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Published by Global Research Publications, New Delhi, India

THE NEW METHOD FOR SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS ARISING IN CONVECTIVE STRAIGHT FINS WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

D. D. Ganji*, M. Sheikholeslami, H. Yahyazadeh, M. Khalili & H. R. Ashorinejad
[a]*Faculty of Mechanical Engineering, Babol University of Technology, Babol, Islamic Republic of Iran*

Received: 01st March 2017, Accepted: 12th July 2017

Abstract: *In this study, fin efficiency of convective straight fins with temperature-dependent thermal conductivity is solved using an analytical method called the Parameterized Perturbation Method (PPM). The concept of Parameterized Perturbation Method is briefly introduced and employed to derive solutions of nonlinear equation. The obtained results from PPM are compared with those from the Numerical Method and also Homotopy Analysis Method (HAM) and Variational Iteration Method (VIM) to verify the accuracy of the proposed method. The results reveal that the Parameterized Perturbation Method can achieve suitable results in predicting the solution of such problems. After this verification, the effects of some physical applicable parameters have been analyzed in this problem such as thermo-geometric fin parameter and thermal conductivity parameter.*

Keywords: *Fin efficiency, Temperature dependent, Thermal conductivity, Parameterized Perturbation Method (PPM), Numerical Method (NM), Homotopy Analysis Method (HAM), Variational Iteration Method (VIM).*

Nomenclature

A_c	cross-sectional area of the fin (m^2)
b	fin length (m)
HAM	Homotopy Analysis Method
h	heat transfer coefficient ($W m^{-1} K^{-1}$)
k	thermal conductivity of the fin material ($W m^{-1} K^{-1}$)
k_a	thermal conductivity at the ambient fluid temperature ($W m^{-1} K^{-1}$)
k_b	thermal conductivity at the base temperature ($W m^{-1} K^{-1}$)
NM	Numerical Method
PPM	Parameterized Perturbation Method
P	fin perimeter (m)

***Corresponding Author: *ddg davood@yahoo.com*

Q	heat-transfer rate (W)
r	Constant parameter
T_a	temperature of surface a (K)
T_b	temperature of surface b (K)
VIM	Variational Iteration Method
x	distance measured from the fin tip (m)

Greek symbols

ζ	dimensionless coordinate
η	fin efficiency
β	dimensionless parameter describing variation of thethermal conductivity
λ	the slope of the thermal conductivity–temperaturecurve (K^{-1})
ψ	thermo-geometric fin parameter
θ	dimensionless temperature
δ	Constant normalized thickness

1. INTRODUCTION

A growing number of engineering applications are concerned with energy transport by requiring the rapid movement of heat. To increase the heat transfer rate on a surface, fin assembly is commonly used. The heat transfer mechanism of fin is to conduct heat from heat source to the fin surface by its thermal conduction, and then dissipate heat to the air by the effect of thermal convection. These extended surfaces are extensively used in various industrial applications. An extensive review on this topic is presented by Kern and Krause [1] and Aziz [2]. Also a considerable amount of research has been conducted about the variable thermal parameters which are associated with fins operating in practical situations. In the present paper, the resulting nonlinear differential equation is solved by PPM to evaluate the temperature distribution within the fin and compared with exact solution [3, 4] and numerical solutions. Using the temperature distribution, the efficiency of the fins is expressed through a term called thermo-geometric fin parameter (ψ) and thermal conductivity parameter (β), describing the variation of the thermal conductivity. All these problems and phenomena are modeled by ordinary or partial differential equations. In most cases, these problems do not admit analytical solution, so these equations should be solved using special techniques. In recent years some researchers used new methods to solve these kinds of problem [5-10]. Integral transform methods such as the Laplace and the Fourier transform methods are widely used in engineering problems. These methods transform differential equations into algebraic equations which are easier to deal with. However, integral transform methods are more complex and difficult when applying to nonlinear problems. Perturbation methods depend on a small parameter which is difficult to be found for real-life nonlinear problems. Parameterized Perturbation Method (PPM) [11-14] helps us to overcome this shortcoming and improves the accuracy of solution.

In this study, analytical solution of fin efficiency of convective straight fins with temperature-dependent thermal conductivity has been studied by Differential Transformation Method. For this purpose, after description of the problem and brief introduction for DTM, we applied DTM to find the approximate solution. Obtaining the analytical solution of the model and comparing with numerical results, HAM, VIM reveal the capability, effectiveness, convenience and high accuracy of this method.

2. MATHEMATICAL FORMULATION

To analyze the fin problem, the heat transfer performance of the extended surface refers to two approaches. One is based on thermal convection through fin surface area where the heat transfer rate of fin is a product of three factors which are heat transfer coefficients, fin surface area, and temperature difference between fin surface and surrounding. The other one is based on thermal conduction through fin cross-section where the main factors that govern the heat removal of fin are thermal conductivity, fin cross-section area, and temperature gradient along heat flow direction.

Consider a straight fin with a temperature-dependent thermal conductivity, arbitrary constant cross-sectional area A_c; perimeter P and length b (see Fig. 1).

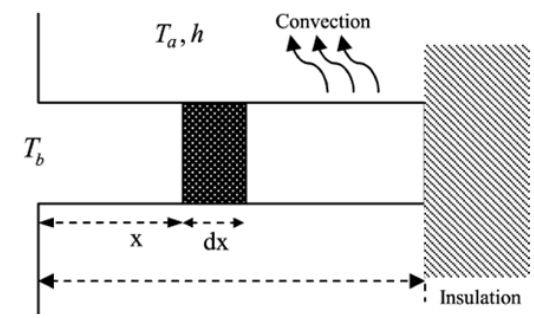

Figure 1: Geometry of a Straight Fin

The fin is attached to a base surface of temperature T_b, extends into a fluid of temperature T_a, and its tip is insulated. The one-dimensional energy balance equation is given:

$$A_c \frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] - Ph(T_b - T_a) = 0\,. \tag{2.1}$$

The thermal conductivity of the fin material is assumed to be a linear function of temperature according to:

$$k(T) = k_a[1 + \lambda(T - T_a)]\,. \tag{2.2}$$

Where k_a is the thermal conductivity at the ambient fluid temperature of the fin and k is the parameter describing the thermal conductivity variation.

Employing the following dimensionless parameters [15]:

$$\theta = \frac{T - T}{T_b - T_a}, \qquad \zeta = \frac{x}{b}, \qquad \beta = \lambda(T_b - T_a), \quad \psi = \left(\frac{hPb^2}{k_a A_c}\right)^{0.5} \tag{2.3}$$

The problem formulation reduces to:

$$\frac{d^2\theta}{d\zeta^2}+\beta\theta\frac{d^2\theta}{d\zeta^2}+\beta\left(\frac{d\theta}{d\zeta}\right)^2-\psi^2\theta=0 \tag{2.4}$$

$$\begin{aligned}&\frac{d\theta}{d\zeta}=0 \qquad \text{at} \quad \zeta=0\\ &\theta=1 \qquad \text{at} \quad \zeta=1\end{aligned} \tag{25}$$

The heat transfer rate from the fin is found by using Newton's law of cooling

$$Q=\int_a^b P(T-T_a)\,dx. \tag{2.6}$$

The ratio of actual heat transfer from the fin surface to the other side while whole fin surface is at the same temperature, we commonly called it the fin efficiency.

$$\eta=\frac{Q}{Q_{ideal}}=\frac{\int_a^b P(T-T_a)\,dx}{P(T_b-T_a)}=\int_0^1 \theta(\zeta)\,d\zeta. \tag{2.7}$$

3. ANALYSIS OF THE PARAMETERIZED PERTURBATION METHOD

The parameterized perturbation method was first proposed in 1999 in [11]. According to [11], an expanding parameter is introduced by a linear trans-formation:

$$\theta=\varepsilon v+r \tag{3.1}$$

where ε is the perturbation parameter, by substituting Eq. (1) into an original equation in order to have no secular term in the equation; we can obtain the un-known constant parameter r, Then, the solution is expanded in the form:

$$v=\sum_{i=0}^{n}\varepsilon^i v_i=v_0+\varepsilon v_1+\varepsilon^2 v_2+\ldots. \tag{3.2}$$

Here ε is an artificial bookkeeping parameter. Unlike traditional perturbation methods, we keep

$$v_0(0)=v(0), \quad \text{and} \quad \sum_{i=1}^{n} v_i=0.$$

4. APPLICATION OF PPM

Considering Eqs. (2.4) and (2.5), by subsisting Eq. (3.1) we can obtained that:

$$\frac{d^2v}{d\zeta^2}+\beta(\varepsilon v)\frac{d^2v}{d\zeta^2}+\beta\varepsilon\left(\frac{dv}{d\zeta}\right)^2-\psi^2v+\frac{r}{\varepsilon}(\beta-\psi^2)=0 \tag{4.1}$$

$$\frac{dv}{d\zeta} = 0 \qquad \text{at} \quad \zeta = 0$$
$$\varepsilon v + r = 1 \qquad \text{at} \quad \zeta = 1 \tag{4.1a}$$

In order to have no secular term in the equation we can obtain the un-known constant parameter *b*:

$$(\beta - \psi^2)\, r = 0\,. \tag{4.2}$$

By solving Eq. (4.2) constant parameter *r* can be obtained:

$$r = 0\,. \tag{4.3}$$

Then Eq. (3.1) must be subsisted into Eq. (4.1), after some simplification and rearranging based on powers of ε-terms, we have:

$$\varepsilon^0 : \frac{d^2 v_0}{d\zeta^2} - \psi^2 v_0 = 0,$$
$$\frac{dv_0}{d\zeta} = 0, \quad v_0 = \frac{1}{\varepsilon}, \tag{4.4}$$

$$\varepsilon^1 : \beta v_0 \left(\frac{d^2 v_0}{d\zeta^2} \right) + \frac{d^2 v_1}{d\zeta^2} + \beta \left(\frac{dv_0}{d\zeta} \right)^2 - \psi^2 v_1 = 0,$$
$$\frac{dv_1}{d\zeta} = 0, \quad v_1 = 0,$$

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. (4.5)

Solving Eqs. (4.4), (4.5) with boundary conditions, we have:

$$v_0 = \frac{e^{-\psi\zeta} + e^{\psi\zeta}}{\varepsilon(e^{-\psi} + e^{\psi})}, \tag{4.6}$$

$$\begin{aligned} v_1 = \frac{2}{3}\, &\frac{1}{\varepsilon^2 (e^{-\psi} + 3e^{\psi} + 3e^{3\psi} + e^{5\psi})(1 + 2e^{2\psi} + e^{4\psi})} (\beta (e^{-\psi\zeta} + 2e^{-\psi(\zeta-2)} \\ &+ 2e^{-\psi(\zeta-4)} + 2e^{-\psi(\zeta-6)} + e^{\psi(\zeta)} + 2e^{\psi(\zeta+2)} + 2e^{\psi(\zeta+4)} + 2e^{\psi(\zeta+6)} \\ &+ e^{\psi(\zeta+8)} - e^{-\psi(2\zeta-1)} - 3e^{-\psi(2\zeta-3)} - 3e^{-\psi(2\zeta-5)} - e^{-\psi(2\zeta-7)} \\ &- 3e^{\psi(2\zeta+3)} - 3e^{\psi(2\zeta+5)} - e^{-\psi(2\zeta+7)})) \end{aligned}$$

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. (4.7)

According to Eq. (3.2) we can obtain v, by subsisting into Eq. (3.1) we can find v:

$$\theta(\zeta) = \frac{e^{-\psi\zeta} + e^{\psi\zeta}}{(e^{-\psi} + e^{\psi})} + \frac{2}{3}\frac{1}{(e^{-\psi} + 3e^{\psi} + 3e^{3\psi} + e^{5\psi})(1 + 2e^{2\psi} + e^{4\psi})}$$
$$(\beta(e^{-\psi\zeta} + 2e^{-\psi(\zeta-2)} + 2e^{-\psi(\zeta-4)} + 2e^{-\psi(\zeta-6)} + e^{\psi(\zeta)} + 2e^{\psi(\zeta+2)}$$
$$+ 2e^{\psi(\zeta+4)} + 2e^{\psi(\zeta+6)} + e^{\psi(\zeta+8)} - e^{-\psi(2\zeta-1)} - 3e^{-\psi(2\zeta-3)}$$
$$- 3e^{-\psi(2\zeta-5)} - e^{-\psi(2\zeta-7)} - 3e^{\psi(2\zeta+3)} - 3e^{\psi(2\zeta+5)} - e^{-\psi(2\zeta+7)})) + \ldots \quad (4.8)$$

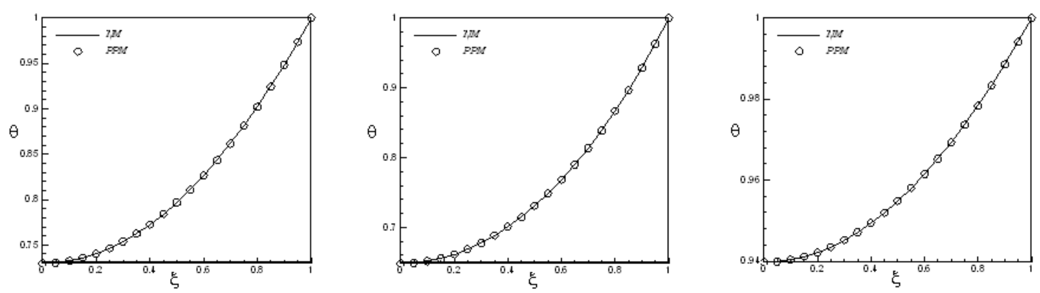
5. RESULT AND DISCUSSION

In the present study PPM is applied to obtain an explicit analytic solution of fin efficiency for convective straight fins with temperature dependent thermal conductivity (Fig. 1). In this study, the percentage of Error is introduced as fallow:

$$\%\ Error = \left| \frac{f(\eta)_{NM} - f(\eta)_a}{f(\eta)_{NM}} \right| \times 100$$

where $f(\eta)_a$ is the amount obtained using analytical methods.

In Fig. 2 the comparison of the DTM and exact solutions is represented. The results were well matched with the results carried out by numerical method (fourth-order Runge–Kutta).



(a) (b) (c)

Figure 2: Comparison between Numerical Results and PPM when (a) $\beta = -0.5$, $\psi = 1$, (b) $\beta = 0$, $\psi = 1$ and (c) $\beta = 0.5$, $y = 1$.

Comparison between the obtained results showed that PPM is more acceptable and accurate than HAM and VIM as we can see in Fig. 3.

Figure 4 depicts the effective temperature difference with various values of thermo-geometric fin parameter (ψ) from 0.5 to 0.65. This figure displays that for all amount of thermal conductivity, increasing in the values of thermo-geometric fin parameter, results the decrease in the values of dimensionless temperature.

The dimensionless temperature distributions along the fin surface with β varying from -0.2 to 0.2 are depicted in Fig. 5 for different values of $\psi = 0.5$ and $\psi = 1$, respectively. If the thermal conductivity of the fin's material increases with the temperature, the dimensionless temperature increases, too.

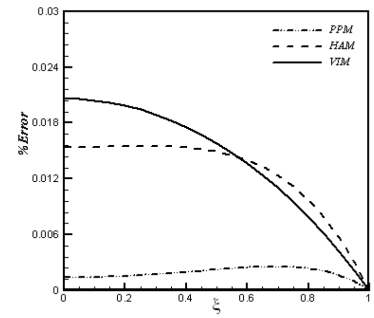
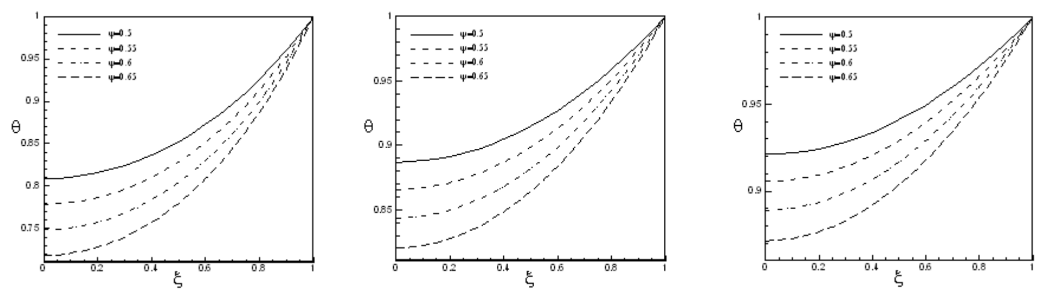
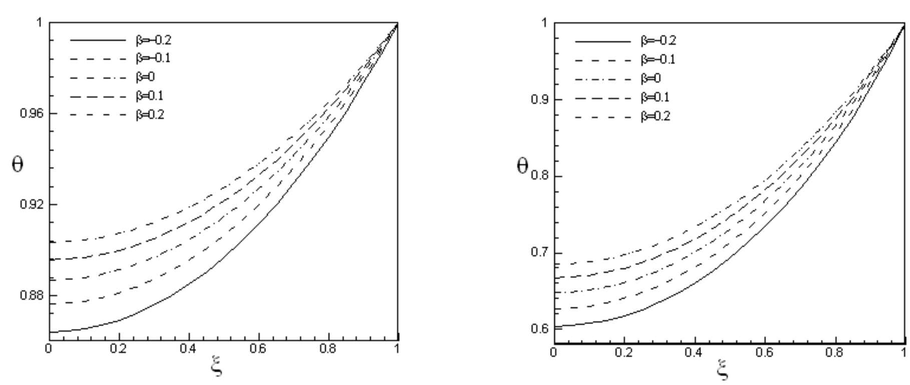


Figure 3: Comparison between % Errors of PPM, HAM and VIM for θ when β = 0.4, ψ = 1.



(a) (b) (c)

Figure 4: Temperature Distribution in Convective Fins with Variable Thermal Conductivity. For Various ψ when (a) β = –0.5, (b) β = 0 and (c) β = 0.5.



(a) (b)

Figure 5: Temperature Distribution in Convective Fins with Variable Thermal Conductivity. (a) When ψ = 0.5, (b) when ψ = 1.

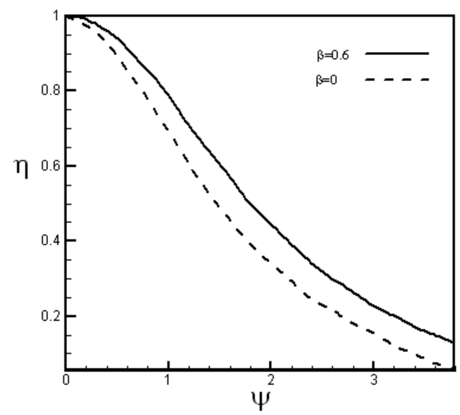


Figure 6: Variation of the Fin Efficiency with the Thermo-Geometric Fin Parameter (ψ) for Different Values of the Thermal Conductivity Parameter

Figure 6 shows the fin efficiency as a function of the thermo-geometric fin parameter for different values of the thermal conductivity parameter. This figure shows that fin efficiency increased as the result of thermo-geometric fin parameter reduction and thermal conductivity parameter increment.

6. CONCLUSION

In this study, we presented the definition and operation of Parameter Perturbation (PPM) method. Using the parameterized perturbation method a small parameter was created in the differential equation which is a difficulty of perturbation method. This method has been applied to solve a nonlinear differential equation arising in convective straight fins with temperature dependent thermal conductivity problem. The results of this method are compared to exact solution. The figures clearly show high accuracy of PPM to solve heat transfer problems in engineering.

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