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SOLUTION OF STRONGLY NONLINEAR OSCILLATORS USING MODIFIED VARIATIONAL ITERATION METHOD

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Abstract: *In [T. A. Abassy, M. A. El-Tawil, H. El-Zoheiry, Exact solutions of some nonlinear partial differential equations using the variational iteration method linked with Laplace transforms and the Padé technique, Comput. Math. Appl. 54 (2007) 940-954] an efficient modification of the variational iteration method (VIM) was presented by using Laplace transforms and Padé approximants. In this paper, standard and modified variational iteration methods are applied to solve the strongly nonlinear oscillators, which has rich mathematical structures and many important applications in physics and mathematics. In some cases, the solution of VIM is adequate only in a small region when the exact solution is not reached. To overcome the drawback, Laplace transforms and Padé approximants, are applied to the approximate solution to improve the accuracy and enlarge the convergence domain. By using this modified, the solution of the strongly nonlinear oscillator is constructed with better accuracy and better convergence than by using the VIM alone. The current results are compared with those derived from the established Runge-Kutta method in order to verify the accuracy of the modified VIM. Numerical and figurative illustrations show that it is a promising tool for solving strongly nonlinear oscillators.*

Keywords: *Nonlinear oscillators, Approximate solutions, Variational iteration method, Laplace transform, Padé approximants.*

1. INTRODUCTION

In science and engineering there exist many nonlinear differential equations and even strongly nonlinear problems which are still very difficult to solve either analytically or numerically. Nonlinear oscillation in physics and applied mathematics has been a topic to intensive research for many years. There are many approaches for approximating solutions to strongly nonlinear oscillators. Some of these well-known methods are such as: harmonic balance method [1], multiple scales method [2], Krylov-Bogoliubov-Mitropolsky method [3, 4], modified Lindstedt-Poincare method [5], linearized perturbation method [6], energy balance method [7, 13, 14], iteration perturbation method [8], bookkeeping parameter perturbation method [9], amplitude frequency formulation [10] max-min approach [11, 15], Mickens iteration procedure [12],

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rational harmonic balance method [17], adomian decomposition method [18], variational iteration method [19, 16], homotopy perturbation method [20, 16], and etc.

In this paper we consider the following general second-order nonlinear oscillator differential equation:

$$y''(t) + \varepsilon F(y(t), y'(t), t) = 0, \tag{1}$$

with initial conditions

$$y(0) = A, \quad y'(0) = B, \tag{2}$$

where ε need not be small, and $F(y(t), y'(t), t)$ is nonlinear analytic function of the displacement $y(t)$, the velocity $y'(t)$, and the time t.

In [22], a modified variational iteration method (VIM) has been presented by using taylor series method [21], laplace transform [23] and padé approximates [24]. The purpose of this paper is to consider the numerical solution of strongly nonlinear oscillators by using standard and modified variational iteration methods.

The organization of this paper is as follows. In Section 2 we describe the standard variational iteration method and briefly discuss Padé approximants. In Section 3, we present the modification technique of variational iteration method. In Section 4, the methods are applied to a variety of examples to show the efficiency and simplicity of the methods. At the end of the paper, there is a summary of the main conclusions.

2. VARIATIONAL ITERATION METHOD (VIM)

The variational iteration method is a method for solving linear and nonlinear problems. This method is introduced by the Chinese researcher He [25] by modifying the general Lagrange multiplier method [26]. The method constructs an iterative sequence of functions converging to exact solution. In the case of linear problems by determining exact Lagrange multiplier, approximate solution turns into exact solution and is available by only one iteration. To illustrate the method, let us consider the following nonlinear equation:

$$L(y(t)) + N(y(t)) = g(t), \tag{3}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytical function. According to the variational iteration method, we can construct the following correction functional:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\xi)\,(L(y_n(\xi)) + N(\tilde{y}_n(\xi)) - g(\xi))\,d\xi, \tag{4}$$

where λ is a general Lagrange multiplier which can be identified via variational theory, $y_0(t)$ is an initial approximation with possible unknowns, and $\tilde{y}_n$ is considered as restricted variation [27], i.e. $\delta\tilde{y}_n = 0$. Consequently, the solution is given by

$$y(t) = \lim_{n\to\infty} y_n(t). \tag{5}$$

According to the variational iteration method, we can construct the correction functional of (1) as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\xi)\,(y_n''(\xi) + \varepsilon F(\tilde{y}_n(\xi), \tilde{y}_n'(\xi), \xi))\,d\xi, \tag{6}$$

where λ is general Lagrange multiplier, $\tilde{y}_n$ and $\tilde{y}'_n$ denote restricted variation, i.e. $\delta\tilde{y}_n = \delta\tilde{y}'_n = 0$. Making the above correction functional as the initial guess, we can begin with the following stationary conditions:

$$1 - \lambda'\big|_{\xi = t} = 0, \tag{7}$$

$$\lambda\big|_{\xi = t} = 0, \tag{8}$$

$$\lambda'' = 0. \tag{9}$$

This in turn gives

$$\lambda = \xi - t. \tag{10}$$

Substituting this value of the Lagrangian multiplier into functional (6) gives the iteration formula

$$y_{n+1}(t) = y_n(t) + \int_0^t (\xi - t)\,(y''_n(\xi) + \varepsilon F\,(y_n(\xi), y'_n(\xi), \xi))\, d\xi\,, \tag{11}$$

According to (2) we start with initial approximation $y_0(t) = A + Bt$, and using (11) we obtain the following successive approximations:

$$y_0(t) = A + Bt,$$

$$y_1(t) = A + Bt + \int_0^t (\xi - t)\,(y''_0(\xi) + \varepsilon F\,(y_0(\xi), y'_0(\xi), \xi))\, d\xi\,,$$

$$y_2(t) = y_1(t) + \int_0^t (\xi - t)\,(y''_1(\xi) + \varepsilon F\,(y_1(\xi), y'_1(\xi), \xi))\, d\xi\,,$$

$$\vdots$$

$$y_{n+1}(t) = y_n(t) + \int_0^t (\xi - t)\,(y''_n(\xi) + \varepsilon F\,(y_n(\xi), y'_n(\xi), \xi))\, d\xi\,.$$

Recall that

$$y(t) = \lim_{n \to \infty} y_{n+1}(t)\,. \tag{12}$$

2.1. Padé Approximants

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(t)$. The $[L/M]$ Padé approximants to a function $u(t)$ is given by [24, 28]:

$$\left[\frac{L}{M}\right] = \frac{P_L(t)}{Q_M(t)}\,, \tag{13}$$

where $P_L(t)$ is a polynomial of degree at most L and $Q_M(t)$ is a polynomial of degree at most M. The formal power series

$$u(t) - \frac{P_L(t)}{Q_M(t)} = 0(t^{L+M+1})\,, \tag{14}$$

$$u(t) = \sum_{i=0}^{\infty} a_i t^i, \tag{15}$$

determine the coefficients of $P_L(t)$ and $Q_M(t)$.

Since we can obviously multiply the numerator and denominator by constant and leave $[L/M]$ unchanged, we impose the normalization condition

$$Q_M(t) = 1. \tag{16}$$

Finally we require that $P_L(t)$ and $Q_M(t)$ have no common factors.

If we write the coefficients of $P_L(t)$ and $Q_M(t)$ as

$$\begin{aligned} P_L(t) &= p_0 + p_1 t + p_2 t^2 + ... + p_L t^L, \\ Q_M(t) &= q_0 + q_1 t + q_2 t^2 + ... + p_M t^M. \end{aligned} \tag{17}$$

Then by (16) and (17) we may multiply (14) by $Q_M(t)$, which linearizes the coefficient equations. We can write out (14) in more detail as

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0, \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0, \\ \vdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0, \end{cases} \tag{18}$$

$$\begin{cases} a_0 = p_0, \\ a_1 + a_0 q_1 = p_1, \\ a_2 + a_1 q_1 + a_0 q_2 = p_2, \\ \vdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L. \end{cases} \tag{19}$$

Once, the q‘s are known from (18), (19) can be solved easily. If (18) and (19) are nonsingular, then they can be solved directly as follows:

$$\left[\frac{L}{M}\right] = \frac{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} t^j & \sum_{j=M-1}^{L} a_{j-M+1} t^j & \cdots & \sum_{j=0}^{L} a_j t^j \end{bmatrix}}{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ t^M & t^{m-1} & \cdots & 1 \end{bmatrix}}. \tag{20}$$

To obtain diagonal Padé approximants of different order like [2/2], [4/4] or [6/6] MAPLE can be efficiently used.

2. THE MODIFICATION OF VARIATIONAL ITERATION METHOD

In some cases, the solution of variational iteration method converges in a limited interval, and outside it, its error is high. Furthermore this solution does not exhibit the periodic behavior which is characteristic of oscillator equations. In order to improve the accuracy of VIM, we have used the method given in [22] for solving nonlinear oscillators. This method can be done by using the following algorithm:

Algorithm:

Step 1: Solve the differential equation using VIM and convert the obtained solution to Taylor series.

Step 2: Take the Laplace transform of the truncated series.

Step 3: Find the Padé approximation of the step 2.

Step 4: Take the inverse Laplace transform.

Remark 3.1: The main reason for utilizing this modified method in solution of nonlinear oscillator equations are the use of Padé approximation and the Laplace transform. Because, Padé approximation gives fractional expressions and moreover the inverse Laplace transform of such expressions usually generates trigonometric or exponential expressions. So this modified method generates approximate periodic or damping solutions. These instances will become evident in the examples given in the next section.

4. ILLUSTRATIVE EXAMPLES

In this section three examples are given to demonstrate the applicability and accuracy of our methods. All the results are calculated by MAPLE 13 with digits precision on a Personal Computer.

Example 4.1: Consider the following Duffing equation [29, 30]:

$$y''(t) + y(t) + \varepsilon y^3(t) = 0, \qquad 0 \le t \le 100, \tag{21}$$

subject to the initial conditions

$$y(0) = A, \qquad y'(0) = 0. \tag{22}$$

The solution of (21) is only defined for $\varepsilon A^2 > -1$. We set the parameter $\varepsilon = -0.1$ and $A = 0.5$ for this example. Now, we will solve (21) by using the above algorithm as follows:

Step 1: Solve the differential equation (21) using VIM.

From (11) and (21), we obtain following iteration formula

$$y_{n+1}(t) = y_n(t) + \int_0^t (\xi - t)\,(y_n''(\xi) + y_n(\xi) + \varepsilon y_n^3(\xi))\,d\xi\,. \tag{23}$$

According to (22) we start with initial approximation $y_0(t) = 0.5$, and we can get the following successive approximations:

$$y_0(t) = 0.500000,$$

$$y_1(t) = 0.500000 - 0.243750\,t^2,$$

$$y_2(t) = 0.500000 - 0.243750\,t^2 + 0.0187891\,t^4 + 0.000297070\,t^6 - 0.0000258610\,t^8,$$

$$\vdots$$

and so on. In a similar manner the rest of the components can be obtained by using (23). After ten iterative the approximate solution for (21) is given by

$$y(t) \approx y_{10}(t) = 0.500000 - 0.243750\,t^2 + 0.0187891\,t^4 + 0.000297070\,t^6 - 0.0000258610\,t^8 + 0.00000510722\,t^{10} - 0.000000245471\,t^8 + O(t^{14}). \quad (24)$$

This series solution does not exhibit the periodic behavior which is characteristic of the oscillatory equation (21). Comparison of the approximate solution (24) and the solution obtained by the fourth-order Runge-Kutta method in Fig. 1.

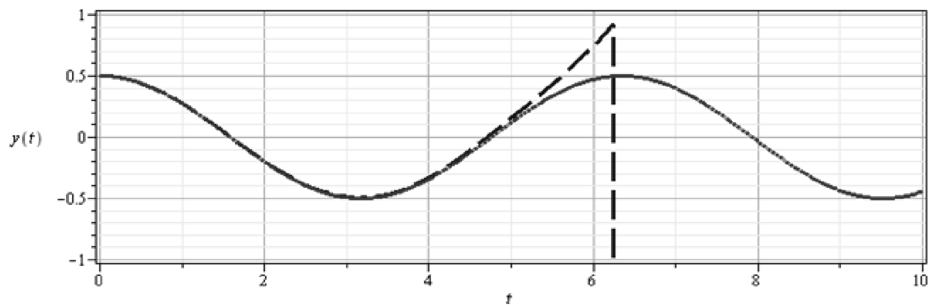

Figure 1: Plots of Displacement *y* versus Time *t*. Solid Line: Runge-Kutta Method; Dashed Line: VIM.

Step 2: Take the Laplace transform of the equations (24).

Applying the Laplace transformation to the series solution (24) yields

$$\pounds[y(t)] \approx \frac{0.500000}{s} - \frac{0.487500}{s^3} + \frac{0.450938}{s^5} + \frac{0.203227}{s^7} - \frac{18.5331}{s^{11}} - \frac{117.581}{s^{13}}. \quad (25)$$

Step 3: Find the Padé approximation of the equation (25).

For simplicity, let $s = \frac{1}{t}$, then from (25) we have

$$\pounds[y(t)] \approx 0.500000\,t - 0.487500\,t^3 + 0.450938\,t^5 - 0.0203227\,t^7 - 1.84398\,t^9 + 18.5331\,t^{11} - 117.581\,t^{13}. \quad (26)$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.500000\,t + 4.36250\,t^3}{1.0 + 9.6999\,t^2 + 8.55562\,t^4}. \quad (27)$$

Recalling $t = \frac{1}{s}$, we obtain in terms of

$$\left[\frac{4}{4}\right] = \frac{0.500000\,s^3 + 4.362450\,s}{1.0\,s^4 + 9.6999\,s^2 + 8.55562}. \quad (28)$$

Step 4: Take the inverse Laplace transform of the equation (28).

By using the inverse Laplace transformation to the Padé approximation, we obtain

$$y(t) \approx 0.500407\cos(0.990604\,t) - 0.000406818\cos(2.95275\,t). \quad (29)$$

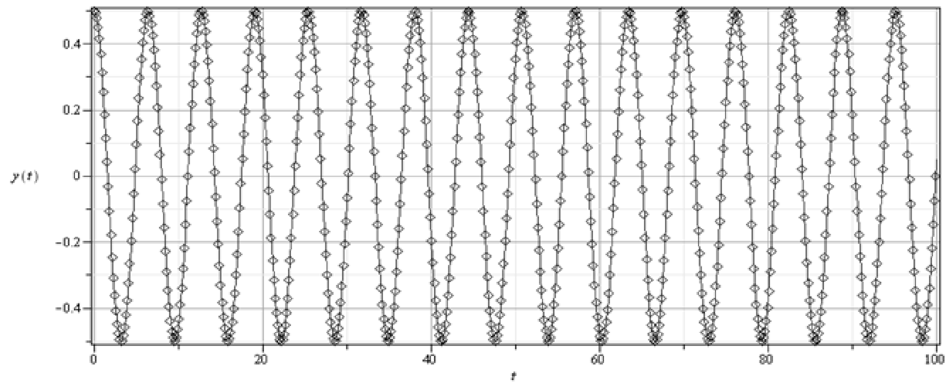

Figure 2: Plots of Displacement y versus Time t for Example 4.1. Solid Line: Presented Method; Diamond: RK4

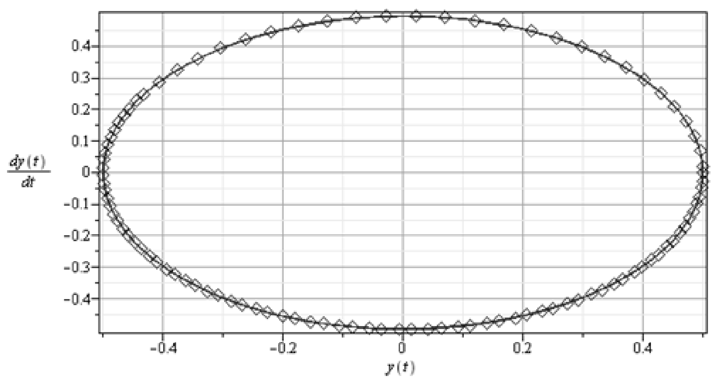

Figure 3: Plots of Phase Plane Diagram for Example 4.1. Solid Line: Presented Method; Diamond: RK4

The graphs of the displacement $y(t)$ and phase diagram are sketched in Figs. 2 and 3 and are compared with the numerical solution of the fourth-order Runge-Kutta method with time step $\Delta t = 0.001$.

Example 4.2: Consider the following Van der Pol oscillator [31]

$$y''(t) + \varepsilon (y^2(t) - 1) y'(t) + y(t) = 0, \qquad 0 \le t \le 100, \tag{30}$$

subject to the initial conditions

$$y(0) = 0.1, \qquad y'(0) = 0. \tag{31}$$

We choose $\varepsilon = -0.1$ for this example.

Step 1: Solve the differential equation (30) using VIM.

From (11) and (30), we obtain following iteration formula

$$y_{n+1}(t) = y_n(t) + \int_0^t (\xi - t)\,(y_n''(\xi) + \varepsilon\,(y_n^2(\xi) - 1)\,y_n'(\xi) + y_n(\xi))\,d\xi\,. \tag{32}$$

According to (31) we start with initial approximation $y_0(t) = 0.1$, and we can get the following successive approximations:

$$y_0(t) = 0.100000,$$
$$y_1(t) = 0.100000 - 0.0500000\,t^2,$$
$$y_2(t) = 0.100000 - 0.0500000\,t^2 + 0.00165000\,t^3 + 0.00416667\,t^4 - 0.000005\,t^5 - 5.95238 \times 10^{-7}\,t^7,$$
$$\vdots$$

and so on. In a similar manner the rest of the components can be obtained by using (32). After five iterative the approximate solution for (30) is given by

$$y(t) \approx y_5(t) = 0.100000 - 0.0500000\,t^2 + 0.00165000\,t^3 + 0.00412583\,t^4 - 0.000159191\,t^5 - 0.000135175\,t^6 + 4.52110 \times 10^{-6}\,t^7 + 2.44519 \times 10^{-6}\,t^8 + 5.40753 \times 10^{-8}\,t^9 + O(t^{10}). \quad (33)$$

Comparison of the approximate solution (33) and the solution obtained by the fourth-order Runge-Kutta method in Fig. 4.

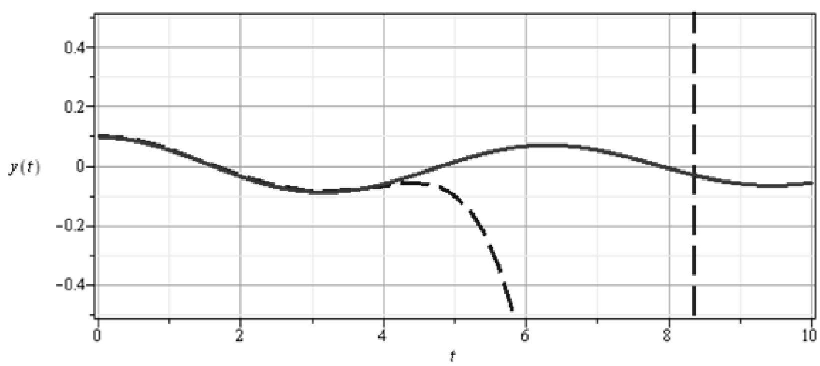

Figure 4: Plots of Displacement y versus Time t. Solid Line: Runge-Kutta Method; Dashed Line: VIM

Step 2: Take the Laplace transform of the equations (33).

Applying the Laplace transformation to the series solution (33) yields

$$£[y(t)]; \frac{0.100000}{s} - \frac{0.100000}{s^3} + \frac{0.00990000}{s^4} + \frac{0.0999199}{s^5} + \frac{0.0191029}{s^6} - \frac{0.0973267}{s^7} - \frac{0.0227864}{s^8} + \frac{0.0985904}{s^9} + \frac{0.0196229}{s^{10}}. \quad (34)$$

Step 3: Find the Padé approximation of the equation (34).

Setting $s = \frac{1}{t}$ in (34) and calculating the [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right] = \frac{0.100000\,t + 0.0329999\,t^2 + 0.902963\,t^3 + 0.0897669\,t^4}{1.0 + 0.329999\,t + 10.0296\,t^2 + 1.12867\,t^3 + 9.00676\,t^4}. \quad (35)$$

Recalling $t = \frac{1}{s}$, we obtain in terms of s

$$\left[\frac{4}{4}\right] = \frac{0.100000\, s^3 + 0.0329999\, s^2 + 0.032999\, s + 0.0897669}{s^4 + 0.0329999\, s^3 + 10.0296\, s^2 + 1.12867\, s + 9.00676} \tag{36}$$

Step 4: Take the inverse Laplace transform of the equation (36).

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain

$$\begin{aligned} y(t) \approx & -1.53099 \times 10^{-7}\, e^{(-0.115124 - 2.99889\, i)\, t} + 1.55508 \times 10^{-6}\, i\, e^{(-0.115124 - 2.99889\, i)\, t} \\ & -1.53099 \times 10^{-7}\, e^{(-0.115124 + 2.99889\, i)\, t} - 1.55508 \times 10^{-6}\, i\, e^{(-0.115124 + 2.99889\, i)\, t} \\ & + 0.0500002\, e^{(-0.0498751 - 0.998761\, i)\, t} + 0.00249217\, i\, e^{(-0.098751 - 0.998761\, i)\, t} \\ & + 0.0500002\, e^{(-0.0498751 - 0.998761\, i)\, t} + 0.00249217\, i\, e^{(-0.098751 - 0.998761\, i)\, t} \end{aligned} \tag{37}$$

The graphs of the displacement $y(t)$ and phase diagram are sketched in Figs. 5 and 6 and are compared with the numerical solution of the fourth-order Runge-Kutta method with time step $\Delta t = 0.001$.

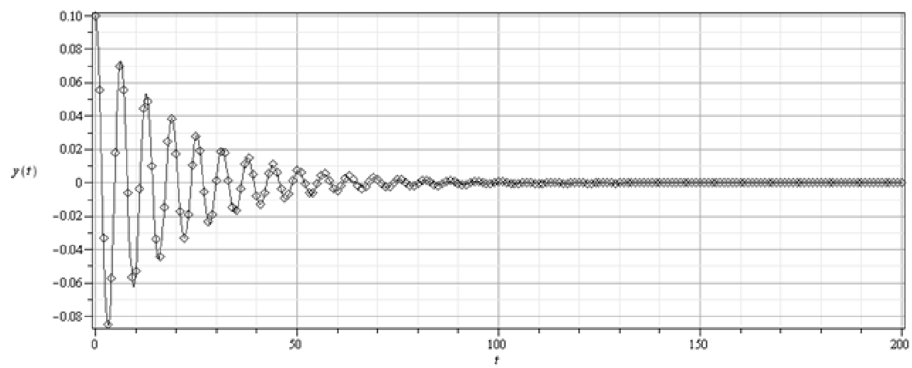

Figure 5: Plots of Displacement y versus Time t for Example 4.2. Solid Line: Presented Method; Diamond: RK4

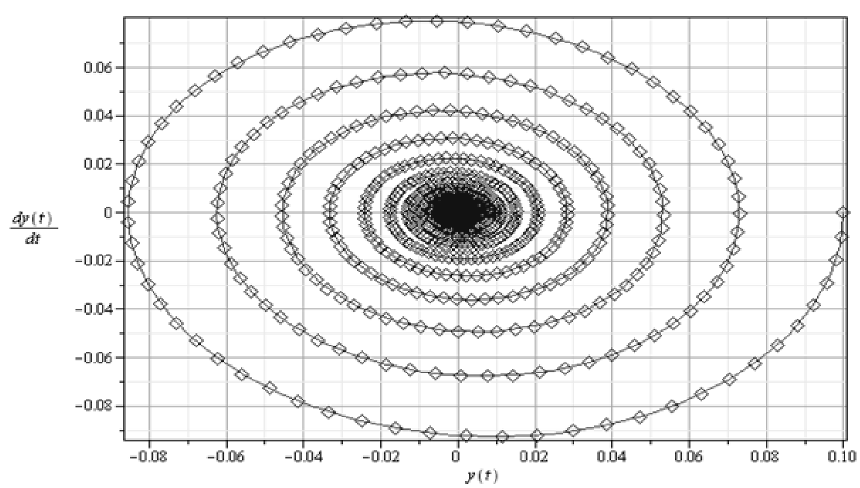

Figure 6: Plots of Phase Plane Diagram for Example 4.2. Solid Line: Presented Method; Diamond: RK4

Example 4.3: Consider the following initial-value problem [32]:

$$y''(t) + \frac{y(t)}{1 + \varepsilon y^2(t)} = 0, \qquad 0 \le t \le 100, \tag{38}$$

subject to the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{39}$$

We rewrite equation (38) in the form

$$y''(t) + y(t) + \varepsilon y''(t) y^2(t) = 0, \tag{40}$$

and set $\varepsilon = 0.1$ for this example.

Step 1: Solve the differential equation (40) using VIM.

From (11) and (40), we obtain following iteration formula

$$y_{n+1}(t) = y_n(t) + \int_0^t (\xi - t)\,(y_n''(\xi) + y_n(\xi) + \varepsilon\, y_n''(\xi)\, y_n^2(\xi))\, d\xi\,. \tag{41}$$

According to (39) we start with initial approximation $y_0(t) = 1$, and we can get the following successive approximations:

$$y_0(t) = 1.00000,$$
$$y_1(t) = 1.00000 - 0.500000\,t^2,$$
$$y_2(t) = 1.00000 - 0.450000\,t^2 + 0.033333\,t^4 + 0.000833333\,t^6$$
$$\vdots$$

and so on. In a similar manner the rest of the components can be obtained by using (41). After five iterative the approximate solution for (40) is given by

$$y(t) \approx y_5(t) = 1.00000 - 0.454550\,t^2 + 0.0281619\,t^4 + 0.000803118 - 0.000159429\,t^8 - 3.80710\,t^6 \times 10^{-6}\,t^{10} + 1.39159 \times 10^{-8}\,t^{12} + O(t^{14}). \tag{42}$$

Comparison of the approximate solution (42) and the solution obtained by the fourth-order Runge-Kutta method in Fig. 7.

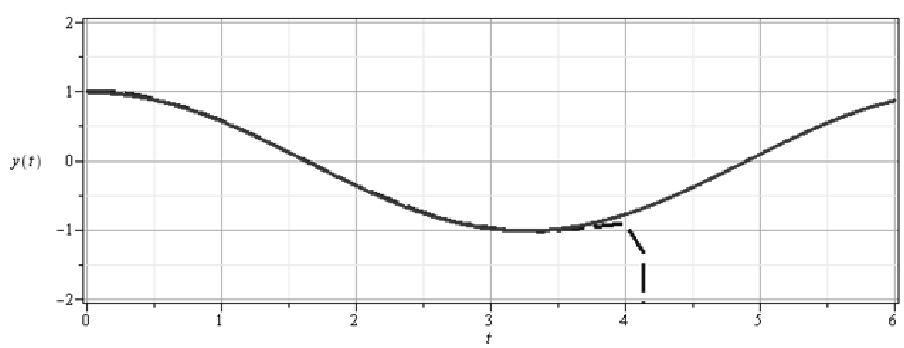

Figure 7: Plots of Displacement y versus Time t. Solid Line: Runge-Kutta Method; Dashed Line: VIM

Step 2: Take the Laplace transform of the equations (42).

Applying the Laplace transformation to the series solution (42) yields

$$£[y(t)]\approx\frac{1.00000}{s}-\frac{0.909100}{s^3}+\frac{0.675886}{s^5}+\frac{0.578245}{s^7}-\frac{6.42819}{s^9}-\frac{13.8152}{s^{11}}+\frac{666.575}{s^{13}}. \quad (43)$$

Step 3: Find the Padé approximation of the equation (43).

Setting $s=\frac{1}{t}$ in (43) and calculating the [4/4] Padé approximant gives

$$\left[\frac{4}{4}\right]=\frac{1.0t+7.001172t^3}{1.0+7.92082t^2+6.52493t^4}. \quad (44)$$

Recalling $s=\frac{1}{t}$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right]=\frac{s^3+7.01172s}{s^4+7.92082s^2+6.52493}. \quad (45)$$

Step 4: Take the inverse Laplace transform of the equation (45).

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain

$$y(t)\approx 1.00409\cos(0.966372t)-0.000409292\cos(2.64328t). \quad (46)$$

The graphs of the displacement and phase diagram are sketched in Figs. 8 and 9 and are compared with the numerical solution of the fourth-order Runge-Kutta method with time step $\Delta t = 0.001$.

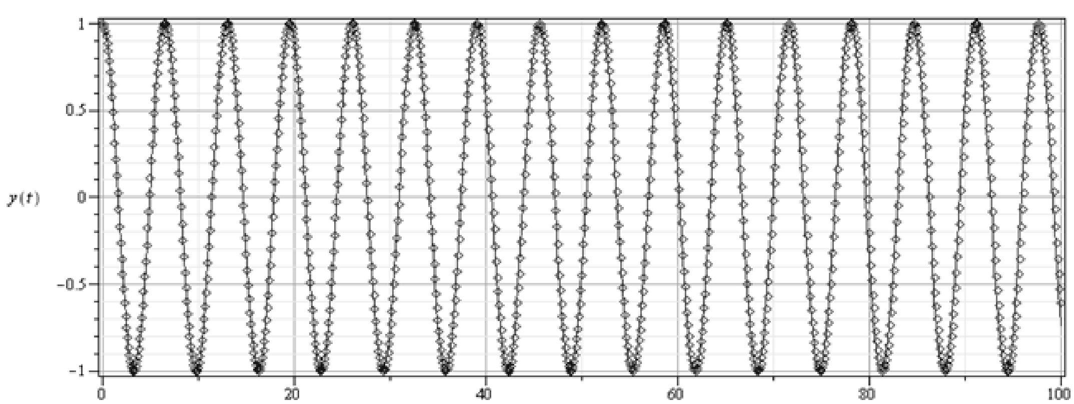

Figure 8: Plots of Displacement y versus Time t for Example 4.3. Solid Line: Presented Method; Diamond: RK4

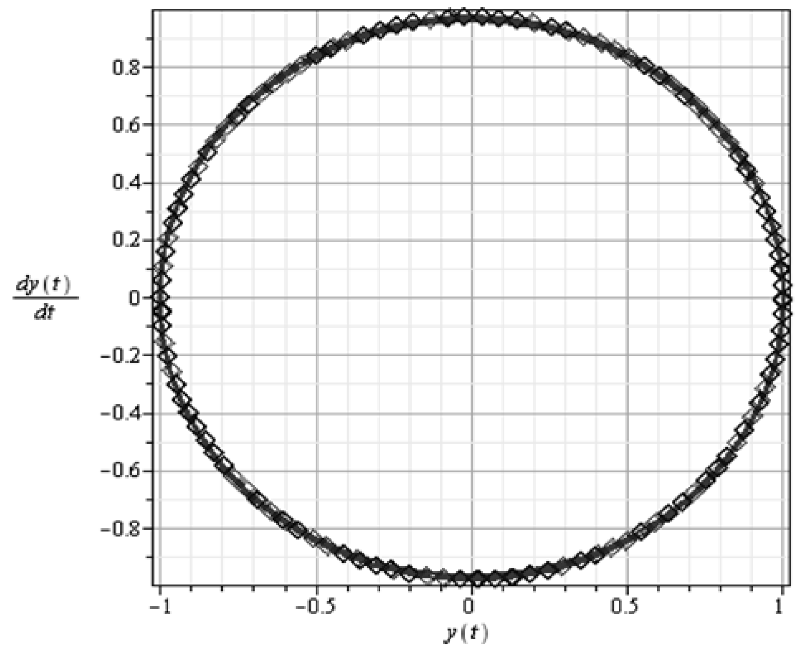

Figure 9: Plots of Phase Plane Diagram for Example 4.3. Solid Line: Presented Method; Diamond: RK4

5. CONCLUSION

In this work, we used a simple but effective modification of the variational iteration method to handle strongly nonlinear oscillators. Some examples were given to illustrate the effectiveness and convenience of this method. The results anticipated were compared with the standard variational iteration method and the fourth-order Runge-Kutta (RK4). The obtained results show that:

1. The modification of the variational iteration method is more accurate than the standard one,
2. This modified method is valid for larger region,
3. This modified method generates approximate periodic or damping solutions.

Moreover, the present work can be used as paradigms for many other applications in searching for periodic solutions of nonlinear oscillations and so can be found widely applicable in engineering and science.

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