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APPLICATION OF VARIATIONAL HOMOTOPY-PERTURBATION METHOD TO NONLINEAR HEAT TRANSFER EQUATIONS

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Abstract: *In a recent paper Noor and Mahyud-Din (Hindawi publishing corporation, Mathematical Problems in Engineering, Volume 2008) proposed the Variational Homotopy Perturbation Method (VHPM) for solving higher dimensional initial boundary value problems. In this research, the Variational Homotopy Perturbation Method (VHPM) which is a combination of Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) has been applied to different equations in steady and unsteady heat transfer equations to show the capability of this method in solving nonlinear equations. The simple procedure has been presented in three different examples and the results are compared with other methods.*

Keywords: *Variational homotopy perturbation method (VHPM), Variational iteration method (VIM), Homotopy perturbation method (HPM), Heat transfer.*

1. INTRODUCTION

Most scientific problems and phenomena such as heat transfer occur nonlinearly. Except in a limited number of these problems, we have difficulty in finding their exact solutions; therefore, approximate analytical solutions were introduced. Perturbation method is one of the well-known methods to solve the nonlinear equations which were studied by a large number of researchers such as Bellman [1], Cole [2] and O'Malley [3]. Due to some limitations with the common perturbation method other approximate analytical solutions were presented, among which Variational Iteration Method (VIM) [4-8] and Homotopy-Perturbation Method (HPM) [9-17] are very effective and convenient ones for both weakly and strongly nonlinear equations.

The VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory, and the initial approximations can be freely chosen with unknown constants. The HPM deforms a difficult problem into a simple problem which can be easily solved. For more information one may follow Refs. [18, 19] to see a concise comparison between VIM, HPM. As mentioned before, the VHPM [20-24] is a technique which results from combination of Variation Iteration and Homotopy Perturbation Methods.

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The VHPM provides the solution in a rapid convergent series which may lead the solution in a closed form. It is worth mentioning that the VHPM is applied without any discretization, restrictive assumption, or transformation and is free from round-off errors. Also the VHPM provides an analytical solution by using the initial conditions only and the boundary conditions can be used only to justify the obtained result.

In this paper, we have introduced the applications of Variational Homotopy Perturbation Method (VHPM) which is presented by Noor, for solving different nonlinear equations in heat transfer. It is worth mentioning that we experienced that the run time of each iterate was very shorter than VIM. More over the results are compared with previous studies.

2. VARIATIONAL HOMOTOPY-PERTURBATION METHOD

To explain this method, let us consider the following function:

$$L(u) + N(u) = g(x) \tag{1}$$

where L and N represent linear and nonlinear parts of the equation and $g(x)$ is an inhomogeneous term, considering variation iteration method we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{Lu_n(\tau) + N\widetilde{u_n(\tau)} - g(\tau)\}\,d\tau \tag{2}$$

where $\lambda(\tau)$ is a Lagrange multiplier which can be identified optimally via the variational theory. For more details about Lagrange multiplier see reference [25]. The subscripts n denote the n^{th} approximation, $\widetilde{u_n(\tau)}$ is considered as a restricted variation. e. $\delta(\tilde{u}_n) = 0$.

Having applied homotopy perturbation method, we have:

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p\int_0^x \lambda(\tau)\left\{N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right\}d\tau - p\int_0^x \lambda(\tau)\,g(\tau)\,d\tau. \tag{3}$$

Eq. (3) is formulated by the coupling of variational iteration method and Adomian's polynomials. A comparison of like powers of p gives solutions of various.

3. APPLICATIONS

3.1. Fin Temperature Distribution

Consider a prismatic fin with a temperature-dependent thermal conductivity, arbitrary constant cross-sectional area, A_c, perimeter P and length b (see Ref. [26]). The fin is attached to a base surface of temperature, T_b extends into a fluid of temperature T_a, with adiabatic tip. The one-dimensional energy balance equation is given:

$$A_c \frac{d}{dx}\left[K(T)\frac{dT}{dx}\right] - ph(T_b - T_a) = 0. \tag{4}$$

The thermal conductivity of the fin material is assumed to be a linear function of temperature according to:

$$K(T) = K_a[1 + \lambda(T_b - T_a)]. \tag{5}$$

Where K_a is the thermal conductivity of the ambient fluid and λ is a constant parameter which describes the variation of the thermal conductivity. Employing the following dimensionless parameters:

$$\theta = \frac{T - T_a}{T_b - T_a} \qquad \xi = \frac{x}{b} \qquad \beta = \lambda (T_b - T_a) \qquad \psi = \sqrt[z]{\frac{hPb^2}{K_a A_c}} . \tag{6}$$

The formulation of the problem reduces to:

$$\frac{d^2\theta}{d\xi^2} + \beta\theta \frac{d^2\theta}{d\xi^2} + \beta \left(\frac{d\theta}{d\xi} \right)^2 - \psi^2\theta = 0 . \tag{7}$$

And the boundary conditions are:

$$\begin{aligned} \theta'(0) &= 0, \\ \theta(0) &= 1. \end{aligned} \tag{8}$$

Its stationary conditions can be obtained as follows:

$$\lambda''(\tau) = 0, \qquad -\lambda'(\tau) + 1 = 0, \qquad \lambda(\tau)\big|\tau = \xi = 0 . \tag{9}$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = \tau - \xi . \tag{10}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$\theta_0(\xi) = C . \tag{11}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} \theta_0 + p\theta_1 + p^2\theta_2 + \cdots = C + p\int_0^{\xi} (\tau - \xi)(\beta[(\theta_0 + p\theta_1 + p^2\theta_2 + \cdots)(\theta_0'' + p\theta_1'' + p^2\theta_2'' + \cdots) \\ + (\theta_0' + p\theta_1' + p^2\theta_2' + \cdots)^2] - (\psi^2)(\theta_0 + p\theta_1 + p^2\theta_2 + \cdots))\, d\tau. \end{aligned} \tag{12}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} &p^0 : \theta_0(\xi) = C \\ &p^1 : \theta_1(\xi) = \frac{1}{2}\psi^2 C \xi^2 \\ &p^2 : \theta_2(\xi) = \frac{1}{2} C^2 \psi^2 \beta \xi^2 + \frac{1}{2}\xi^4 \psi^4 C \\ &p^3 : \theta_3(\xi) = -\frac{5}{24}\xi^4 \beta \psi^4 C^2 + \frac{1}{2} C^3 \psi^2 \beta^2 \xi^2 + \frac{1}{2}\xi^6 \psi^6 C \\ &\quad\vdots \end{aligned} \tag{13}$$

Increasing n number of iterations causes more exact results. Thus the summation of $\theta_0(\tau)$ to $\theta_n(\tau)$ is the answer.

$$\theta(\xi) = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots$$
$$= C + \frac{1}{2}\psi^2 C\xi^2 - \frac{1}{2}C^2\psi^2\beta\xi^2 + \frac{1}{24}\xi^4\psi^2 C - \frac{5}{24}\xi^4\beta\psi^4 C^2$$
$$+ \frac{1}{2}C^3\psi^2\beta^2\xi^2 + \frac{1}{720}\xi^6\psi^6 C + \cdots \tag{14}$$

3.2. Cooling of a lumped System with Variable Specific Heat

Consider the cooling of a lumped system [27] in which the system have volume V, surface area A, density ρ, specific heat c and initial temperature T_i. At time $t = 0$, the system is exposed to a convective environment at temperature T_a with convective heat transfer coefficient h. Assume that the specific heat c is a linear function temperature of the form:

$$c = c_a[1 + \beta(T - T_a)]. \tag{15}$$

Where c_a is the specific heat, at temperature T_a and β is a constant. The one dimensional energy balance equation is given as:

$$\rho Vc\frac{dT}{dt} + hA(T - T_a) = 0, \qquad T(0) = T_i. \tag{16}$$

Applying the following variables to Eq. (16):

$$\theta = \frac{T - T_a}{T_b - T_a} \qquad \tau = \frac{t}{\rho V {c}/{hA}} \qquad \varepsilon = \beta(T - T_a). \tag{17}$$

The dimensionless form of Eq. (16) can be obtained as follows:

$$(1 + \varepsilon\theta)\frac{d\theta}{dt} + \theta = 0, \qquad \theta(0) = 1. \tag{18}$$

Its stationary conditions can be obtained as follows:

$$\lambda'(t) - \lambda(t) = 0, \qquad 1 + \lambda(t)\big|t = \tau = 0 \rightarrow \lambda = e^{t-\tau} \tag{19}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$\theta_0(\tau) = e^{-\tau}. \tag{20}$$

Applying the Variational Homotopy Perturbation method, we have:

$$\theta_0 + p\theta_1 + p^2\theta_2 + \cdots = e^{-\tau} + p\int_0^\tau (e^{t-\tau})\varepsilon(\theta_0 + p\theta_1 + p^2\theta_2 + \cdots)(\theta_0' + p\theta_1' + p^2\theta_2' + \cdots)\,dt. \tag{21}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
&p^0: \theta_0(\tau) = e^{-\tau}\\
&p^1: \theta_1(\tau) = \frac{\varepsilon}{e^{\tau}} - \frac{\varepsilon}{e^{\tau^2}}\\
&p^2: \theta_2(\tau) = \frac{1}{2}\frac{\varepsilon^2}{e^{\tau}} + \frac{3}{2}\frac{\varepsilon^2}{(e^{\tau})^3} - 2\frac{\varepsilon^2}{(e^{\tau})^2}\\
&p^3: \theta_3(\tau) = \frac{1}{6}\frac{\varepsilon^3}{e^{\tau}} - \frac{8}{3}\frac{\varepsilon^2}{(e^{\tau})^4} - 2\frac{\varepsilon^3}{(e^{\tau})^2} + \frac{9}{2}\frac{\varepsilon^3}{(e^{\tau})^3}\\
&\quad\vdots
\end{aligned} \tag{22}$$

It is worth mentioning that with more iteration we can reach more exact solutions. The summation of $\theta_0(\tau)$ to $\theta_n(\tau)$ is the answer.

$$\begin{aligned}
\theta(\tau) &= \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots\\
&= e^{-\tau} + \varepsilon e^{-\tau} - \varepsilon e^{-2\tau} + \frac{1}{2}\varepsilon^2 e^{-\tau} + \frac{3}{2}\varepsilon^2 e^{-3\tau} - 2\varepsilon^2 e^{-2\tau}\\
&\quad + \frac{1}{2}\varepsilon^3 e^{-\tau} - \frac{8}{3}\varepsilon^3 e^{-4\tau} - 2\varepsilon^3 e^{-2\tau} + \frac{9}{2}\varepsilon^3 e^{-3\tau}
\end{aligned} \tag{23}$$

The results of these two examples which are solved with HPM and VIM, have been presented in ref. [28].

3.3. Nonlinear Heat Equation with Cubic Nonlinearity

Finally, we consider the nonlinear heat transfer equation with cubic nonlinearity in the form of [29]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u^3. \tag{24}$$

With the initial condition of

$$u(x,0) = \frac{1+2x}{x^2+x+1}. \tag{25}$$

In order to find the Lagrangian multiplier, we have studied the stationary conditions:

$$\lambda'(\tau) = 0, \qquad 1 + \lambda(t)\big|\tau = t = 0 \to \lambda = -1 \tag{26}$$

Having considered on initial condition, the starting approximation can be as follows:

$$u(x,t) = x. \tag{27}$$

Applying the Variational Homotopy Perturbation method, we have:

$$u_0 + pu_1 + p^2u_2 + \cdots = \frac{1+2x}{x^2+x+1} - p\int_0^t (u_0'' + pu_1'' + p^2u_2'' + \cdots) - (u_0 + pu_1 + p^2u_2 + \cdots)^3\, d\tau. \tag{28}$$

Here, ($'$) dentes derivative with respect to x. Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 &: u_0(x,t) = x \\ p^1 &: u_1(x,t) = \frac{2x+1}{x^2+x+1} \\ p^2 &: u_2(x,t) = \left(-\frac{6}{(x^2+x+1)^2} - \frac{12x}{(x^2+x+1)^2}\right)t \\ p^3 &: \theta_3(\tau) = \left(\frac{36}{(x^2+x+1)^3} - \frac{72x}{(x^2+x+1)^3}\right)t^2 \\ &\;\;\vdots \end{aligned} \tag{29}$$

So the answer will be:

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \\ &= \frac{2x+1}{x^2+x+1} + \left(-\frac{6}{(x^2+x+1)^2} - \frac{12x}{(x^2+x+1)^2}\right)t \\ &\quad + \left(\frac{36}{(x^2+x+1)^3} - \frac{72x}{(x^2+x+1)^3}\right)t^2 + x + \cdots \end{aligned} \tag{30}$$

4. CONCLUSION

In this study we presented an efficient and reliable treatment of the Variational Homotopy Perturbation method (VHPM) for the nonlinear Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). This method is based on Lagrange multipliers for identification of optimal value of parameters in a functional and Homotopy Perturbation Method. First we introduce the applied method then different kinds of heat transfer equations are formulated and solved with VHPM in which the results are in best agreements with previous studies.

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