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ANALYTICAL SOLUTION OF THE FALKNER–SKAN EQUATION FOR WEDGE

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Abstract: *Homotopy Analysis Method (HAM) is a powerful analytical method to solve a nonlinear coupled differential equation system of the two dimensional laminar boundary layer of Falkner–Skan equation for wedge. This work aims at the solution of momentum and energy equation in the case of accelerated flow and decelerated flow with separation, which gives us a vast freedom to choose the answer type. In this article Homotopy Analysis Method (HAM) is applied to solve nonlinear equation of a Solution of the Falkner–Skan equation for wedge.*

Keyword: *Homotopy analysis method, Nonlinear differential equations, Falkner–Skan, Wedge, Separation, Heat transfer.*

1. INTRODUCTION

Historically, the steady laminar flow passing a fixed wedge was first analyzed in the early 1930s by Falkner and Skan [1] to illustrate the application of Prandtl's boundary layer theory. With a similarity transformation the boundary layer equation is reduced to an ordinary differential equation, which is well known as the Falkner-Skan equation. This equation includes non-uniform flow, i.e. outer flows which, when evaluated at the wall, takes the form $ue(x) = ax^m$, where x is the coordinate measured along the wedge and m is a constant.

Researchers such as Hartree [2], Howarth [3], Asaithambi [4], Cebeci and Keller [5], and Sher and Yakhot [6], have numerically investigated the solutions of the Falkner–Skan equation owing to the difficulties in obtaining an exact solution to the problem considered in a closed form. As far as the heat transfer analysis of the Falkner–Skan wedge flow is concerned, Lin and Lin [7] has introduced a similarity solution method for the forced convection heat transfer from isothermal or uniform-flux surfaces to fluids of any Prandtl number and then solved the resulting similarity equations by the Runge–Kutta scheme. Hsu *et al.*, [8] has studied the temperature and flow fields of the flow past a wedge by the series expansion method, Runge–Kutta integration and the shooting method. Kuo [9] has investigated the temperature field associated with the Falkner–Skan boundary-layer problem by converting it into a pair of initial value problems with the usage of the differential transformatio method, and then calculating it numerically. In particular, Liao's analysis

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has applied a new analytical method, the so-called homotopy analysis method.This method is one of the well-known methods used to solve the nonlinear equations which was expressed by Liao [10-15] and studied by a large number of researchers such as Ganji [16-19] and many others such as Abbasbandy [20-21] and Hayat [22].

Based on homotopy of topology, the validity of HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions of perturbation methods [23].

The HAM also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation and avoidance of physically unrealistic assumptions.

Furthermore, the HAM always provides us with a family of solution expressions in the auxiliary parameter λ, the convergence region and rate of each solution might be determined conveniently by the auxiliary parameter λ. Besides, the HAM is more general and contains the homotopy perturbation method (HPM) [23], the Adomian decomposition method (ADM) [24] and the δ-expansion method.

In recent years, the homotopy analysis method has been successfully employed to solve many types of nonlinear problems such as the nonlinear equations arising in heat transfer [20], the nonlinear model of diffusion and reaction in porous catalysts [21], the chaotic dynamical systems [25], the nonhomogeneous Blasius problem [26], the generalized three-dimensional MHD flow over a porous stretching sheet [27], the wire coating analysis using MHD Oldroyd 8-constant fluid [28], the axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet [29], the MHD flow of a second grade fluid in a porous channel [30], the generalized Couette flow [31], the squeezing flow between two infinite plates [32], the Glauert-jet problem [33], the Burger and regularized long wave equations [34], the laminar viscous flow in a semi-porous channel in the presence of a uniform magnetic field [35], the nano-boundary layer flows [36], the two dimensional steady slip flow in microchannels [37], and other problems. All of these successful applications verified the validity, effectiveness and flexibility of the HAM.

In this paper, the basic idea of HAM is described, and then it is applied to the nonlinear equation of the two dimensional laminar boundary layer of Falkner–Skan equation for wedge, after that The auxiliary parameter validity is different for each prandtl and also is different for the second part of the coupled system of differential equation. Then, the velocity profiles in boundary layer are obtained and Results show a good accuracy compared to the Adomian Decomposition Method and exact solution.

2. THE BASIC IDEA OF HOMOTOPY ANALYSIS METHOD

Suppose that we are concerned with a general kth order nonlinear ordinary differential equation and associated nonlinear differential operator $N : C^k \to R$, where C^K is the space of real-valued functions possessing derivatives up to kth order. In order to solve such a nonlinear differential equation, we seek to understand the kernel of such a map N, which is simply the set of all C^K functions $u(x)$ such that $N[u(x)] = 0$ for all x in the domain of interest. Thus, a solution $u(x)$ to such a nonlinear differential equation will satisfy $N[u(x)] = 0$ for all x in the domain of interest. In practice, obtaining an exact solution $u(x)$ to the relation $N[u(x)] = 0$ is not easy, and more likely impossible for an arbitrary N However, the Homotopy Analysis Method allows us to obtain approximate series solutions to a wide variety of nonlinear differential equations. In this method, we construct a homotopy

$$(1-q)L[\varphi(x;q)-g_0(x)] = ghH(x)N[\varphi(x;q)], \tag{2.1}$$

through the homotopy embedding parameter q, between the nonlinear operator N and an auxiliary linear operator L. Here, $h \neq 0$ is the convergence control parameter, while $H(x)$ is the auxiliary function.

Without loss of generality, we may set the auxiliary function $H(x) = 1$.

Also, $g_0(x)$ serves as an initial approximation to the solution of the nonlinear differential equation. We see that when $q = 0$, we have $L[\varphi(x;q) - g_0(x)] = 0$, while when $q = 1$, we have, so that any such functions $\varphi(x; 1)$ satisfy the nonlinear differential equation of interest. When $q = 0$, we may take $\varphi(x; 0) = g_0(x)$, so that the function $\varphi(x; q)$ agrees with the initial approximation at $q = 0$ and with a solution to the nonlinear differential equation of interest when $q = 1$. In this regard, (2.1) serves as the zeroth order deformation equation.

In order to obtain a solution to the nonlinear differential equation $N[u] = 0$, Liao [12] proposed a perturbation solution in which one regards the homotopy embedding parameter q as the parameter about which we expand the solution. Expanded as a Taylor series, this is given by

$$\varphi(x;q) = g_0(x) + \sum_{m=1}^{\infty} g_m(x) q^m \,. \tag{2.2}$$

According to the theory of Taylor series, this power series is unique as one regards q as a small parameter. Since we have freedom to select the initial approximation, auxiliary linear operator, auxiliary function, and the convergence control parameter, we must assume that they are properly chosen so that:

(i) The solution $\varphi(x; q)$ to the zeroth order deformation (2.1) exists for all $q \in [0, 1]$ and

(ii) the series solution (2.2) converges at $q = 1$.

When these two assumptions hold, the series solution (2.2) gives a relation between the initial guess $g_0(x)$ and the exact solution. Further, the exact solution will be given by

$$u(x) = g_0(x) + \sum_{m=1}^{\infty} g_m(x) \tag{2.3}$$

over the region of convergence for this representation. To obtain the $g_m(x)$'s, one recursively solves what are known as the mth order deformation equations, given by

$$L[g_m(x) - x_m g_{m-1}(x)] = hR_m(g_{m-1}(x), x) \tag{2.4}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{2.5}$$

and

$$R_m(g_{m-1}(x), x) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\varphi(x; q)]}{\partial q^{m-1}} \right|_{q=0}$$

$$= \frac{1}{(m-1)!} \left. \left(\frac{\partial^{m-1}}{\partial q^{m-1}} N \left[\sum_{m=0}^{\infty} g_m(x) q^m \right] \right) \right|_{q=0} . \tag{2.6}$$

Let us define the partial sum $S_M(x)$ as

$$S_M(x) = g_0(x) + \sum_{m=1}^{M} g_m(x) . \tag{2.7}$$

Then $S_M(x)$ will serve as the M^{th} order approximation to the solution (2.3). Our nonlinear operator can always be decomposed into linear and nonlinear components. Let $N = N_1 + N_2$, where N_1 is a linear differential operator and N_2 is a nonlinear differential operator. The mth order deformation (2.4) is then written in the much more useful form

$$L[g_m(x) - \chi_m g_{m-1}(x)] = hN_1[g_{m-1}(x)] + hN_2[S_{M-1}(x)] . \tag{2.8}$$

As L is a linear operator, we see that

$$L[g_m(x)] = \chi_m L[g_{m-1}(x)] + hN_1[g_{m-1}(x)] + hN_2[S_{M-1}(x)] . \tag{2.9}$$

Hence, we have an expression for $g_m(x)$, in terms of all lower order terms $g_j(x)$, for $j = 0, 1, ..., m - 1$. As such, we may in principle solve for the $g_m(x)$'s sequentially, to obtain the approximate solutions in (2.7) or even for the exact solution in (2.3). We note that, just like any series solution , the results may not converge over the entire domain of the problem However, the Homotopy Analysis Method does allow us to have some control over the domain of convergence via the choice of initial guess $g_0(x)$, auxiliary linear operator L, and convergence control parameter h.

These will be discussed in subsequent sections.

We further note that all terms on the right-hand side of (2.9) are known, as are all lower order terms $g_j(x)$, for $j = 0, 1, ..., m - 1$. Thus, in order to obtain $g_m(x)$, we have to solve an inhomogeneous linear differential equation.

$$L[g_m(x)] = Z(g_{m-1}(x), ..., g_0(x), x) = z_m(x) . \tag{2.10}$$

This in general is much simpler than solving a nonlinear differential equation. By solving this linear differential equation, subject to the relevant initial and or boundary conditions, one may in principle obtain the expression

$$g_m(x) = I(z_m(x)) + J(x) , \tag{2.11}$$

where J is the homogeneous contribution from the linear operator L, and I is the inhomogeneous contribution due to $Z_m(x)$.

3. FLOW ANALYSIS

The flow problem can be restated here as the conservation of mass and momentum at every point in a p_∞ = constant boundary layer:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3.1}$$

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\nu\frac{\partial^2 u}{\partial y^2} \tag{3.2}$$

$$\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{1}{pr}\frac{\partial^2 T}{\partial T^2} \tag{3.3}$$

It was proved that the similar solution exist when the velocity of the potential flow is proportional to a power of the length coordinate, x, that is measured from the stagnation point. For a wedge with angle $\pi\varepsilon$, as is shown in Fig. 1, in the neighborhood of the leading edge the potential velocity distribution is $U(x) = u_1 x^m$ [26]. x, y, u, v are dimensionless using reference length x_0 and the inherent characteristic velocity u_∞, respectively. By using dimensionless and new define variables; we have the following equation

Using the similarity variables:

$$\frac{u}{U}=f'(\eta) \tag{3.4}$$

$$\eta=\sqrt{\frac{m+1}{2}}\sqrt{\frac{U}{\nu x}}\,y \tag{3.5}$$

$$\psi=\sqrt{\frac{2}{m+1}}\sqrt{U\nu x}\;f(\eta) \tag{3.6}$$

$$\nu=\eta=\sqrt{\frac{m+1}{2}}\sqrt{\frac{\nu U}{x}}\left[f(\eta)+\frac{m-1}{m+1}\eta f'(\eta)\right] \tag{3.7}$$

$$m=\frac{\varepsilon}{2-\varepsilon}=\frac{2m}{m+1}=\varepsilon \tag{3.8}$$

$$\theta(\eta)=\frac{T-T_0}{T\infty-T_0}\,. \tag{3.4}$$

When ψ is the stream function defined by $u=\frac{\partial\psi}{\partial x}$ and $u=\frac{\partial\psi}{\partial y}$, f and θ are the similarity functions dependent on η, Eqs [26] transformed to:

$$f'''(\eta)+f(\eta)\,f''(\eta)+\varepsilon\,(1-(f(\eta))^2)=0 \tag{3.10}$$

$$\theta''(\eta)+prf(\eta)\theta'(\eta)=0\,. \tag{3.11}$$

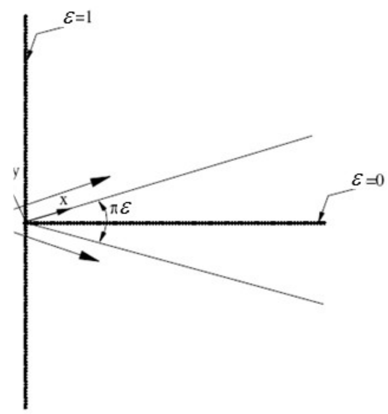

Figure 1: Physical Model and Coordinate System

Subject to the boundary conditions:

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1 \qquad (3.12)$$
$$\theta(0) = 0, \quad \theta(\infty) = 1$$

3.1 HAM Solut0ion

To seek the explicit analytical solution of Eqs. (3.10) and (3.11) by using HAM, the initial guess approximation and the auxiliary linear operator are

$$f_0(x) = -1 + x + e^{(-x)}, \quad L_1(f) = f''' - f'$$
$$g_0(x) = e^{(-x)}, \quad L_2(\theta) = \theta'' - \theta \qquad (3.13)$$

which satisfies

$$L_1(c_1 + c_2 e^{n} + c_3 e^{-n}) = 0$$
$$L_2(c_4 e^{n} + c_5 e^{-n}) = 0$$

And $c_i\,(i = 1 - 5)$ are constants Let $p \in [0, 1]$ denotes the embedding parameter and $\hbar_1, \hbar_2$ indicate the none-zero auxiliary parameters. We then construct the following problems

Zeroth-order deformation problems

$$(1-p)L_1[f(\eta;p) - f_0(\eta)] = p\hbar_1 N_1[f(\eta;p)]$$
$$f(0;p) = 0, \quad f'(0;p) = 0, \quad f'(\infty;p) = 1 \qquad (3.14)$$
$$(1-p)L_1[f(\eta;p) - f_0(\eta)] = p\hbar_1 N_1[f(\eta;p), \theta(\eta;p)]$$
$$\theta(0;p) = 0, \quad \theta(\infty;p) = 1$$
$$N_1[f(\eta,p)] = \frac{\partial^3 f(\eta,p)}{\partial\eta^3} + \left(f(\eta,p)\frac{\partial f(\eta,p)}{\partial\eta^2}\right) + \varepsilon\,(1-(f(\eta,p))^2)$$
$$N_2[f(\eta,p),\theta(\eta,p)] = \frac{\partial^2\theta(\eta,p)}{\partial\eta^2} + pr\left(f(\eta,p)\frac{\partial\theta(\eta,p)}{\partial\eta}\right). \qquad (3.15)$$

When p increases from 0 to 1 then $f(\eta, p)$ and $\theta(\eta, p)$ vary from $f_0(\eta)$ and $\theta_0(\eta)$ to $f(\eta)$ and $\theta(\eta)$. Due to tailors' series with respect to p, we have

$$f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)\, p^m, \qquad f_m(\eta) = \frac{1}{m!}\frac{\partial^m (f(\eta; p))}{\partial p^m}$$

$$\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)\, p^m, \qquad \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m (\theta(\eta; p))}{\partial p^m} \tag{3.16}$$

m^{th}-order deformation problems

$$L_1[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_1 R_m^f(\eta) \qquad f_m(0) = f_m'(0) = f_m'(\infty) = 0$$

$$L_2[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_2 R_m^\theta(\eta) \qquad \theta_m(0) = \theta_m(\infty) = 0\,. \tag{3.17}$$

$$R_m^f = f_{m-1}''' + \sum_{i=0}^{m-1} f_{m-1-i}\, f_i'' + \varepsilon\left(1 - \sum_{i=0}^{m-1} f_i\, f_{m-1-i}\right)$$

$$R_m^\theta = \theta_{m-1}'' + pr \sum_{i=0}^{m-1} f_{m-1-i}\, \theta_i'\,. \tag{3.18}$$

Where

$$\chi_m \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

We have found the answer by maple analytic solution device. The following terms are solutions for $pr = 1$, for first deformation of the coupled solution.

$$f_1(\eta) = 4.0742\times 10^{-8} \times e^{\eta} \times h_1 + 0.5417 \times e^{(-\eta)} \times h_1 - \frac{0.275}{e^{\eta}} + 0.55 \times h_1\left(-\frac{x}{e^{\eta}} - \frac{1}{e^{\eta}}\right)$$
$$-\frac{0.1333}{(e^{\eta})^2} - 0.25 \times h_1\left(-\frac{\eta^2}{e^{\eta}} - \frac{2\eta}{e^{\eta}} - \frac{2}{e^{\eta}}\right) - 0.0833 \times h_1$$

$$\theta_1(\eta) = -h_2 \times e^{-\eta} \times \eta + 0.5018 \times h_2\, e^{-\eta} + 0.5 \times h_2 \times e^{-\eta} \times \eta^2 - 0.5 \times h_2 \times e^{-2\eta} - 0.0018 \times h_2$$

The solutions $f_2(\eta)$ and $\theta_2(\eta)$ were too long to be mentioned here, therefore, they are shown graphically. But it is necessary to remind that both auxiliary parameters of $\hbar_1$ and $\hbar_2$ appear in other terms of energy equation.

4. CONVERGENCE OF THE HAM SOLUTION

As mentioned in introduction, HAM provides us with great freedom in choosing the solution of a nonlinear problem by different base functions. This has a great effect on the convergence region because the convergence region and rate of a series are chiefly determined by the base functions used to express the solution. Therefore, we can approximate a nonlinear problem more efficiently by choosing a proper set of base functions and ensure its convergency. On the other hand, as pointed out by Liao, the convergence and rate of approximation for the HAM solution strongly depends on the value of auxiliary parameters $\hbar$s. Even, if the initial approximations $f_0(\eta)$ and $\theta_0(\eta)$, the auxiliary linear operator $\mathcal{L}$, and the auxiliary function $H(\eta)$ are given, we still have great freedom to choose the value of the auxiliary parameters $\hbar_1$ and $\hbar_2$. So, the auxiliary parameters provide us with an additional way to conveniently adjust and control the convergence region and rate of solution series. By means of the so-called $\hbar$-curves, it is easy to find out the so-called valid regions of auxiliary parameters to gain a convergent solution series. When the important physical parameters such as: $f''(0)$ and $\theta'(0)$ considering

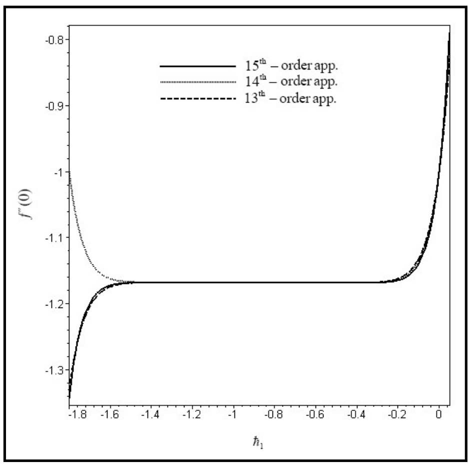

Figure 2: The $\hbar_1$-Validity for $\varepsilon = 1.0$, and Pr = 1.0

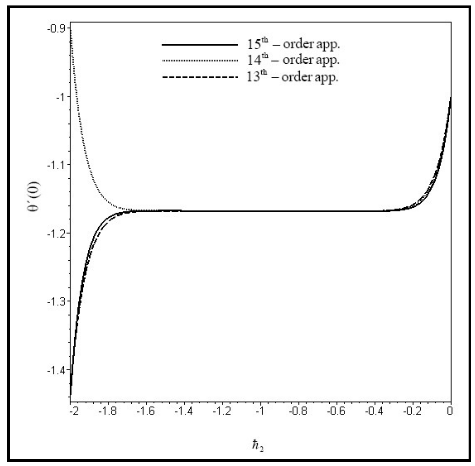

Figure 3: The $\hbar_2$-Validity for $\varepsilon = 1.0$, and Pr = 1.0

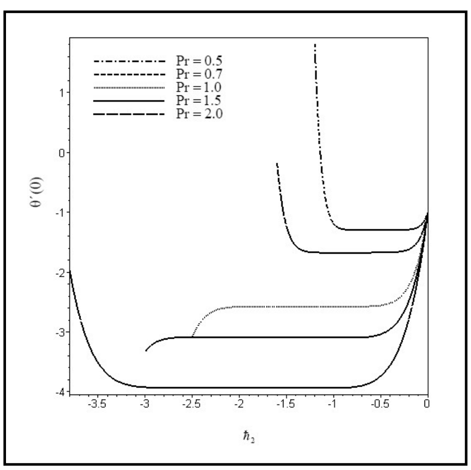

Figure 4: The $\hbar_2$-Validity for Various Pr when $\varepsilon = 2.0$ and Pr = 1.0

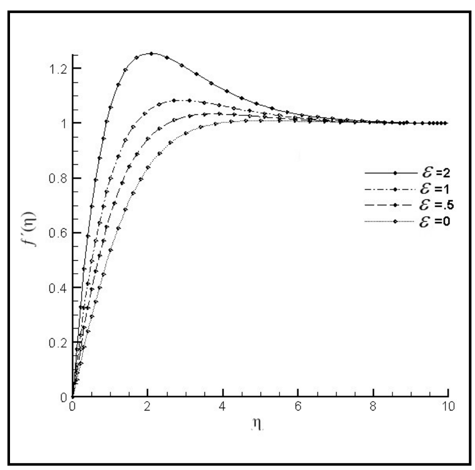

Figure 5: The $f'(\eta)$ by Homotopy Analysis Method for Various ε when, Pr = 1.0 and $\hbar_1 = -0.9$

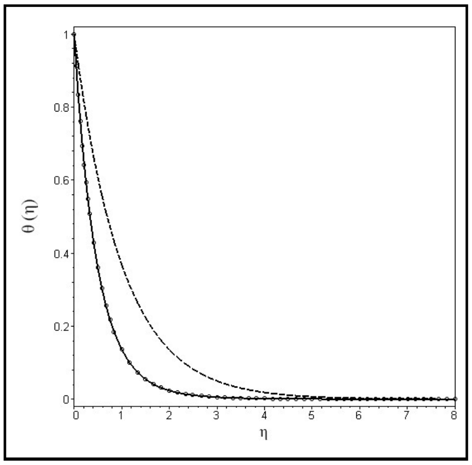

Figure 6: The Analytic Approximation for $\theta(\eta)$, Solid Curve: 14-Order Approximate; Symbols: 15-Order Approximate; Dotted Curve: Initial Approximation of by HAM for Pr = 1.0, $\varepsilon = 2.0$, $\hbar_1 = -0.9$ and $\hbar_2 = -1.0$

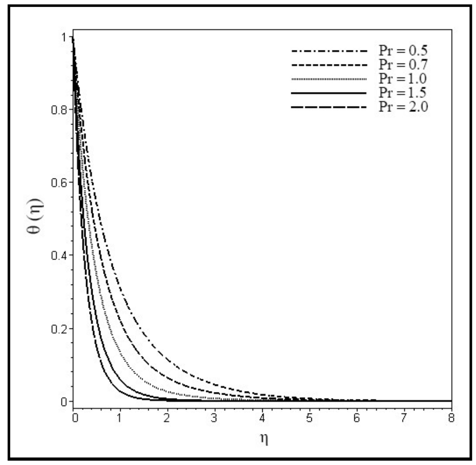

Figure 7: The $\theta(\eta)$ by Homotopy Analysis Method for Various Pr when, $\varepsilon = \frac{1}{2}$ and $\hbar_1 = -1$

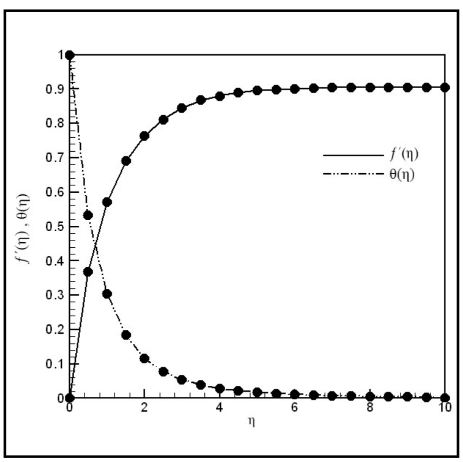

Figure 8: The $f'(\eta)$ and $\theta(\eta)$ by Homotopy Analysis Method when Pr = 1.0, $\varepsilon = 1.0$, $\hbar_1 = -1$ and Point Symbol is Numerical Method

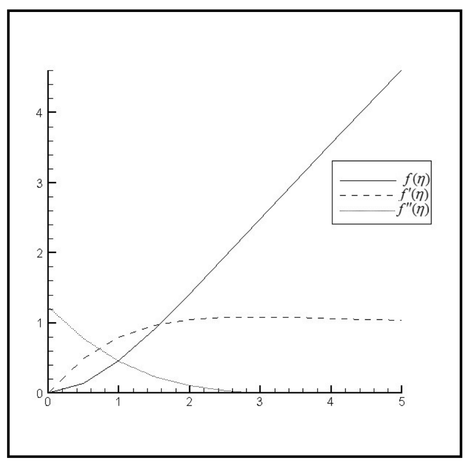

Figure 9: The $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ by Homotopy Analysis Method when Pr = 1.0, $\varepsilon = 1.0$, $\hbar_1 = -1$

auxiliary parameters, if they do not change, corresponds region is known as the convergence region. In our case study, according to Figs (2) and (3), the acceptable range of auxiliary parameters for $\varepsilon = 2$, Pr = 1 are $-1.4 < \hbar_1 < -0.3$ and $-1.4 < \hbar_2 < -0.4$. Figures (4) show how auxiliary parameters varied with changing Pr. According to Fig. (4) by increasing Pr, the range of convergency is decreased. Figures (7) and (8) indicate the analytic approximation for $f'(\eta)$ and θ in which Solid curve is 14-order approximate, symbols show the 15-order approximate and dotted curve is initial approximation of by HAM for Pr = 1, $\varepsilon = 2$, $\hbar_1 = -0.9$ and $\hbar_2 = -1$.

5. NUMERICAL METHOD

The best approximate for solving Eq. (3.10) that can be used is fourth order Runge-Kutta method. It is often utilized to solve differential equation systems. Third order differential equations can be usually changed into second order equations and then first order. After that, it can be solved through Runge-Kutta method. For solving Eq. (3.11) we used second order Runge-Kutta method.

6. CONCLUSIONS

In this paper, the homotopy analysis method (HAM) is applied to obtain the approximate solution of the nonlinear coupled differential equation system of the two dimensional laminar boundary layer of Falkner–Skan equation for wedge. The HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between the HAM and other methods. Solutions of HAM can be expressed with different functions and therefore they can be originated from the nature of the problems. According to the figures this method provides highly accurate analytic solutions for nonlinear problems in comparison with other methods.

Figure (5) shows $f'(\eta)$ that are obtained by using homotopy analysis method for various values of ε when Pr = 1.0 and $\hbar_1 = -0.9$, $\hbar_2 = -1.0$. And Fig. (7) illustrates $\theta(\eta)$ for various values of Pr in $\hbar_2 = -1.0$ and $\hbar_1 = -0.9$. Figure (7) shows the effect of the Prandtl number (Pr) on the thermal boundary layer's thickness respectively. The figures show that increasing Prandtl number (Pr) decreases the velocity boundary layer thickness and thermal boundary layer's thickness. As shown in Fig. (8), it has been attempted to show the accuracy, capabilities and wide-range applications of the homotopy analysis method in comparison with the numerical solution of heat transfer over an unsteady stretching permeable surface with prescribed wall temperature, and finally we plot in Fig. (9) the solution curves of the Falkner-skan equation that comparisons between our analytical solution and those reported in previous literature Fig. 4. [38], which clearly show that our results are in excellent accordance with existed ones.

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