dR} = 0 \quad \text{at} \quad R = \lambda - 1. \tag{15}$$

3. BASIC IDEA OF THE METHODS

To illustrate the basic concepts of the methods, we consider the following differential equation:

$$\left\{\begin{matrix} Lu + Nu = Au \\ Au = f(r) \end{matrix}\right\} \Rightarrow Lu + Nu = f(r) \tag{16}$$

where L is a linear differential operator, N is a nonlinear analytic operator, and $f(r)$ an inhomogeneous term.

3.1. Homotopy Perturbation Method

Now we will apply HPM to the nonlinear equation (16) as follows:

$$u = u_0 + p^1 u_1 + p^2 u_2 + \cdots, \tag{17}$$

$$\theta = \lim_{p \to 1} u, \tag{18}$$

$$H(u, p) = (1 - p)\,[L(u) - L(\theta_0)] + p\,[A(u) - f(R)] = 0 \tag{19}$$

where $L(u)$ is the linear part of the equation and $L(\theta_0)$ is the initial approximation.

Substituting Eq. (18) into Eq. (19), we have:

$$H(p,u) = (1-p)\left[\frac{d^2}{dR^2}u(R) - \frac{2Bi}{\delta}u(R)\right] + p\left\{\frac{d^2}{dR^2}u(R) + \beta\left[\frac{d}{dR}u(R)\right]^2 \right.$$
$$\left. + \beta u(R)\left[\frac{d^2}{dR^2}u(R)\right] + \frac{\beta}{(1+R)}u(R)\left[\frac{d}{dR}u(R)\right] + \frac{1}{R+1}\left[\frac{d}{dR}u(R)\right] - \frac{2Bi}{\delta}u(R)\right\} = 0. \tag{20}$$

Substituting (17) in (20) and collecting like powers of p, we obtain the following sequence of expressions for coefficients of the parameter p:

$$p^0) = \frac{\left(\frac{d^2}{dR^2}u_0(R)\right)R}{1+R} + \frac{\frac{d^2}{dR^2}u_0(R)}{1+R} - \frac{2Biu_0(R)}{(1+R)\,\delta} - \frac{2Biu_0(R)\,R}{(1+R)\,\delta} \tag{21}$$

$$R = 0 \rightarrow u_0(R) = 1 \tag{22}$$

$$R = \lambda - 1 \rightarrow \frac{d}{dR}u(R) = 0 \tag{23}$$

$$p^1) = \frac{\frac{d^2}{dR^2}u_1(R)}{1+R} + \frac{\beta\left(\frac{d}{dR}u_0(R)\right)^2}{1+R} + \frac{\beta\left(\frac{d}{dR}u_0(R)\right)u_0(R)}{1+R} + \frac{\frac{d}{dR}u_0(R)}{1+R}$$
$$-\frac{2Biu_1(R)R}{(1+R)\delta} + \frac{\beta\left(\frac{d^2}{dR^2}u_0(R)\right)u_0(R)R}{1+R} + \frac{\left(\frac{d^2}{dR^2}u_1(R)\right)}{1+R}$$
$$+\frac{\beta\left(\frac{d}{dR}u_0(R)\right)^2 R}{1+R} - \frac{2Biu_1(R)}{(1+R)\delta} + \frac{\beta\left(\frac{d^2}{dR^2}u_0(R)\right)u_0(R)}{1+R} \tag{24}$$

$$R = 0 \to u_0(R) = 0 \tag{25}$$

$$R = \lambda - 1 \to \frac{d}{dR}u(R) = 0 \tag{26}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition: The solutions of Eqs. (21, 24) are:

$$u_0(R) = \frac{e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\lambda}}}\, e^{\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}} + e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}} + \frac{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}\, e^{-\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}} + e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}} \tag{27}$$

$$u_1(R) = \frac{1}{2}\sinh\left(\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}\right)\sqrt{2} + \cdots \tag{28}$$

And finally:

$$\theta = u_0 + u_1, \tag{29}$$

$$\theta = \frac{e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}\, e^{\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}} + e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}} + \frac{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}\, e^{-\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}} + e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}} + \frac{1}{2}\sinh\left(\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}\right)\sqrt{2}\,. \tag{30}$$

3.2. Variational Iteration Method

According to the VIM, we can construct a correction functional as follows:

$$\theta = \sum_{n=0}^{\infty} u_n \tag{31}$$

$$u_{n+1}(R) = u_n(R) + \int_0^R \lambda[Lu_n(\tau) + N\tilde{u}_n(\tau) - f(\tau)]\, dt \tag{32}$$

where λ is a general Lagrange multiplier, which can be identify optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e., $\tilde{u}_n$.

And also according to the VIM, we can construct the correction functional of (14) as follows:

$$u_{n+1}(R) = u_{n+1}(R) + \int_0^R \lambda\left[\frac{d^2}{d\tau^2}u_n(\tau) + \beta\left(\frac{d}{d\tau}u_n(\tau)\right)^2 + \beta u_n(\tau)\left(\frac{d^2}{d\tau^2}u_n(\tau)\right)\right.$$
$$\left. + \frac{\beta}{(1+\tau)}\left(\frac{d}{d\tau}u_n(\tau)\right) + \frac{1}{(1+\tau)}\left(\frac{d}{d\tau}u_n(\tau)\right) - \frac{2Bi}{\delta}u_n(\tau)\right]. \tag{33}$$

Its stationary conditions can be obtained as follows:

$$\frac{d^2}{d\tau^2}\lambda(\tau) - \frac{2Bi\,\lambda(\tau)}{\delta} = 0 \tag{34}$$

$$-\left[\frac{d}{d\tau}\lambda(\tau)\right] + 1 = 0 \tag{35}$$

$$\lambda(R) = 0. \tag{36}$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = \frac{1}{\sqrt{Bi}}\left(\left(\frac{1}{4}e^{\frac{\sqrt{2}\sqrt{Bi}\,\tau}{\sqrt{\delta}} - \frac{\sqrt{2}\sqrt{Bi}\,t}{\sqrt{\delta}}} - \frac{1}{4}e^{-\frac{\sqrt{2}\sqrt{Bi}\,\tau}{\sqrt{\delta}} + \frac{\sqrt{2}\sqrt{Bi}\,t}{\sqrt{\delta}}}\right)\sqrt{\delta}\sqrt{2}\right). \tag{37}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(R) = \left[u_n(R) + \int_0^R \frac{1}{\sqrt{Bi}}\left(\left(\frac{1}{4}e^{\frac{\sqrt{2}\sqrt{Bi}\,\tau}{\sqrt{\delta}} - \frac{\sqrt{2}\sqrt{Bi}\,t}{\sqrt{\delta}}} - \frac{1}{4}e^{-\frac{\sqrt{2}\sqrt{Bi}\,\tau}{\sqrt{\delta}} + \frac{\sqrt{2}\sqrt{Bi}\,t}{\sqrt{\delta}}}\right)\sqrt{\delta}\sqrt{2}\right)\right.$$
$$+ \frac{d^2}{d\tau^2}u_n(\tau) + \beta\left(\frac{d}{d\tau}u_n(\tau)\right)^2 + \beta u_n(\tau)\left(\frac{d^2}{d\tau^2}u_n(\tau)\right)$$
$$\left. + \frac{\beta}{(1+\tau)}u_n(\tau)\left(\frac{d}{d\tau}u_n(\tau)\right) + \frac{1}{(1+\tau)}\left(\frac{d}{d\tau}u_n(\tau)\right) - \frac{2Bi}{\delta}u_n(\tau)\right]d\tau. \tag{38}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(R)=\frac{e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}\,e^{\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}+e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}}+\frac{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}\,e^{-\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}+e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}} \tag{39}$$

Using the above variational formula (39), we have:

$$u_{n+1}(R)=u_n(R)+\int_0^R\lambda\left[\frac{d^2}{d\tau^2}u_n(\tau)+\beta\left(\frac{d}{d\tau}u_n(\tau)\right)^2+\beta u_n(\tau)\left(\frac{d^2}{d\tau^2}u_n(\tau)\right)\right.$$
$$\left.+\frac{\beta}{(1+\tau)}u_n(\tau)\left(\frac{d}{d\tau}u_n(\tau)\right)+\frac{1}{(1+\tau)}\left(\frac{d}{d\tau}u_n(\tau)\right)-\frac{2Bi}{\delta}u_n(\tau)\right].$$

$$u_1(R)=u_0(R)+\int_0^R\lambda\left[\frac{d^2}{d\tau^2}u_0(\tau)+\beta\left(\frac{d}{d\tau}u_0(\tau)\right)^2+\beta u_0(\tau)\left(\frac{d^2}{d\tau^2}u_0(\tau)\right)\right.$$
$$\left.+\frac{\beta}{(1+\tau)}u_0(\tau)\left(\frac{d}{d\tau}u_0(\tau)\right)+\frac{1}{(1+\tau)}\left(\frac{d}{d\tau}u_0(\tau)\right)-\frac{2Bi}{\delta}u_0(\tau)\right]d\tau. \tag{40}$$

Substituting Eq. (39) into Eq. (40) and after some simplifications, we have:

$$u_1(R)=\frac{1}{4}\sqrt{2}\left(\int_0^x\left(-\frac{1}{1+R}\left(\left(-e^{\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}}+e^{-\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}}\right)\left(4\beta Bi\,e^{-\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}}\right.\right.\right.\right.$$
$$+4\beta Bi\,e^{\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}}+4\beta Bi\,e^{-\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}}R+4\beta Bi\,e^{\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}}R$$
$$+\beta\sqrt{2}\sqrt{Bi}\sqrt{\delta}\,e^{-\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}}-\beta\sqrt{2}\sqrt{Bi}\sqrt{\delta}\,e^{\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}}+\sqrt{2}\sqrt{Bi}\sqrt{\delta}\,e^{-\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}}$$
$$+\sqrt{2}\sqrt{Bi}\sqrt{\delta}\,e^{-\frac{\sqrt{2}\sqrt{Bi}(2\lambda-2-R)}{\sqrt{\delta}}}-\sqrt{2}\sqrt{Bi}\sqrt{\delta}\,e^{-\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}}$$
$$\left.\left.\left.\left.-\sqrt{2}\sqrt{Bi}\sqrt{\delta}\,e^{\frac{\sqrt{2}\sqrt{Bi}(2\lambda-2-R)}{\sqrt{\delta}}}\right)\right)\right)dR\right). \tag{41}$$

And so on. In the same way the rest of the components of the iteration formula can be obtained.

Finally, we obtain following formula:

$$\theta = u_0 + u_1, \tag{42}$$

$$\begin{aligned}
\theta = {} & \frac{e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}\, e^{\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}} + e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}} + \frac{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}\, e^{-\frac{\sqrt{2}\sqrt{Bi}R}{\sqrt{\delta}}}}{e^{\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}} + e^{-\frac{\sqrt{2}\sqrt{Bi}(\lambda-1)}{\sqrt{\delta}}}} \\
& + \frac{1}{4}\sqrt{2}\left(\int_0^x \left(-\frac{1}{1+R}\left(\left(-e^{\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}} + e^{-\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}}\right)\left(4\beta Bi\, e^{-\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}}\right.\right.\right.\right. \\
& + 4\beta Bi\, e^{\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}} + 4\beta Bi\, e^{-\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}} R + 4\beta Bi\, e^{\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}} R \\
& + \beta\sqrt{2}\sqrt{Bi}\sqrt{\delta}\, e^{-\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}} - \beta\sqrt{2}\sqrt{Bi}\sqrt{\delta}\, e^{\frac{2\sqrt{2}\sqrt{Bi}(\lambda-1-R)}{\sqrt{\delta}}} + \sqrt{2}\sqrt{Bi}\sqrt{\delta}\, e^{-\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}} \\
& + \sqrt{2}\sqrt{Bi}\sqrt{\delta}\, e^{-\frac{\sqrt{2}\sqrt{Bi}(2\lambda-2-R)}{\sqrt{\delta}}} - \sqrt{2}\sqrt{Bi}\sqrt{\delta}\, e^{-\frac{\sqrt{2}\sqrt{Bi}\,R}{\sqrt{\delta}}} \\
& \left.\left.\left.\left. - \sqrt{2}\sqrt{Bi}\sqrt{\delta}\, e^{\frac{\sqrt{2}\sqrt{Bi}(2\lambda-2-R)}{\sqrt{\delta}}}\right)\right)\right) dR\right).
\end{aligned} \tag{43}$$

4. NUMERICAL METHOD

The analytical solution is verified using direct inegration Runge-Kutta method. To this end, the second order differential equation is expressed in terms of a set of first order differential equation as follows. The next method to be used is the Runge–Kutta method. Second-order differential equations can usually be changed into first-order equations and then it is solved through Runge–Kutta method.

Assuming that $u' = w$, we have:

$$\begin{cases} \theta' = w = f(X, \theta, w) \\ w' = -F(X, \theta, w) = g(X, \theta, w) \\ w(X_0) = \alpha \\ w(X_0) = \beta \end{cases} \tag{44}$$

Then, the system of first order ordinary differential equations are solved by using Runge-Kutta numerical integration scheme. Therefore, the system of equations can be solved through the Runge–Kutta method.

5. RESULTS AND DISCUSSION

In this section we will compare the two applied methods. The results show that, the shape of temperature contour changes with the variation of β. If $\beta < 0$ then the concavity of temperature contour will be upturned, while β is reaching (going to be) zero, the concavity will change to zero too, and at last for $\beta > 0$ the concavity will be down turned, (See Fig. 2,3).

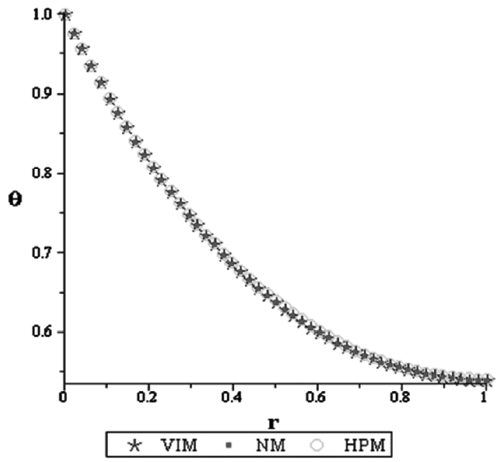

Figure 2: The Comparison of the Three Methods for Temperature Distribution, at $\lambda = 2$, $Bi = 1.5$, $\delta = 0.15$, $\beta = -0.3$

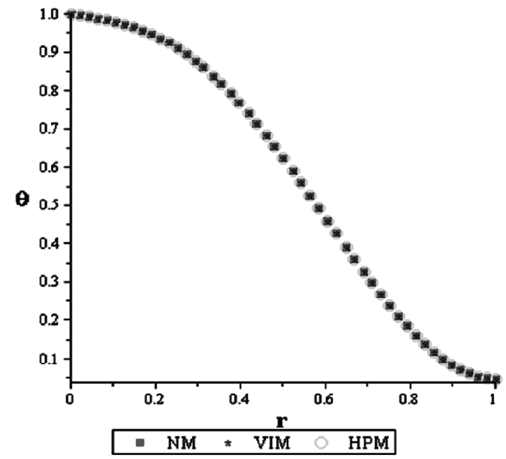

Figure 3: The Comparison of the Three Methods for Temperature Distribution, at $\lambda = 2$, $Bi = 1.5$, $\delta = 0.15$, $\beta = 0.3$

Figures 4 to 5 show $\theta(R)$ that is obtained by using Homotopy perturbation method (HPM) and Variational iteration method (VIM) for various values of *Bi* and δ when $\lambda = 2$ and $\beta = -0.3$. Finally, as shown in Table 1, it has been attempted to show the accuracy, capabilities, and wide-range applications of the HPM and VIM in comparison with the numerical solution of nonlinear temperature distribution of annular fin with temperature-dependent thermal conductivity.

Table 1
The Comparison Between HAM and ADM with Numerical Method for $\theta(R)$ for $\lambda = 2$, $\delta = 0.15$, and $Bi = 1.5$

$\theta(R)$							
$\beta = 0.3$				$\beta = -0.3$			
R	*HPM*	*VIM*	*NM [26]*	*R*	*HPM*	*VIM*	*NM [26]*
0	1	1	1	0	1	1	1
0.1	0.9479	0.9473	0.9477	0.1	0.9150	0.9152	0.9157
0.2	0.9038	0.9032	0.9036	0.2	0.8479	0.8481	0.8483
0.3	0.8664	0.8661	0.8668	0.3	0.7940	0.7942	0.7943
0.4	0.8369	0.8364	0.8365	0.4	0.7508	0.7510	0.7512
0.5	0.8118	0.8115	0.8119	0.5	0.7178	0.7175	0.7172
0.6	0.7922	0.7920	0.7927	0.6	0.6903	0.6909	0.6911
0.7	0.7780	0.7779	0.7782	0.7	0.6711	0.6717	0.6719
0.8	0.7685	0.7681	0.7682	0.8	0.6580	0.6583	0.6587
0.9	0.7629	0.7622	0.7624	0.9	0.6502	0.6515	0.6511
1	0.7603	0.7601	0.7607	1	0.6483	0.6490	0.6487

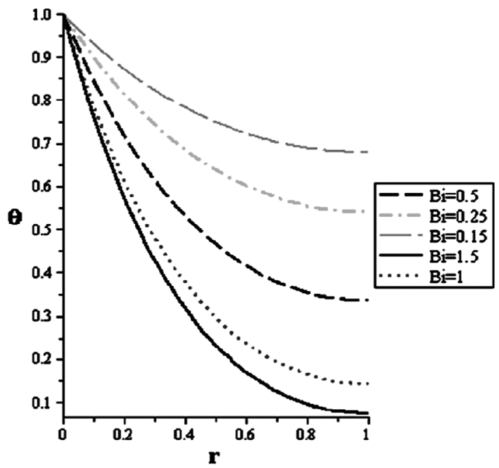

Figure 4: Temperature Distribution $\theta(\xi)$ by HPM for Various *Bi* when $\lambda = 2$, $\delta = 0.15$, $\beta = -0.3$

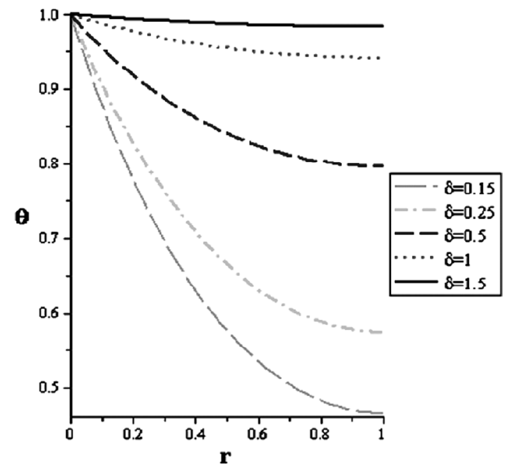

Figure 5: Temperature Distribution $\theta(\xi)$ by VIM for Various *Bi* when $\lambda = 2$, $Bi = 1.5$, $\beta = -0.3$

6. CONCLUSIONS

In this survey, the authors have studied a nonlinear equation through Variational iteration method (VIM) and Homotopy perturbation method (HPM). We have solved the nonlinear heat transfer equation of annular fins by using HPM and VIM methods. We have verified the results of analytical approximation methods with direct numerical solution of the governing differential equation obtained by employing Runge-Kutta method.

The results show that these two methods are capable of solving a large class of nonlinear equations with rapid convergent successive approximations without any restrictive assumptions or transformations that may change the physical behavior of the problem and also adding up the number of iterations cause one to attain the exact solution of the problem if it exists. Also the methods can be applied to the nonlinear equations with boundary or initial conditions defined in different points just with developing the correction functional using the extra parameters, as used in this Letter.

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