ijndes
Published by Global Research Publications, New Delhi, India

ANALYTICAL INVESTIGATION OF NONLINEAR TREATMENT OF OSCILLATOR ARISING MICROELECTROMECHANICAL SYSTEM (MEMS)

A. Khatami. Jouybari[b], M. Jouya[a*] & M. Azimi[a]
[a]*Babol University of Technology, Department of Mechanical Engineering, Babol, Iran.*
[b]*Azad University of Jouybar, Department of Mechanical Engineering, Jouybar, Iran.*

Received: 10th January 2018, Accepted: 12th March 2018

Abstract: *In this paper, two methods are applied to determine solutions for a nonlinear oscillator arising in the micro-beam based microelectromechanical system (MEMS). One method applies Modified Homotopy Perturbation Method (MHPM) and other one is max-min approach. The first Method is Modified case of Homotopy Perturbation Method (HPM) and presents an effective tool for nonlinear equations. In the second method (which is called MMA), an approximate solution of the nonlinear equation can be easily deduced by finding Maximal and minimal solution thresholds of this nonlinear problem. What we understood is that both methods, works properly and scales down the deal of the work. Compare conclusions with the results from fourth-order Runge-Kutta method and energy balance method (EBM) shows that obtained results are of high accuracy and convenient.*

Keywords: *Modified homotopy perturbation method (MHPM); Max-min approach (MMA); Microelectromechanical system.*

1. INTRODUCTION

Almost every natural event can be modeled with nonlinear equations; this is due to nature of the phenomena in the world. Although it is easy to find solution of some problems by means of computers, it is still difficult to solve nonlinear equations either numerically or analytically. Nonlinearity can be found in oscillators, and also nonlinear oscillation is one of the most important subjects for many researches. One of those devices in which nonlinear oscillators are used is microelecteromechanical systems [1-5].

Microelectromechanical systems (MEMS) are electrically actuated systems and require few mechanical components and small voltage levels for actuation and also have been developed during last decade. MEMS devices have gained a great deal of interest over the years because of their small size, low power consumption, ability to be batch fabricated, and also ability to be integrated with on-chip electronics; Capacitive sensors, switches and accelerometers. Electrostatic actuation, large deflections and damping are the most effective parameters which can cause nonlinearity in MEMS and also make difficulties in computations. According to

[*] Corresponding Author: *mohsen.jouya@gmail.com*

these difficulties, some analytic techniques are available for nonlinear problems of MEMS such as perturbation techniques.

The perturbation method is one of the most well-known methods for solving nonlinear equations. But, since using the common perturbation method is based on the existence of a small parameter, Therefore many different new methods have recently presented some ways to eliminate the small parameter, such as Adomian decomposition [6-8], homotopy perturbation [9-11] and it's modified case [12-15], variational iteration [16-18], max-min approach [19-21], amplitude frequency formulation [22-24], and energy balance method [25-27].

Modified homotopy perturbation method (MHPM) and Max-Min approach (MMA), suggested by J. H. He are simple and effective methods to solve nonlinear oscillatory equations. Answers of solutions obtained by these methods are valid for not only weakly nonlinear equations, but also strong ones. In this paper, MHPM and MMA are used to study a nonlinear oscillator arising in the micro-beam based MEMS where the mid plane stretching effect and distributed electrostatic force are both considered.

2. PROBLEM DESCRIPTION

As it is illustrated in Fig. 1 and Fig. 2, two electrodes are connected to the voltage source. Changing voltage of electrodes and inducing this fluctuated voltage to the microbeam which is connected to the ground, the oscillation arises in the system. This type of oscillator can be considered as a double-sided driven clamped-clamped microbeam-based electromechanical resonator:

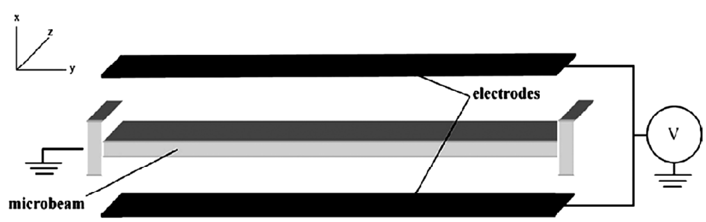

Figure 1: Schematics of a 3-D Fixed Microbeam-Based Electromechanical Resonator

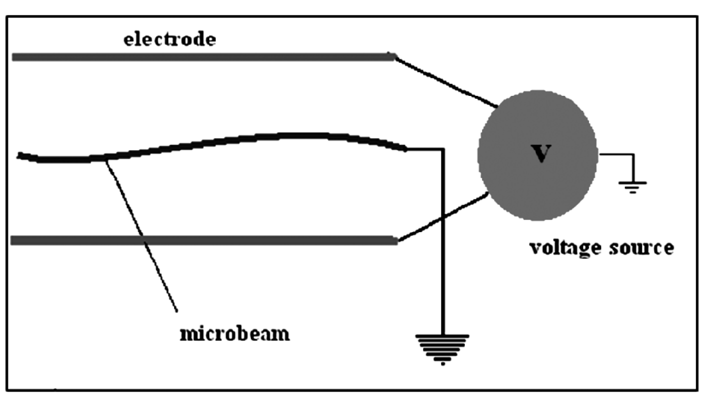

Figure 2: Schematics of a 2-D Fixed Microbeam-Based Electromechanical Resonator

This kind of oscillator arising in MEMS has been analyzed before [28-29]. They also normalized the equation of motion that governs the transverse to the following form:

$$\ddot{u}(a_1u^4 + a_2u^2 + a_3) + a_4u + a_5u^3 + a_6u^5 + a_7u^7 = 0 \tag{2.1}$$

Where:

$$\begin{aligned}
&a_1 = \int_0^1 \phi^6.d\xi, \quad a_2 = -2\int_0^1 \phi^4.d\xi, \quad a_3 = \int_0^1 \phi^2.d\xi,\\
&a_4 = \int_0^1 \left(\phi''''.\phi - N.\phi''.\phi - V^2.\phi^2\right).d\xi,\\
&a_5 = -\int_0^1 \left(2.\phi.\phi''''.\phi^3 + 2N.\phi''.\phi^3 - \alpha.\phi''.\phi.\int_0^1 (\phi')^2.d\xi\right).d\xi,\\
&a_6 = \int_0^1 \left(\phi''''.\phi^5 - N.\phi''.\phi^5 + 2.\alpha.\phi''.\phi^3.\int_0^1 (\phi')^2.d\xi\right).d\xi,\\
&a_7 = -\int_0^1 \left(2.\alpha.\phi''.\phi^5.\int_0^1 (\phi')^2.d\xi\right).d\xi
\end{aligned} \tag{2.2}$$

And

$$\phi = 16.\xi^2.(1-\xi^2). \tag{2.3}$$

Here, an overdot $(\cdot)$ represents differentiation with respect to the time variable, while a prime $(')$ demonstrates the partial differentiation with respect to the coordinate variable ξ.

3. BASIC IDEA OF MODIFIED HOMOTOPY PERTURBATION METHOD

To indicate the basic ideas of this method, the following equation is considered:

$$\ddot{u} + 1.u = u - N(u, \dot{u}, \ddot{u}, t). \tag{3.1}$$

Therefore, the following homotopy can be obtained:

$$\ddot{u} + 1.u = p[u - N(u, \dot{u}, \ddot{u}, t)], \qquad p \in [0,1]. \tag{3.2}$$

Due to change of homotopy parameter p from zero to unity, when $p = 0$, Eq. (3.2) turns out to be the linearized equation $\ddot{u}_0 + \varpi^2 u = 0$, and when it is equal to one, Eq. (3.2) turns into the primary one (Eq. (3.1)).

By applying MHPM solution, the answer ν will be expanded, and by choosing 1 as a coefficient of u, the series of p will be introduced as follows:

$$u = \sum_{i=1}^{n} p^i.u_i \tag{3.3}$$

$$1 = \varpi^2 - \sum_{i=1}^{n} p^i.\gamma_i \tag{3.4}$$

Combining Eqs. (3.3) and (3.4) with Eq. (3.2) and counting the terms with the equal powers of p, and considering $\Psi(u_0, \dot{u}_0, \ddot{u}_0, t)$ as a function with nonlinear term in Eq. (3.6), series of linear equations can be obtained:

$$p^0 : \ddot{u}_0 + \varpi^2 . u_0 = 0 \tag{3.5}$$

$$p^1 : \ddot{u}_1 + \varpi^2 u_1 = (1+\gamma_1)\, u_0 - \Psi(u_0, \dot{u}_0, \ddot{u}_0, t) \tag{3.6}$$

where the answer of Eq. (3.5) is $u_0 = A\cos(\varpi t)$. By replacing u_0 into Eq. (3.6), following equation can be written:

$$p^1 : \ddot{u}_1 + \varpi^2 u_1 = (1+\gamma_1)\, A\cos\varpi t - \Psi(A\cos\varpi t, -A\sin\varpi t, -A\varpi^2\cos\varpi t, t)\,. \tag{3.7}$$

And the secular term can be achieved by using Fourier expansion series as follows:

$$\Psi(A\cos\varpi t, -A\varpi\sin\varpi t, -A\varpi^2\cos\varpi t, t) = \sum_{n=0}^{\infty} b_{2n+1} A\cos[(2n+1)\varpi t]$$
$$= b_1 A\cos\varpi t + b_3 A\cos 3\varpi t + \ldots \approx b_1 A\cos\varpi t \tag{3.8}$$

Substituting Eq. (3.8) into Eq. (3.7), following equation can be obtained:

$$p^1 : \ddot{u}_1 + \varpi^2 u_1 = (1+\gamma_1 - b_1)\, A\cos\varpi t\,. \tag{3.9}$$

For avoiding secular term the following equation can be written:

$$(1+\gamma_1 - b_1) = 0\,. \tag{3.10}$$

If $p = 1$ in Eq. (3.4), then the following equation is obtained:

$$1 = \varpi^2 - \gamma_1\,. \tag{3.11}$$

Substituting Eq. (3.11) into Eq. (3.10), the first-order frequency of motion which is Eq. (3.6), can be achieved as follows:

$$\varpi_{HHPM} = \sqrt{b_1}\,. \tag{3.12}$$

4. BASIC IDEA OF MAX-MIN APPROACH

In this section, MMA is applied to discuss the problem. To illustrate the basic idea of MMA method, the following nonlinear oscillator is considered:

$$\ddot{u} + N(\ddot{u}, \dot{u}, u, t) = 0 \qquad u(0) = A, \dot{u}(0) = 0\,. \tag{4.1}$$

In order to the fact that small parameters or linear terms are not the requirements of MMA, Eq. (4.1) can be approximately solved by using the MMA. Considering a, b, c and d as the real numbers:

$$\frac{a}{b} < x < \frac{d}{c} \tag{4.2}$$

Then:

$$\frac{a}{b} < \frac{ma+nd}{mb+nc} < \frac{d}{c}. \tag{4.3}$$

Where m and n are weighting factors and x is a rough approximation of:

$$x = \frac{ma+nd}{mb+nc}. \tag{4.4}$$

Eq. (4.1) can be rewritten in the following form:

$$\ddot{u} + u\,.\,f(\ddot{u},\dot{u},u,t) = 0. \tag{4.5}$$

And the frequency can be identified as follows:

$$\frac{a}{b} < \varpi^2 = \frac{ma+nd}{mb+nc} < \frac{d}{c}. \tag{4.6}$$

Then:

$$\ddot{u} + \varpi^2 u = \ddot{u} + N(\ddot{u},\dot{u},u,t) + \rho(\ddot{u},\dot{u},u,t). \tag{4.7}$$

And

$$\rho(\ddot{u},\dot{u},u,t) = 0. \tag{4.8}$$

Here ϖ can be obtained by substitution of $A\cos(\varpi t)$ as initial assumption into Eq. (4.8)

5. APPLICATION OF MHPM

In this section, the MHPM is applied to the nonlinear sector which is a problem of micro-beam.

Eq. (2.1) can be rewritten as follows:

$$\ddot{u} + 1u = \frac{-1}{a_3}(a_1 u^4 \ddot{u} + a_2 u^2 \ddot{u} + (a_4 - a_3)\,u + a_5 u^3 + a_6 u^5 + a_7 u^7). \tag{5.1}$$

Using the homotopy parameter p in Eq. (3.2), Following homotopy can be established as follows:

$$\ddot{u} + 1.u = p\left[\frac{-1}{a_3}(a_1 u^4 \ddot{u} + a_2 u^2 \ddot{u} + (a_4 - a_3)\,u + a_5 u^3 + a_6 u^5 + a_7 u^7)\right]. \tag{5.2}$$

By Substituting Eq. (3.3) and Eq. (3.4) into Eq. (5.2) and expanding it, first two linear equations can be written in the following form:

$$p^0 : \ddot{u}_0 + \varpi^2 . u_0 = 0. \tag{5.3}$$

$$p^1 : \ddot{u}_1 + \varpi^2 u_1 = \alpha_1 u_0 + \frac{1}{a_3}[a_1 \ddot{u}_0 u_0^4 + a_2 u_0^2 \ddot{u}_0 + (a_4 - a_3)\,u_0 + a_5 u_0^3 + a_6 u_0^5 + a_7 u_0^7]. \tag{5.4}$$

Here $u_0 = A\cos(\varpi t)$ can be obtained by solving Eq. (5.3). Substituting u_0 into Eq. (5.4) yields:

$$\ddot{u}_1 + \varpi^2 u_1 = \rho(\varpi t). \tag{5.5}$$

Where:

$$\rho(\varpi t) = \gamma_1 A \cos \varpi t + \frac{1}{a_3}[(a_6 - a_1\varpi^2)A^5 \cos^5 \varpi t + (a_5 - a_2\varpi^2)\, A^3 \cos^3 \varpi t$$
$$+ (a_4 - a_3)\, A \cos \varpi t + a_7 A^7 \cos^7 \varpi t]. \quad (5.5a)$$

By using the following Fourier expansion series:

$$\rho(\varpi t) = \sum_{n=0}^{\infty} \delta_{2n+1} \cos[(2n+1)\,\varpi t] = \delta_1 \cos \varpi t + \delta_3 \cos 3\varpi t + \ldots$$

$$\delta_1 \cong \left(\frac{4}{\pi} \int_0^{\frac{\pi}{2}} \rho(\phi) \cos(\phi)\, d\phi \right) \cos \varpi t$$
$$= \frac{1}{64} \frac{A}{a_3} (-48 a_2 A^2 \varpi^2 - 40 a_1 A^4 \varpi^2 - 64 a_3 + 48 a_5 A^2 \quad (5.6)$$
$$+ 40 a_6 A^4 + 64 a_4 + 35 a_7 A^6 + 64\gamma_1 a_3).$$

Now:

$$\ddot{u}_1 + \varpi^2 u_1 = \frac{1}{64} \frac{A}{a_3} (-48 a_2 A^2 \varpi^2 - 40 a_1 A^4 \varpi^2 - 64 a_3 + 48 a_5 A^2 + 40 a_6 A^4$$
$$+ 64 a_4 + 35 a_7 A^6 + 64\gamma_1 a_3)\, A \cos \varpi t + \sum_{n=1}^{\infty} \delta_{2n+1} \cos[(2n+1)\varpi t]. \quad (5.7)$$

For avoiding secular term the following equation can be set:

$$\delta_1 = 0.$$

Substituting $p = 1$ into Eq. (3.4) gives:

$$1 = \gamma_1 + \varpi^2. \quad (5.9)$$

So the first approximation to the angular frequency is:

$$\varpi_{MHPM} = \frac{\sqrt{2}}{4} \sqrt{\frac{(40 a_6 A^4 + 64 a_4 + 48 a_5 A^2 + 35 a_7 A^6)}{6 a_2 A^2 + 5 a_1 A^4 + 8 a_3}}. \quad (5.10)$$

6. APPLICATION OF MMA

In this section, the MMA is applied to solve nonlinear sector. Eq. (2.1) can be rewritten in the following form:

$$\ddot{u} + \left(\frac{a_1}{a_3} u^3 \ddot{u} + \frac{a_2}{a_3} u \ddot{u} + \frac{a_4}{a_3} + \frac{a_5}{a_3} u^2 + \frac{a_6}{a_3} u^4 + \frac{a_7}{a_3} u^6 \right) u = 0. \quad (6.1)$$

In addition, boundary conditions are considered. Here, If $u(t) = A \cos(\varpi t)$ be chosen as a trial function, then:

$$\frac{a_4}{a_3} < \frac{a_1}{a_3} u^3 \ddot{u} + \frac{a_2}{a_3} u \ddot{u} + \frac{a_4}{a_3} + \frac{a_5}{a_3} u^2 + \frac{a_6}{a_3} u^4 + \frac{a_7}{a_3} u^6$$
$$< \frac{-a_1 A^4 \varpi^2 - a_2 A^2 \varpi^2 + a_4 + a_5 A^2 + a_6 A^4 + a_7 A^6}{a_3}. \tag{6.2}$$

Where a_4 is the minimum and $-a_1 A^4 \varpi^2 - a_2 A^2 \varpi^2 + a_4 + a_5 A^2 + a_6 A^4 + a_7 A^6$ is the maximum value of $\frac{a_1}{a_3} u^3 \ddot{u} + \frac{a_2}{a_3} u \ddot{u} + \frac{a_4}{a_3} + \frac{a_5}{a_3} u^2 + \frac{a_6}{a_3} u^4 + \frac{a_7}{a_3} u^6$.

Here the frequency value can be identified as follows:

$$\varpi^2 = \frac{n}{(m+n)} \cdot \frac{-a_1 A^4 \varpi^2 - a_2 A^2 \varpi^2 + a_4 + a_5 A^2 + a_6 A^4 + a_7 A^6}{a_3} + \frac{m}{(m+n)} \cdot \frac{a_4}{a_3}$$
$$= \frac{k(-a_1 A^4 \varpi^2 - a_2 A^2 \varpi^2 + a_5 A^2 + a_6 A^4 + a_7 A^6)}{a_3} + \frac{a_4}{a_3}. \tag{6.3}$$

Where m and n are weighting factors, and $k = \frac{n}{m+n}$. So the frequency can be approximated as follows:

$$\varpi = \sqrt{\frac{k(-a_1 A^4 \varpi^2 - a_2 A^2 \varpi^2 + a_5 A^2 + a_6 A^4 + a_7 A^6)}{a_3} + \frac{a_4}{a_3}}. \tag{6.4}$$

Then Eq. (4.7) can be rewritten as follows:

$$\ddot{u} + \varpi^2 u = \ddot{u} + \left(\frac{a_1}{a_3} u^3 \ddot{u} + \frac{a_2}{a_3} u \ddot{u} + \frac{a_4}{a_3} + \frac{a_5}{a_3} u^2 + \frac{a_6}{a_3} u^4 + \frac{a_7}{a_3} u^6 \right) u + \rho. \tag{6.5}$$

Where:

$$\rho = a_3 \left(\frac{a_1}{a_3} u^3 \ddot{u} + \frac{a_2}{a_3} u \ddot{u} + \frac{a_4}{a_3} + \frac{a_5}{a_3} u^2 + \frac{a_6}{a_3} u^4 + \frac{a_7}{a_3} u^6 \right) u$$
$$+ (-k a_1 A^4 \varpi^2 - k a_2 A^2 \varpi^2 + a_4 + k a_5 A^2 + k a_7 A^6) u. \tag{6.6}$$

Combining $u(t) = A\cos(\varpi t)$ with ρ, and using Fourier Expansion series gives:

$$\rho(\varpi t) = \sum_{n=0}^{\infty} \delta_{2n+1} \cos[(2n+1)\varpi t] = \delta_1 \cos \varpi t + \delta_3 \cos 3\varpi t + \ldots \cong \delta_1 \cos(\varpi t). \tag{6.7}$$

$$\delta_1 = \left(\frac{4}{\pi} \int_0^{\frac{\pi}{2}} \rho(\phi) \cos(\phi)\, d\phi \right)$$
$$= \frac{8}{315\pi} (-105 k a_1 A^4 \varpi^2 - 105 k a_2 A^2 \varpi^2 + 210 a_4 + 105 k a_5 A^2 + 105 k a_6 A^4$$
$$+ 105 k a_7 A^6 - 72 a_1 A^4 \varpi^2 - 84 a_2 A^2 \varpi^2 + 84 a_5 A^2 + 72 a_6 A^4 + 64 a_7 A^6). \tag{6.8}$$

In order to avoid secular term, the following equation can be set:

$$\delta_1 = 0. \tag{6.9}$$

Then the following equation can be obtained:

$$k = \frac{2}{105}\frac{-36a_1A^4\varpi^2 - 42a_2A^2\varpi + 105a_4 + 42a_5A^2 + 36a_6A^4 + 32a_7A^6}{A^2(-A^2a_1\varpi^2 - a_2\varpi^2 + a_5 + A^2a_6 + A^4a_7)}. \tag{6.10}$$

Substituting Eq. (6.10) into Eq. (6.4), yields:

$$\varpi_{MMA} = \frac{\sqrt{3}}{3}\sqrt{\frac{84a_5A^2 + 64a_7A^6 + 72a_6A^4 + 105a_4}{24a_1A^4 + 35a_3 + 28a_2A^2}}. \tag{6.11}$$

The results of modified homotopy perturbation method and max-min approach are plotted in **Fig. 3** and **Fig. 4**. The results of energy balance [29] and numerical fourth-order Runge-Kutta method are plotted in these figures in order to have a better comparison**.** Herein the values of parameters are taken as $N = 15$, $\alpha = 10$. Comparison of errors is available via **Table 1.** and **Table 2.**

Table 1
Comparison Between the Results of MHPM , MMA and EBM with the Numerical Solution for A = 0.2, V = 10

t	0.4	0.8	1.2	1.6	2
MMA	−0.1942	0.1772	−0.1499	0.114	−0.0715
MHPM	−0.1942	0.1771	−0.1498	0.1138	−0.0712
EBM	−0.1971	0.1886	−0.1746	0.1556	−0.1321
Runge-Kutta	−0.1942	0.1772	−0.1498	0.1138	−0.0712
ErrorPercentage (MMA)	0.004	0.022	0.0597	0.137	0.326
ErrorPercentage (MHPM)	0.002	0.007	0.015	0.025	0.043
ErrorPercentage (EBM)	1.5	6.439	16.511	36.661	85.423

Table 2
Comparison Between the Results of MHPM , MMA and EBM with the Numerical Solution for A = 0.4, V = 10

t	0.4	0.8	1.2	1.6	2
MMA	−0.3974	0.3895	−0.3765	0.3585	−0.3358
MHPM	−0.3967	0.3868	−0.3706	0.3483	−0.3202
EBM	−0.3951	0.3805	−0.3565	0.3239	−0.2832
Runge-Kutta	−0.3970	0.3880	−0.3731	0.3526	−0.3266
ErrorPercentage (MMA)	0.09	0.39	0.89	1.68	2.82
ErrorPercentage (MHPM)	0.074	0.29	0.68	1.23	1.98
ErrorPercentage (EBM)	0.48	1.94	4.45	8.15	13.29

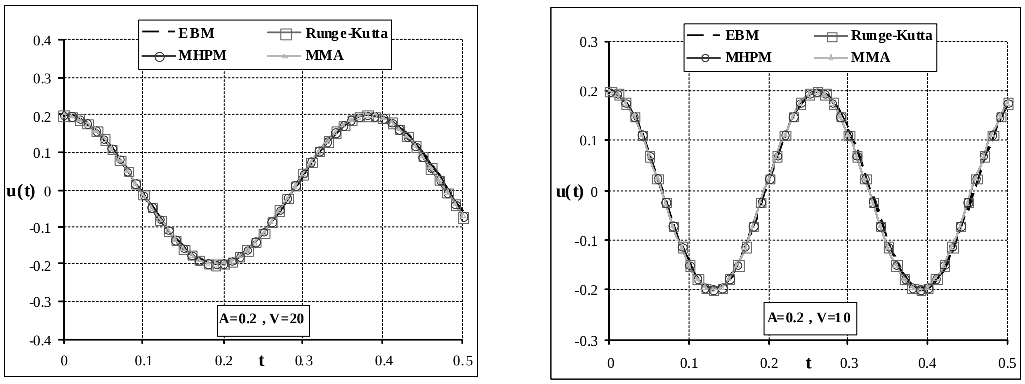

Figure 3: The Results of MMA, MHPM, EBM and Fourth Order Runge-Kutta Method for A = 0.2, N = 15

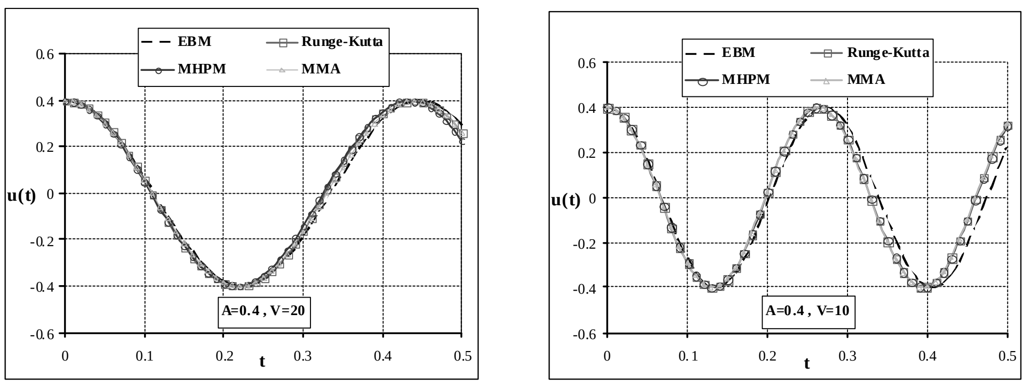

Figure 4: The Results of MMA, MHPM, EBM and Fourth Order Runge-Kutta Method for A = 0.4, N = 15

7. CONCLUSION

In this paper, the main purpose was to illustrate the application of modified homotopy perturbation method (MHPM) and max-min approach (MMA) in solving nonlinear oscillator arising in the microbeam based MEMS. Also, the capabilities and facile applications of these methods have been demonstrated in comparison with the numerical solution and energy balance method. By intensifying the value of Voltage and Initial Condition A, it has been considered that the disagreement between the answers obtained by MHPM, MMA and Runge-Kutta is very insignificant which indicates that these methods provide highly precise answers for nonlinear equations.

REFERENCES

[1] S. Lee, R. Ramadoss, M. Buck, V. M. Bright, K. C. Gupta, and Y. C. Lee, Reliability Testing of Flexible Printed Circuit-Based RF MEMS Capacitive Switches, *Microelectronics Reliability*, 44, (2004), 245-250.

[2] Jeffrey F. Rhoads, Steven W. Shaw, and Kimberly L. Turner, The Nonlinear Response of Resonant Microbeam Systems with Purely-Parametric Electrostatic Actuation, *Journal of Micromechanics and Microengineering*, 16, (2006), 890-899.

[3] Ingo Kuehne, Alexander Frey, Djordje Marinkovic, Gerald Eckstein, and Helmut Seidel, Power MEMS – A Capacitive Vibration-to-Electrical Energy Converter with Built-In Voltage, *Sensors and Actuators A.*, 142, (2008), 263-269.

[4] P. Czarnecki , X. Rottenberg, P. Soussan, P. Ekkels, P. Muller, P. Nolmans, W. De Raedt, H. A. C. Tilmans, R. Puers, L. Marchand, and I. DeWolf, Effect of Substrate Charging on the Reliability of Capacitive RF MEMS Switches, *Sensors and Actuators A.*, 154, (2009), 261-268.

[5] A. Tocchio, G. Langfelder, A. Longoni, and E. Lasalandra, In-Plane and Out-of-Plane MEMS Motion Sensors Based on Fringe Capacitances, *Procedia Engineering*, 5, (2010), 1392-1395.

[6] M. S. M. N. Hashim, and B. Batiha, A Note on the Adomian Decomposition Method for the Generalized Huxley Equation, *Applied Mathematics and Computation*, 181, (2006), 1439-1445.

[7] A. Sadighi, and D. D. Ganji, Analytic Treatment of Linear and Nonlinear Schrödinger Equations: A Study with Homotopy-Perturbation and Adomian Decomposition Methods, *Physics Letters A.*, 372, (2008), 465-469.

[8] M. Safari, D. D. Ganji, and M. Moslemi, Application of He's Variational Iteration Method and Adomian's Decomposition Method to the Fractional KdV_Burgers_Kuramoto Equation, *Computers and Mathematics with Applications*, 58, (2009), 2091-2097.

[9] Ji-Huan He, An Elementary Introduction to the Homotopy Perturbation Method, *Computers and Mathematics with Applications*, 57, (2009), 410-412.

[10] M. Jalaal, M. G. Nejad, P. Jalili, M. Esmaeilpour, H. Bararnia, E. Ghasemi, Soheil Soleimani, D. D. Ganji, and S. M. Moghimi, Homotopy Perturbation Method for Motion of a Spherical Solid Particle in Plane Couette Fluid Flow, *Computers and Mathematics with Applications*, 61, (2011), 2267-2270.

[11] S. M. Moghimi, D. D. Ganji, H. Bararnia, M. Hosseini, and M. Jalaal, Homotopy Perturbation Method for Nonlinear MHD Jeffery–Hamel Problem, *Computers and Mathematics with Applications*, 61, (2011), 2213-2216.

[12] A. M. Siddiqui, T. Haroon, and S. Irum, Torsional Flow of Third Grade Fluid Using Modified Homotopy Perturbation Method, *Computers and Mathematics with Applications*, 58, (2009), 2274-2285.

[13] Junfeng Lu, An Analytical Approach to the Sine_Gordon Equation Using the Modified Homotopy Perturbation Method, *Computers and Mathematics with Applications*, 58, (2009), 2313-2319.

[14] M. Shaban, D. D. Ganji, and M. M. Alipour, Nonlinear Fluctuation, Frequency and Stability Analyses in Free Vibration of Circular Sector Oscillation Systems, *Current Applied Physics*, 10, (2010), 1267-1285.

[15] Xinlong Feng, and Yinnian He, Modified Homotopy Perturbation Method for Solving the Stokes Equations, *Computers and Mathematics with Applications*, 61, (2011), 2262-2266.

[16] Ji-Huan He, and Xu-Hong Wu, Variational Iteration Method: New Development and Applications, *Computers and Mathematics with Applications*, 54, (2007), 881-894.

[17] Saeed Rastegar, Bahram Azizollah Ganji, Mojtaba Varedi, and Mehran Erza, Application of He's Variational Iteration Method to the Estimation of Diaphragm Deflection in MEMS Capacitive Microphone, *Measurement*, 44, (2011), 113-120.

[18] S. Ghafoori, M. Motevalli, M. G. Nejad, F. Shakeri, D. D. Ganji, and M. Jalaal, Efficiency of Differential Transformation Method for Nonlinear Oscillation, *Comparison with HPM and VIM, Current Applied Physics*, 11, (2011), 965-971.

[19] J. H. He, Max-Min Approach to Nonlinear Oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, 9, (2008), 207-210.

[20] S. S. Ganji, A. Barari, and D. D. Ganji, Approximate Analysis of Two-Mass-Spring Systems and Buckling of a Column, *Computers and Mathematics with Applications*, 61, (2011), 1088-1095.

[21] Guo-hua Chen, Zhao-Ling Tao, and Jin-Zhong Min, Notes on a Conservative Nonlinear Oscillator, *Computers and Mathematics with Applications*, 61, (2011), 2120-2122.

[22] L. Geng, and X. C. Cai, He's Frequency Formulation for Nonlinear Oscillators, *European Journal of Physics*, 28, (2007), 923-931.

[23] J. H. He, Comment on 'He's Frequency Formulation for Nonlinear Oscillators, *European Journal of Physics*, 29, (2008), L19-L22.

[24] H. L. Zhang, Application of He's Amplitude_Frequency Formulation to a Nonlinear Oscillator with Discontinuity, *Computers & Mathematics with Applications*, 58, (2009), 2197-2198.

[25] Ji-Huan He, Preliminary Report on the Energy Balance for Nonlinear Oscillations, *Mechanics Research Communications*, 29, (2002), 107-111.

[26] H. L. Zhang, Y. G. Xu, and J. R. Chang, Application of He's Energy Balance Method to a Nonlinear Oscillator with Discontinuity, *International Journal of Nonlinear Sciences and Numerical Simulation*, 10, (2009), 207-214.

[27] I. Mehdipour, D. D. Ganji, and M. Mozaffari, Application of the Energy Balance Method to Nonlinear Vibrating Equations, *Current Applied Physics*, 10, (2010), 104-112.

[28] Romesh C. Batra, Maurizio Porfiri, and Davide Spinello, Electromechanical Model of Electrically Actuated Narrow Microbeams, *Journal of Microelectromechanical Systems*, 15, (2006), 1175-1189.

[29] Yiming Fu, Jin Zhang, and Liangjiao Wan, Application of the Energy Balance Method to a Nonlinear Oscillator Arising in the Microelectromechanical System (MEMS), *Current Applied Physics*, 11, (2011), 482-485.