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RECONSTRUCTION OF VARIATIONAL ITERATION METHOD FOR BOUNDARY VALUE PROBLEMS IN STRUCTURAL ENGINEERING AND FLUID MECHANICS

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Abstract: *Reconstruction of Variational iteration method (RVIM) is applied to solve linear and nonlinear boundary value problems with particular significance in structural engineering and fluid mechanics. Particularly a range of fourth-order equations appears as boundary condition in the analysis of beam deformation. In this article according to importance of these kinds of equations in structure analysis specifically beams. We discussed some of these equations as examples to present a new method to obtain an accurate approximate solutions. This new method that is based on Laplace transform reduces the size of calculations and tends to an accurate answer with very good speed. RVIM is able to solve many linear and Non-linear equations. Choosing easy initial conditions and high accuracy even in first steps are the advantages of this method. Comparison with exact solution revealed that RVIM is very effective, powerful and high accuracy.*

1. INTRODUCTION

The non-stopping continuing search for a better and easy to use tool for the solution of nonlinear equations illuminating the nonlinear phenomena of our life has been the focus of recent studies in the literature. Most of the scientific phenomena are modeled by ordinary or partial differential equations, large class of nonlinear equations do not have a precise analytic solution, so numerical methods have widely been used to handle these equations. There are also some analytic techniques for nonlinear equations. Some of the classic analytic methods are the Lyapunov's artificial small parameter method, perturbation techniques and Expansion method. In the last two decades, some new analytic methods have been proposed to handle functional equations, among them are Adomian decomposition method (ADM) [1, 2], homotopy analysis method (HAM) [3-6], variational iteration method (VIM) [7-9] and homotopy perturbation method (HPM) [10-13].One of the newest analytical methods to solve nonlinear equations is RVIM, Our analysis indicates that RVIM yields a variational scheme made by Laplace Transform.

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In this method we applied it to avoid using Adomian Polynomials and calculating Lagrange Multiplier which in many cases are unsolvable and too complicated. Also this method is independent of any small parameters.

Besides, RVIM reduces size of calculation and decreases time of solution. The ability and power of variation analysis is further confirmed by the numerical results.

This paper discusses the analytical approximate solution for fourth-order equations with nonlinear boundary conditions involving third-order derivatives. The general form of two equations for a fixed positive integer n, $n \geq 2$, is a differential equation of order $2n$:

$$y^{(2n)} + f(x, y) = 0. \tag{1}$$

Subject to the boundary conditions

$$y^{(2j)}(a) = A_{2j}, \quad y^{(2j)}(b) = B_{2j}, \quad j = 0\,(1)\,n - 1. \tag{2}$$

Where $-\infty < a \leq x \leq b < \infty$, A_{2j}, B_{2j}, $j = 0\,(1)\,n - 1$ are finite constants.

It is assumed that y is sufficiently differentiable and that a unique solution of (1) exists. Problems of this kind are commonly encountered in plate-deflection theory and in fluid mechanics for modeling viscoelastic and inelastic flows [14-16]. Usmani [14,15] discussed sixth order methods for the linear differential equation $y^{(4)} + P(x)y = q(x)$ subject to the boundary conditions $y_{(a)} = a_0$, $y''_{(A)} = a_2$, $y_{(b)} = B_0$, $y''_{(b)} = B_2$.. The method described in [14] leads to five diagonal linear systems and involves p', p'', q', q', at a and b, while the method described in [15] leads to nine diagonal linear systems.

In this paper we consider iterative solutions for fourth-order equations with nonlinear boundary conditions involving third-order derivatives. This kind of problem appears naturally in viscoelastic and inelastic flows, deformation of beams, and plate deflection and the study of deformations of elastic beams on elastic bearings. Plates are employed in many engineering structures and thin-walled structural components play fundamental role in a diverse range of engineering applications. So it's necessary to solve these problems with higher accuracy that RVIM can provide it.

2. DESCRIPTION OF THE METHOD

In the following section, an alternative method for finding the optimal value of the Lagrange multiplier by using of the Laplace transform [17] will be investigated a large number of problems in science and engineering involve the solution of partial differential equations. Suppose x, t are two independent variables; consider t as the principal variable and x as the secondary variable. If $u\,(x, t)$ is a function of two variables x and t, when the Laplace transform is applied with as a variable, definition of Laplace transform is

$$\mathbb{L}[u\,(x, t); s] = \int_0^\infty e^{(-st)} u\,(x, t)\,dt \tag{3}$$

We have some preliminary notations as

$$\mathbb{L}\left[\frac{\partial u}{\partial t}; s\right] = \int_0^\infty e^{(-st)}\,\frac{\partial u}{\partial t}\,dt = sU\,(x, s) - u\,(x, 0) \tag{4}$$

$$\mathbb{L}\left[\frac{\partial^2 u}{\partial t^2}; s\right] = s^2 U(x, s) - su(x, 0) - u_t(x, 0) \tag{5}$$

where

$$U(x, s) = \mathbb{L}[u(x, t); s]. \tag{6}$$

We often come across functions which are not the transform of some known function, but then, they can possibly be as a product of two functions, each of which is the transform of a known function. Thus we may be able to write the given function as $U(x, s)$, $V(x, s)$ where $U(s)$ and $V(s)$ are known to the transform of the functions $u(x, t)$, $v(x, t)$ respectively. The convolution of $u(x, t)$ and $v(x, t)$ is written $u(x, t) * v(x, t)$. It is defined as the integral of the product of the two functions after one is reversed and shifted.

Convolution Theorem: if $U(x, s)$, $V(x, s)$ are the Laplace transform of $u(x, t)$, $v(x, t)$ when the Laplace transform is applied to as a variable, respectively; then $U(x, s).V(x, s)$ is the Laplace Transform of $\int_0^t u(x, t-\mathcal{E})\, v(x, \mathcal{E})\, d\mathcal{E}$

$$\mathbb{L}^{-1}[U(x, s).V(x, s)] = \int_0^t u(x, t-\mathcal{E})\, v(x, \mathcal{E})\, d\mathcal{E}. \tag{7}$$

To facilitate our discussion of Reconstruction of Variational Iteration Method, introducing the new linear or nonlinear function $h(u(t, x)) = f(t, x) - N(u(t, x))$ and considering the new equation, rewrite $h(u(t, x)) = f(t, x) - N(u(t, x))$ as

$$L(u(t, x)) = h(t, x, u). \tag{8}$$

Now, for implementation the correctional function of VIM based on new idea of Laplace transform, applying Laplace Transform to both sides of the above equation so that we introduce artificial initial conditions to zero for main problem, then left hand side of equation after transformation is featured as

$$\mathbb{L}[L\{u(x, t)\}] = U(x, s)P(s). \tag{9}$$

Where $P(s)$ is polynomial with the degree of the highest order derivative of the selected linear operator.

$$\mathbb{L}[L\{u(x, t)\}] = U(x, s)P(s) = \mathbb{L}[h\{(x, t, u)\}]. \tag{10}$$

Then

$$U(x, s) = \frac{\mathbb{L}[h\{(x, t, u)\}]}{P(s)}. \tag{11}$$

Suppose that $D(s) = \frac{1}{P(s)}$, and $\mathbb{L}[h\{(x, t, u)\}] = H(x, s)$. Therefore using the convolution theorem we have

$$U(x, s) = D(s).H(x, s) = \mathbb{L}\{(d(t) * h(x, t, u))\}. \tag{12}$$

Taking the inverse Laplace transform on both side of Eq.

$$u(x, t) = \int_0^t d(t-\mathcal{E})\, h(x, \mathcal{E}, u)\, d\mathcal{E}. \tag{13}$$

Thus the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x, t) = u_0(x, t) + \int_0^t d(t-\mathcal{E})\, h(x, \mathcal{E}, u_n)\, d\mathcal{E}. \tag{14}$$

And $u_0(x, t)$ is initial solution with or without unknown parameters. In absence of unknown parameters, $u_0(x, t)$ should satisfy initial/boundary conditions.

3. THE APPLICATIONS OF RVIM METHOD

In this section, the RVIM technique is applied to two different forms of the fourth-order boundary value problems introduced as follow:

Example 1: Consider the following linear boundary value problem:

$$u^{(4)}(x) = 4e^x + u(x), \quad 0 < x < 1 \tag{15}$$

subject to the boundary conditions:

$$u(0) = 1, \quad u'(0) = 2, \quad u(1) = 2e, \quad u'(1) = 3e. \tag{16}$$

The exact solution for this problem is

$$u(x) = (1 + x)e^x. \tag{17}$$

At first rewrite Eq. (15) based on selective linear operator as

$$L\{u(x)\} = u^{(4)}(x) = \overbrace{(4e^x + u(x))}^{h(x,u)}. \tag{18}$$

Now Laplace Transform is implemented with respect to independent variable x on both sides of Eq. (18) and by using the new artificial initial conditions (which all of them are zero) we have

$$s^4U(x) = L\{h(u, x)\} \tag{19}$$

$$U(x) = \frac{\mathbb{L}\{h(x, u)\}}{s^4}. \tag{20}$$

And whereas Laplace inverse Transform of $\frac{1}{s^4}$ is as follows:

$$\mathbb{L}^{-1}\left[\frac{1}{4}\right] = \frac{1}{6}t^3. \tag{21}$$

By using the Laplace inverse Transform and convolution theorem, it is concluded that

$$u(x) = \int_0^x \frac{1}{6}(x - \mathcal{E})^3 h(u, \mathcal{E})\, d\mathcal{E}. \tag{22}$$

Hence, we arrive at the following iterative formula for the approximate solution of (15) subject to the initial condition (16),

$$u_{n+1}(x) = u_0(x) + \int_0^x \frac{1}{6}(x - \mathcal{E})^3 [4e^{\mathcal{E}} + u_n(\mathcal{E})]\, d\mathcal{E}. \tag{23}$$

According to above equation, for first order approximation we have:

$$u_1(x) = u_0(x) + \int_0^x \frac{1}{6}(x - \mathcal{E})^3 [4e^{\mathcal{E}} + u_0(\mathcal{E})]\, d\mathcal{E}. \tag{24}$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d, \tag{25}$$

where a, b, c, and d are unknown constants to be further determined.

By the iteration formula (23), the following first-order approximation may be written:

$$u_1(x) = ax^3 + bx^2 + cx + d\int_0^x \frac{1}{6}(x-\mathcal{E})^3\,[4e^{\mathcal{E}} + u_0(\mathcal{E})]\,d\mathcal{E}$$

$$= ax^3 + bx^2 + cx + d - \left(\frac{2}{3}\right)x^3 - 4 - 2x^2 - 4x - 4e^x$$

$$+\left(\frac{1}{360}\right)bx^6 + \left(\frac{1}{120}\right)cx^5 + \left(\frac{1}{840}\right)ax^7 + \left(\frac{1}{24}\right)dx^4. \quad (26)$$

Incorporating the boundary conditions (16), into $u_1(x)$, the following coefficients can be Obtained:

$$\left[\left[a = -\frac{2289756}{301681} + \left(\frac{916440}{301681}\right)e,\ b = \frac{4575063}{301681} - \left(\frac{1516680}{301681}\right)e,\ c = 2,\ d = 1\right]\right]. \quad (27)$$

Therefore, the following first-order approximate solution is derived:

$$u_1(x) = \left(-\frac{2289756}{301681} + \left(\frac{916440}{301681}e\right)\right)x^3 + \left(\frac{4575063}{301681} - \left(\frac{1516680}{301681}e\right)\right)x^2$$

$$-2x - 3 - \left(\frac{2}{3}\right)x^3 - 2x^2 + 4e^x + \left(\frac{1}{360}\left(\frac{4575063}{301681} - \left(\frac{1516680}{301681}e\right)\right)\right)x^6$$

$$+\left(\frac{1}{60}\right)x^5 + \left(\frac{1}{840}\left(-\frac{2289756}{301681} + \left(\frac{916440}{301681}e\right)\right)\right)x^7 + \left(\frac{1}{24}\right)x^4. \quad (28)$$

Comparison of the first-order approximate solution with exact solution is tabulated in Table 1, Showing a remarkable agreement.

Table 1
Comparison of the First Order Approximate Solution with Exact Solution

X	u_E	u_1	*Error*
0	1	1	0
0.1	1.21568800988322	1.21568152520360	0.000006484679615
0.2	1.46568330979220	1.46566089134464	0.000022418447564
0.3	1.75481644984880	1.75477392118332	0.00004252866548
0.4	2.08855457669778	2.08849297781809	0.00006159887969
0.5	2.47308190605020	2.47300726421803	0.00007464183217
0.6	2.91539008062482	2.91531273581669	0.00007734480813
0.7	3.42337960269982	3.42331259597951	0.00006700672031
0.8	4.00597367128645	4.00592940403031	0.00004426725614
0.9	4.67324591119820	4.67322989169107	0.00001601950713
1	5.43656365691810	5.43656365691807	1 E-14

Similarly, the following second-order approximation is obtained:

$$u_2(x) = u_0(x) + \int_0^x \frac{1}{6}(x-\mathcal{E})^3[4e^{\mathcal{E}} + u_1(\mathcal{E})]\,d\mathcal{E}$$

$$= ax^3 + bx^2 + cx + d - 8 - 8x - 4x^2 - \left(\frac{4}{3}\right)x^3 - \left(\frac{1}{8}\right)x^4 + 8e^x - \left(\frac{1}{60}\right)x^5$$

$$+\left(\frac{3971701}{108605160}\right)x^6 + \left(\frac{747263}{76023612}\right)x^7 + \left(\frac{1}{40320}\right)x^8 + \left(\frac{1}{181440}\right)x^9$$

$$+\left(\frac{1525021}{182456668800}\right)x^{10} - \left(\frac{27259}{23893135200}\right)x^{11} - \left(\frac{4213}{301681}e\right)x^6$$

$$+\left(\frac{1091}{301681}e\right)x^7 - \left(\frac{4213}{1520472240}e\right)x^{10} + \left(\frac{1091}{2389313520}e\right)x^{11} \tag{29}$$

$$\begin{bmatrix} a = -\dfrac{6948331518407}{376316879400} + \left(\dfrac{117708514171}{16725194640}\right)e, \\ b = \dfrac{12347296666549}{334503892800} - \left(\dfrac{6225330275}{477862704}\right)e, \\ c = 2, \\ d = 1 \end{bmatrix}. \tag{30}$$

Therefore, the second-order approximate solution may be written as:

$$u_2(x) = \left(-\frac{6948331518407}{376316879400} + \left(\frac{1177085141171}{16725194640}e\right)\right)x^3$$

$$+\left(\frac{1234729666549}{334503892800} - \left(\frac{6225330275}{477862704}e\right)\right)x^2 - 6x - 7 - 4x^2 - \left(\frac{4}{3}\right)x^3$$

$$-\left(\frac{1}{8}\right)x^4 + 8e^x - \left(\frac{1}{60}\right)x^5 + \left(\frac{3971701}{108605160}\right)x^6 - \left(\frac{747263}{76023612}\right)x^7 + \left(\frac{1}{40320}\right)x^8$$

$$+\left(\frac{1}{181440}\right)x^9 + \left(\frac{1525021}{18245666800}\right)x^{10} - \left(\frac{27259}{23893135200}\right)x^{11} - \left(\frac{4213}{301681}e\right)x^6$$

$$+\left(\frac{1091}{301681}e\right)x^7 - \left(\frac{4213}{1520472240}e\right)x^{10} + \left(\frac{1091}{2389312520}e\right)x^{11}. \tag{31}$$

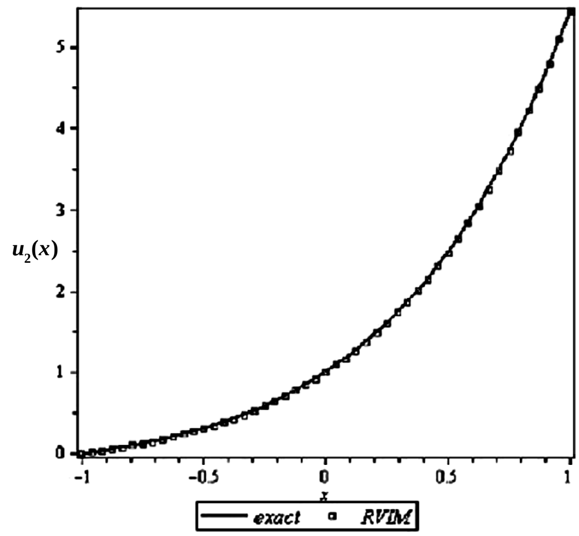

Figure 1: Comparison between Different Solutions

Example 2: Consider the following linear boundary value problem:

$$u^{(4)}(x) = u(x) + u''(x) + e^x(x-3), \quad 0 < x < 1 \tag{32}$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = -e. \tag{33}$$

The exact solution for this problem is

$$U(x) = (1-x). \tag{34}$$

At first, rewrite Eq. (32) based on selective linear operator as

$$L\{u(x)\} = u^{(4)}(x) = \overbrace{(u(x) + u''(x) + e^x(x-3))}^{h(x,u)}. \tag{35}$$

Now Laplace Transform is implemented with respect to independent variable on both sides of Eq. (35) and by using the new artificial initial conditions (which all of them are zero),we have

$$s^4 U(x) = L\{h(u, x)\} \tag{36}$$

$$U(x) = \frac{\mathbb{L}\{h(x,u)\}}{s^4}. \tag{37}$$

And whereas Laplace inverse Transform of $\frac{1}{s^4}$ is as follows:

$$\mathbb{L}^{-1}\left[\frac{1}{4}\right] = \frac{1}{6}t^3. \tag{38}$$

By using the Laplace inverse Transform and convolution theorem it is concluded that

$$u(x) = \int_0^x \frac{1}{6}(x-\mathcal{E})^3 h(u,\mathcal{E})\, d\mathcal{E}. \tag{39}$$

Hence, we arrive at the following iterative formula for the approximate solution of (32) subject to the initial condition (33),

$$u_{n+1}(x) = u_0(x) + \int_0^x \frac{1}{6}(x-\mathcal{E})^3\,[(\mathcal{E}-3)\,e^4 + u_n''(\mathcal{E}) + u(\mathcal{E})]\,d\mathcal{E}. \tag{40}$$

According to above equation, for first order approximation we have:

$$u_1(x) = u_0(x) + \int_0^x \frac{1}{6}(x-\mathcal{E})^3\,[(\mathcal{E}-3)\,e^4 + u_0''(\mathcal{E}) + u_0(\mathcal{E})]\,d\mathcal{E}. \tag{41}$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d, \tag{42}$$

where a, b, c, and are unknown constants to be further determined.

By the iteration formula (40), the following first-order approximation may be written:

$$\begin{aligned} u_1(x) &= ax^3 + bx^2 + cx + d \int_0^x \frac{1}{6}(x-\mathcal{E})^3\,[(\mathcal{E}-3)\,e^4 + u_0''(\mathcal{E}) + u_0(\mathcal{E})]\,d\mathcal{E} \\ &= ax^3 + bx^2 + cx + d + 7 + 6x + \left(\frac{5}{2}\right)x^3 - 7e^x + xe^x \\ &\quad + \left(\frac{1}{360}\right)bx^6 + \left(\frac{1}{120}\right)cx^5 + \left(\frac{1}{840}\right)ax^7 + \left(\frac{1}{24}\right)dx^4. \end{aligned} \tag{43}$$

Incorporating the boundary conditions (33), into $u_1(x)$, the following coefficients can be Obtained:

$$\left[\left[a = -\frac{6501670}{301681} + \left(\frac{2446080}{301681}\right)e,\ b = \frac{11668425}{301681} - \left(\frac{4247280}{301681}\right)e,\ c = 0,\ d = 1\right]\right]. \tag{44}$$

Therefore, the following first-order approximate solution is derived:

$$\begin{aligned} u_1(x) &= \left(\frac{6501670}{301681} - \left(\frac{2446080}{301681}\right)e\right)x^3 + \left(-\frac{11558425}{301681} + \left(\frac{4247280}{301681}\right)e\right)x^2 \\ &\quad + 8 + 6x + \left(\frac{5}{2}\right)x^2 + \left(\frac{2}{3}\right)x^3 - 7e^x + xe^x + \left(\frac{1}{360}\left(\frac{11668425}{301681} + \left(\frac{4247280}{301681}\right)e\right)\right)x^6 \\ &\quad + \left(\frac{1}{840}\left(\frac{6501670}{301681} - \left(\frac{2446080}{301681}\right)e\right)\right)x^7 + \left(\frac{1}{24}\right)x^4. \end{aligned} \tag{45}$$

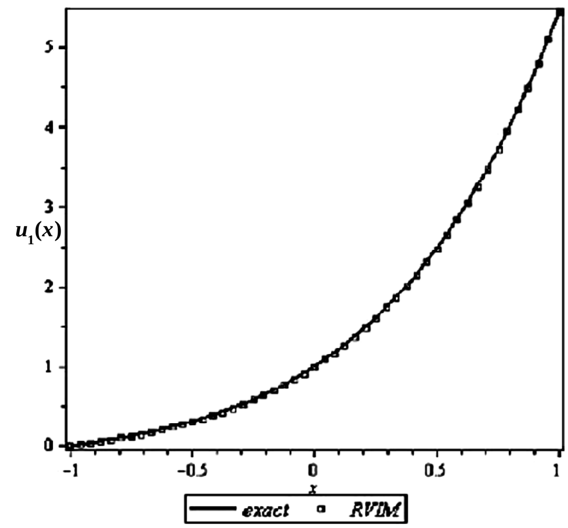

Figure 2: Comparison between Different Solutions

Table 2
Comparison of the First Order Approximate Solution with Exact Solution

X	u_E	u_2	*Error*
0	1	1.	0.
0.1	1.21568800988322	1.21568799241949	1.74 E-8
0.2	1.46568330979220	1.46568325207598	5.77 E-8
0.3	1.75481644984880	1.75481634654848	1.03 E-7
0.4	2.08855457669778	2.08855443762771	1.39 E-7
0.5	2.47308190605020	2.47308175193473	1.54 E-7
0.6	2.91539008062482	2.91538993703940	1.43 E-7
0.7	3.42337960269982	2.91538993703940	1.09 E-7
0.8	4.00597367128645	4.0059736082074	6.30 E-8
0.9	4.67324591119820	4.6732458916874	1.95 E-8
1	5.43656365691810	5.4365636569182	– 2.85 E -14

4. CONCLUSION

In this paper Reconstruction of variational iteration Method is applied to some fourth order equations used in beam deformation analysis. The accuracy of the method is acceptable and the resulting solution is close to the numerical solution that is shown graphically. Besides, comparison between approximate and exact solution was illustrated in presented tables. In spite of simplicity and requiring less computation, RVIM has a rapid convergence, high accuracy and efficiency and presents acceptable results. We benefited of maple package advantages for our calculations. Finally, the recent appearance of nonlinear differential equations as models in some fields of engineering sciences makes it necessary to investigate methods of solution for such equations (analytical and numerical).

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