A NUMERICAL COMPARISON FOR FRACTIONAL ORDER LOTKA-VOLTERRA EQUATIONS

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Abstract: In this article, homotopy perturbation method (HPM) and variational iteration method (VIM) are implemented to give analytical solutions for fractional order Lotka-Volterra differential equations. The main property of the methods lies in its flexibility and ability to solve nonlinear equations accurately and conveniently. The applications of Lotka-Volterra equations with integer order in many bio-sciences areas such as AIDS, immunity as well as finance and possibly other diverse systems. The analytical solutions of nonlinear two and three dimensional versions are given, which provides a convenient and straightforward approach to calculate the dynamics of the systems. It is shown that there is good agreement between the sets of results.

Keywords: Fractional differential equations, Lotka-Volterra, AIDS, Immunity, Finance.

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1. INTRODUCTION

Nonlinear phenomena occur in a wide variety of scientific applications such as fluid dynamics [1-2], plasma physics, biology, optical fiber and chemical kinetics [3]. These nonlinear phenomena are often related to system of nonlinear differential equations. In order to better understand these phenomena as well as further apply them in practical scientific research, it is important to seek that their exact solutions. The fractional differential equations (FDE) [4-5] appear more and more frequently in the different research areas and engineering applications. The fractional derivative has been occurring in many physical problems such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid etc. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equation of fractional order. Time is discontinuous according to the E-infinity theory (but Hierarchical), and the fractional derivative models are the best candidate to describe such problems. Fractional differential equations are therefore valid for discontinuous problems. Now, we consider a well-known Lotka-Volterra model

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$$D_t x_i(t) = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij} x_j(t)\right), \qquad i = 1, 2 \tag{1}$$

$$x_i = c_i, \qquad i = 1, 2 \tag{2}$$

Where x_2 is the number of predators (e.g, wolves), x_1 the number of its prey (e.g, rabbit), and b_i, a_{ij} are the parameters representing the interaction of the two species. The growth of the two populations is discontinuous and a simple modification of the predator-prey equation is to replace D_t by fractional derevatives D_t^{λ}. The populations of the predator-prey may be greatly affected by the fractional order λ.

The generalized Lotka-Volterra equations with fractional derivative extension are

$$D_t^{\lambda} x_i(t) = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij} x_j(t)\right), \quad 0 < \lambda \leq 1, \qquad i = 1, 2, 3, ..., n \tag{3}$$

$$x_i = c_i, \qquad i = 1, 2, 3, ..., n \tag{4}$$

The system (1)-(2) with $\lambda = 1$ has a wide applicability to a variety of different physical [6], chemical [7], and bio-sciences problems [8-9].

Very sophisticated extensions of predator-prey systems are currently used in a wide variety of settings, including physical, chemical, financial considerations and studies of AIDS. In the latter settings, the various strains of AIDS virus [10] are considered the predator against the body's T-cell population. Empirical data from medical studies are used to determined the constants in the differential equations, and the objective is to develop a sufficiently accurate model to be use in developing strategies for treatment. An example of Lotka-Volterra equations are in the phenomenon of immunity: for many infections, particularly those due to viruses, once you have been exposed to the diseases your body continues to produce high levels of antibodies to the disease for the rest of your life, even in the absence of any further stimulation from the virus.

Another kind of feedback structure of Lotka-Volterra system from finance [12-13], in which two species competing for the same food and territory. Both species has a negative impact on the growth rate of other.

The homotopy perturbation method first proposed by He [14] and was further developed and improved by some authors [15-17]. The method, which a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem, which is easily solve. Also, the variational iteration method proposed by He [18-21] will be used to study the linear and nonlinear problems [22-26]. The ADM [27] suffers for the complicated algorithms used to calculate Adomian polynomials that are necessary problems. The VIM and HPM have no specific requirements, such as linearization, small perturbation etc. This shows that the great potential of Dr. He's methodologies for nonlinear problems of sciences.

In this work we implement HPM and VIM for a special kind of system of differential equations with fractional order. We will highlight briefly the main points of the method and

used two most commonly defination are the Riemann-Liouville and Caputo. The fractional extension of a differential equation is obtained by replacing the first time derivative by the fractional derivative D^α of order $0 < \alpha \le 1$, where D^α is the fractional differential operator in the sense of Caputo, defined by $D^\alpha f(t) = J^{m-\alpha} D^m f(t)$. Here D^m is the usual integer differential operator of order, $m - 1 < \alpha \le m$, and J^μ is the Riemann-Liouville integral operator of order $\mu > 0$, defined by

$$J^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \xi)^{\mu - 1} f(\xi)\, d\xi, \quad t > 0 \tag{5}$$

$$J^\alpha J^\beta = J^{\alpha + \beta} f(t),$$

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} t^{\alpha + \gamma},$$

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}. \tag{6}$$

Caputo's defination, which is a modification of the Riemann-Liouville defination.

2. ANALYSIS OF METHODS

2.1. Homotopy Perturbation Method

We write the system in the form

$$D^\lambda u_1(t) = F_1(u_1, u_2, u_3, ..., u_n) + G_1(t)$$
$$D^\lambda u_2(t) = F_2(u_1, u_2, u_3, ..., u_n) + G_2(t)$$
$$. \qquad . \qquad .$$
$$D^\lambda u_n(t) = F_n(u_1, u_2, u_3, ..., u_n) + G_n(t) \tag{7}$$

Where $0 < \alpha \le 1$, subject to the initial conditions

$$u_1(0) = c_1, \qquad u_2(0) = c_2, \qquad u_3(0) = c_3, ..., u_n(0) = c_n. \tag{8}$$

The homotopy perturbation method, which provides an analytical approximate solution is applied on various nonlinear problems. According to HPM technique, we construct a homotopy for system (7)

$$(1 - p)(D^\alpha u_1(t) + D^\alpha u_{1,0}(t)) + p(D^\alpha u_1(t) + F_1(u_1, u_2, u_3, ..., u_n) + G_1(t))$$

$$(1 - p)(D^\alpha u_2(t) + D^\alpha u_{2,0}(t)) + p(D^\alpha u_2(t) + F_2(u_1, u_2, u_3, ..., u_n) + G_2(t))$$

$$. \qquad . \qquad .$$
$$. \qquad . \qquad .$$

$$(1 - p)(D^\alpha u_n(t) + D^\alpha u_{n,0}(t)) + p(D^\alpha u_n(t) + F_n(u_1, u_2, u_3, ..., u_n) + G_n(t)). \tag{9}$$

We can assume that the solution of equations can be written as a power in p, as following

$$u_1(x, t) = u_{1,0} + p\,u_{1,1} + p^2 u_{1,2} + \ldots$$
$$u_2(x, t) = u_{2,0} + p\,u_{2,1} + p^2 u_{2,2} + \ldots$$
$$. \qquad . \qquad .$$
$$. \qquad . \qquad .$$
$$u_n(x, t) = u_{n,0} + p\,u_{n,1} + p^2 u_{n,2} + \ldots \tag{10}$$

2.2. Variational Iteration Method (VIM)

In this section we present the main steps of variational iteration method and construct general formulae. The correctional functionals for the nonlinear system (7) can be approximately constructed as

$$u_1^{k+1}(t) = u_1^k(t) + \int_0^t \eta_1(\xi)\,(D^m u_1^k(\xi) - F_1(\tilde{u}_1^k(\xi), \tilde{u}_2^k(\xi), \tilde{u}_3^k(\xi), \ldots, \tilde{u}_n^k(\xi)) - G_1(\xi))\,d\xi$$

$$u_2^{k+1}(t) = u_2^k(t) + \int_0^t \eta_2(\xi)\,(D^m u_2^k(\xi) - F_2(\tilde{u}_1^k(\xi), \tilde{u}_2^k(\xi), \tilde{u}_3^k(\xi), \ldots, \tilde{u}_n^k(\xi)) - G_2(\xi))\,d\xi$$

$$. \qquad . \qquad .$$

$$u_n^{k+1}(t) = u_n^k(t) + \int_0^t \eta_n(\xi)\,(D^m u_n^k(\xi) - F_2(\tilde{u}_1^k(\xi), \tilde{u}_2^k(\xi), \tilde{u}_3^k(\xi), \ldots, \tilde{u}_n^k(\xi)) - G_n(\xi))\,d\xi \tag{11}$$

Where η_i are the Lagrange multipliers, which can be identified optimally via variation theory; here $\tilde{u}_i$ are considered as restricted variations. Making the above functional stationary, we obtain the Lagrange multipliers

$$\eta_i = -1, \qquad \text{for} \quad m = 1 \tag{12}$$
$$\eta_i = \xi - t, \qquad \text{for} \quad m = 2 \tag{13}$$

Substituting equation (12) into the correction functionals (11) the following iteration formulas

$$u_1^{k+1}(t) = u_1^k(t) - \int_0^t (D^\alpha u_1^k(\xi) - F_1(u_1^k(\xi), u_2^k(\xi), u_3^k(\xi), \ldots, u_n^k(\xi)) - G_1(\xi))\,d\xi$$

$$u_2^{k+1}(t) = u_2^k(t) - \int_0^t (D^\alpha u_2^k(\xi) - F_2(u_1^k(\xi), u_2^k(\xi), u_3^k(\xi), \ldots, u_n^k(\xi)) - G_2(\xi))\,d\xi$$

$$. \qquad . \qquad .$$
$$. \qquad . \qquad .$$

$$u_n^{k+1}(t) = u_n^k(t) - \int_0^t (D^\alpha u_n^k(\xi) - F_2(u_1^k(\xi), u_2^k(\xi), u_3^k(\xi), \ldots, u_n^k(\xi)) - G_n(\xi))\,d\xi \tag{14}$$

2.3. Numerical Experiments

To incorporate our discussion above, we consider system of differential equations (1) with fractional order which are 2 and 3 dimensional versions.

2.4. 2-D Version

$$D_t^\lambda x_1 = x_1 (b_1 + a_{11} x_1 + a_{12} x_2) \tag{15}$$

$$D_t^\lambda x_2 = x_2 (b_2 + a_{21} x_1 + a_{22} x_2) \tag{16}$$

Where $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are constants and subject to the initial conditions

$$x_1(0) = 4, \qquad x_2(0) = 10 \tag{17}$$

We apply the HPM to solve the 2-dimensional version of the problem. According to the HPM, we can construct a homotopy of system of equation (15)-(16) as follows:

$$D_t^\lambda v_1 - D_t^\lambda x_0 + p(D_t^\lambda x_0 - b_1 v_1 - a_{11} v_1^2 - a_{12} v_1 v_2) = 0 \tag{18}$$

$$D_t^\lambda v_2 - D_t^\lambda y_0 + p(D_t^\lambda y_0 - b_2 v_2 - a_{22} v_2^2 - a_{21} v_1 v_2) = 0 \tag{19}$$

The initial approximations are:

$$v_{1,0} = x_0(t) = x_1(0) = 4, \qquad v_{2,0} = y_0(t) = x_2(0) = 10 \tag{20}$$

Assuming that the solution of equations (15)-(16) has the form

$$v_1(t) = v_{1,0} + p v_{1,1} + p^2 v_{1,2} + p^3 v_{1,3} + \dots \tag{21}$$

$$v_2(t) = v_{2,0} + p v_{2,1} + p^2 v_{2,2} + p^3 v_{2,3} + \dots \tag{22}$$

Where $v_{i,j}$, $i = 1, 2$, $j = 1, 2, 3, \dots$ are functions yet to be determined. Substituting (21)-(22) into (18)-(19) and collecting terms the same power of p, we have

$$D_t^\lambda v_{1,1} - b_1 v_{1,0} - a_{11} v_{1,0}^2 - a_{12} v_{1,0} v_{2,0} = 0 \tag{23}$$

$$D_t^\lambda v_{1,2} - b_1 v_{1,1} - a_{12} v_{1,1} v_{2,0} - a_{12} v_{1,0} v_{2,1} = 0 \tag{24}$$

$$D_t^\lambda v_{1,3} - b_1 v_{1,2} - 2a_{11} v_{1,0} v_{1,2} - a_{11} v_{1,1}^2 - a_{12} v_{1,2} v_{2,0} - a_{12} v_{1,1} v_{2,1} - a_{12} v_{2,2} v_{1,0} = 0 \tag{25}$$

$$D_t^\lambda v_{2,1} - b_2 v_{2,0} - a_{21} v_{1,0} v_{2,0} - a_{22} v_{2,0}^2 = 0 \tag{26}$$

$$D_t^\lambda v_{2,2} - b_2 v_{2,1} - a_{21} v_{1,1} v_{2,0} - a_{21} v_{1,0} v_{2,1} - 2a_{22} v_{2,0} v_{2,1} = 0 \tag{27}$$

$$D_t^\lambda v_{2,3} - b_2 v_{2,2} - a_{21} v_{1,2} v_{2,0} - a_{21} v_{1,1} v_{2,1} - a_{21} v_{2,2} v_{1,0} - 2a_{22} v_{2,0} v_{2,2} - a_{22} v_{2,1}^2 = 0 \tag{28}$$

Solving the differential equations (23) and (26), we get

$$v_{1,1} = (4b_1 + 16a_{11} + 40a_{12}) \frac{t^\lambda}{\sqrt{\lambda + 1}} \tag{29}$$

$$v_{2,1} = (10b_2 + 40a_{21} + 100a_{22}) \frac{t^\lambda}{\sqrt{\lambda + 1}} \tag{30}$$

and rest of the components can be obtained by using MATHEMATICA 7. Results for some numerical values are in Table 1.

$$x_1 = \lim_{p \to 1} v_1(t) = \sum_{k=0}^{3} v_{1,k}(t) \tag{31}$$

$$x_2 = \lim_{p \to 1} v_2(t) = \sum_{k=0}^{3} v_{2,k}(t) \tag{32}$$

Table 1
Comparison for Numerical Solutions of 2D Version

X_i	VIM $\lambda = 0.6$	HPM $\lambda = 0.6$	VIM $\lambda = 0.8$	HPM $\lambda = 0.8$	VIM $\lambda = 1$	HPM $\lambda = 1$	NS $\lambda = 1$
$X_1(0.2)$	4.10469	4.14163	4.0891	4.09838	4.06636	4.06636	4.06636
$X_2(0.2)$	10.2106	10.2848	10.1792	10.1979	10.1335	10.1335	10.1335
$X_1(0.4)$	4.19283	4.21646	4.17257	4.17257	4.13362	4.13362	4.13362
$X_2(0.4)$	10.3872	10.4344	10.3341	10.3465	10.2684	10.2684	10.2684
$X_1(0.6)$	4.27153	4.27839	4.23848	4.24047	4.20177	4.20177	4.20177
$X_2(0.6)$	10.5444	10.5577	10.4781	10.4820	10.4046	10.4046	10.4046

The variational formulae for equations (15)-(16) are

$$x_{1,k+1} = x_{1,k} - \int_0^t (D_t^\lambda x_{1,k} - b_1 x_{1,k} - a_{21} x_{1,k} x_{2,k} - a_{11} x_{1,k}^2) \, d\xi \tag{33}$$

$$x_{2,k+1} = x_{2,k} - \int_0^t (D_t^\lambda x_{2,k} - b_2 x_{2,k} - a_{21} x_{2,k} x_{1,k} - a_{22} x_{2,k}^2) \, d\xi \tag{34}$$

$$x_{1,1} = 4 + (4b_1 + 16a_{11} + 40a_{12}) t \tag{35}$$

$$x_{2,1} = 10 + (10b_2 + 40a_{21} + 100a_{22}) t \tag{36}$$

Hence the four component solution of VIM with fractional order derivative extension obtained by MATHEMATICA 7.

$$x_1 = \lim_{k \to 3} x_{1,k+1} \tag{37}$$

$$x_2 = \lim_{k \to 3} x_{2,k+1} \tag{38}$$

2.5. 3 D Version

$$D_t^\lambda x_1 = x_1(1 - x_1 - \alpha x_2 - \beta x_3) \tag{39}$$

$$D_t^\lambda x_2 = x_2(1 - \beta x_1 - x_2 - \alpha x_3) \tag{40}$$

$$D_t^\lambda x_3 = x_3(1 - \alpha x_1 - \beta x_2 - x_3) \tag{41}$$

Where α, β are constants and subject to the initial conditions are taken to be

$$x_1(0) = 0.2, \qquad x_2(0) = 0.3, \qquad x_3(0) = 0.5 \tag{42}$$

We apply the HPM to solve the 3-dimensional version of the problem. According to the HPM, we can construct a homotopy of system of equation (39)-(41) as follows:

$$D_t^\lambda v_1 - D_t^\lambda x_0 + p\,(D_t^\lambda x_0 - v_1 + v_1^2 + \alpha v_1 v_2 + \beta v_1 v_3) = 0 \tag{43}$$

$$D_t^\lambda v_2 - D_t^\lambda y_0 + p\,(D_t^\lambda x_0 - v_2 + v_2^2 + \alpha v_2 v_3 + \beta v_1 v_2) = 0 \tag{44}$$

$$D_t^\lambda v_3 - D_t^\lambda z_0 + p\,(D_t^\lambda x_0 - v_3 + v_3^2 + \alpha v_1 v_3 + \beta v_2 v_3) = 0 \tag{45}$$

The initial approximations are

$$v_{1,0} = x_0(t) = x_1(0) = 0.2, \quad v_{2,0} = y_0(t) = x_2(0) = 0.3, \quad v_{2,0} = y_0(t) = x_2(0) = 0.5 \tag{46}$$

Assuming that the solution of equations (43)-(45) has the form

$$v_1(t) = v_{1,0} + p\,v_{1,1} + p^2 v_{1,2} + p^3 v_{1,3} + \dots \tag{47}$$

$$v_2(t) = v_{2,0} + p\,v_{2,1} + p^2 v_{2,2} + p^3 v_{2,3} + \dots \tag{48}$$

$$v_3(t) = v_{3,0} + p\,v_{3,1} + p^2 v_{3,2} + p^3 v_{3,3} + \dots \tag{49}$$

Substituting equations (47)-(49) in equations (43)-(45), we obtain

$$D_t^\lambda v_{1,1} - v_{1,0} + v_{1,0}^2 + \alpha v_{1,0} v_{2,0} + \beta v_{1,0} v_{3,0} = 0 \tag{50}$$

$$D_t^\lambda v_{1,2} - v_{1,1} + 2v_{1,0} v_{1,1} + \alpha v_{1,0} v_{2,1} + \alpha v_{1,1} v_{2,0} + \beta v_{1,0} v_{3,1} + \beta v_{1,1} v_{3,0} = 0 \tag{51}$$

$$D_t^\lambda v_{1,3} - v_{1,2} + 2v_{1,0} v_{1,2} + v_{1,1}^2 + \alpha v_{1,0} v_{2,2} + \alpha v_{1,1} v_{2,1} + \alpha v_{1,2} v_{2,0}$$
$$+ \beta v_{1,0} v_{3,2} + \beta v_{1,1} v_{3,1} + \beta v_{1,2} v_{3,0} = 0 \tag{52}$$

$$D_t^\lambda v_{2,1} - v_{2,0} + v_{2,0}^2 + \alpha v_{2,0} v_{3,0} + \beta v_{1,0} v_{2,0} = 0 \tag{53}$$

$$D_t^\lambda v_{2,2} - v_{2,1} + 2v_{2,0} v_{2,1} + \alpha v_{2,0} v_{3,1} + \alpha v_{2,1} v_{3,0} + \beta v_{1,0} v_{2,1} + \beta v_{1,1} v_{2,0} = 0 \tag{54}$$

$$D_t^\lambda v_{2,3} - v_{2,2} + 2v_{2,0} v_{2,2} + v_{2,1}^2 + v_{2,1} v_{1,1} + \alpha v_{2,0} v_{3,2} + \alpha v_{2,1} v_{2,1}$$
$$+ \alpha v_{2,2} v_{3,0} + \beta v_{1,0} v_{2,2} + \beta v_{1,2} v_{2,0} = 0 \tag{55}$$

$$D_t^\lambda v_{3,1} - v_{3,0} + v_{3,0}^2 + \alpha v_{1,0} v_{2,0} + \beta v_{2,0} v_{3,0} = 0 \tag{56}$$

$$D_t^\lambda v_{3,2} - v_{3,1} + 2v_{3,0} v_{3,1} + \alpha v_{1,0} v_{3,1} + \alpha v_{1,1} v_{3,0} + \beta v_{2,0} v_{3,1} + \beta v_{2,1} v_{3,0} = 0 \tag{57}$$

$$D_t^\lambda v_{3,3} - v_{3,2} + 2v_{3,0} v_{3,2} + v_{3,1}^2 + \alpha v_{1,0} v_{3,2} + \alpha v_{1,1} v_{3,1} + \alpha v_{1,2} v_{3,0}$$
$$+ \beta v_{2,0} v_{3,2} + \beta v_{2,1} v_{3,1} + \beta v_{2,2} v_{3,0} = 0 \tag{58}$$

The four term approximation, we obtain

$$x_1 = \lim_{p \to 1} v_1(t) = \sum_{k=0}^{3} v_{1,k}(t) \tag{59}$$

$$x_2 = \lim_{p \to 1} v_2(t) = \sum_{k=0}^{3} v_{2,k}(t) \tag{60}$$

$$x_3 = \lim_{p \to 1} v_3(t) = \sum_{k=0}^{3} v_{3,k}(t) \tag{61}$$

The variational formulae for equations (39)-(41) are

$$x_{1,k+1} = x_{1,k} - \int_0^t (D_t^\lambda x_{1,k} - x_{1,k} + \alpha x_{1,k} x_{2,k} + x_{1,k}^2 + \beta x_{1,k} x_{3,k}) \, d\xi \tag{62}$$

$$x_{2,k+1} = x_{2,k} - \int_0^t (D_t^\lambda x_{2,k} - x_{2,k} + \beta x_{2,k} x_{1,k} + x_{2,k}^2 + \alpha x_{2,k} x_{3,k}) \, d\xi \tag{63}$$

$$x_{3,k+1} = x_{3,k} - \int_0^t (D_t^\lambda x_{3,k} - x_{3,k} + \alpha x_{1,k} x_{3,k} + \beta x_{3,k} x_{2,k} + x_{3,k}^2) \, d\xi \tag{64}$$

Hence the four component solution of VIM with fractional order derivative extension obtained by MATHEMATICA 7.

$$x_1 = \lim_{k \to 3} x_{1,k+1} \tag{65}$$

$$x_2 = \lim_{k \to 3} x_{2,k+1} \tag{66}$$

$$x_3 = \lim_{k \to 3} x_{3,k+1} \tag{67}$$

3. NUMERICAL RESULTS AND CONCLUSION

In this paper, the VIM and HPM have been successfully applied to find the analytical solutions [28] of the fractional order Lotka-Volterra differential equations and these equations appear in biological, financial and diverse phenomena. For numerical purposes, we construct the solution for 2 and 3 D versions for fractional order Lotka-Volterra equation. It is quite important to notice that higher numbers of iterations are needed to gain more accuracy. As an advantage of these methods over the other, the methods reduces the computational difficulties and calculations can be made simple manipulations. The results for $\lambda = 1$ are in good agreement with ref. [29] and numerical solution (NS). In our work, we made use of the MATHEMATICA 7 to calculate the series obtained from the HPM and VIM.

Table 2
Comparison for Numerical Solutions of 3D Version

X_i	*VIM* $\lambda = 0.6$	*HPM* $\lambda = 0.6$	*VIM* $\lambda = 0.8$	*HPM* $\lambda = 0.8$	*VIM* $\lambda = 1$	*HPM* $\lambda = 1$	*NS* $\lambda = 1$
$X_1(0.2)$	0.264898	0.264828	0.250783	0.244811	0.230100	0.230106	0.230105
$X_2(0.2)$	0.383417	0.378406	0.365266	0.354896	0.338728	0.337245	0.338730
$X_3(0.2)$	0.591410	0.594885	0.574424	0.566029	0.544285	0.544604	0.544280
$X_1(0.4)$	0.318155	0.303279	0.293516	0.281601	0.262631	0.262723	0.262646
$X_2(0.4)$	0.448370	0.400886	0.417446	0.383952	0.378879	0.367052	0.378881
$X_3(0.4)$	0.661431	0.643382	0.627880	0.614854	0.586548	0.589062	0.586542

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