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Published by Global Research Publications, New Delhi, India

AN ANALYTICAL INVESTIGATION OF SECOND ORDER FLUID FLOW INSIDE A CURVED CIRCULAR PIPE

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Received: 06th May 2016, Accepted: 12th August 2016

Abstract: *In this research, the second order fluid flow in a curved circular pipe is studied analytically using perturbation method to solve the governing equations in the fully developed condition. For calculating drag variations of flow in curved pipes, high order perturbation series have to be considered. Taking into account this point, the drag reduction of fluid flow due to the retardation time of viscoelastic material is affirmed analytically. Also, based on these high order perturbation series, the flow field characteristics such as the axial velocity, stream function of secondary flows and stress components are calculated accurately. Furthermore, the performance mechanism of retardation time on drag reduction and secondary flows enhancement is investigated.*

Keywords: *Second order fluid; Curved pipe; Fluid flow; Perturbation method; Retardation time; Relaxation time*

1. INTRODUCTION

Investigation of fluid flow in a curved pipe has been an interesting subject of many researchers using experimental, numerical and analytical approaches. This kind of flow is very important in medical treatment, industries, engineering systems, etc. The most of previous studies are restricted to investigation of Newtonian fluid flow in curved pipe and few researches have been done on the non-Newtonian fluid flow. It is important to mention that the non-Newtonian fluid flow in a curved pipe as viscoelastic fluid has numerous applications in petrochemical industries, food production, medical treatment, and injection of polymeric materials.

Dean [1, 2] was the first pioneer who find an analytical perturbation solution for fully developed flow of Newtonian fluids in curved pipes of circular cross-section. He found Taylor secondary flows generated due to centrifugal forces resulting from the pipe radius of curvature. Also, he introduced the Dean number specifying the behavior of flow inside the curved pipe. Dean number has direct relation with the intensity, shape of secondary flows, and subsequently

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the stability criteria of flow. Topakoglu [3] improved Dean's work by calculating the higher order terms of perturbation series for flow field parameters and obtained analytical relation for the flow rate of Newtonian fluid in a curved pipe.

Various numerical investigations were carried out on the non-Newtonian fluid flow in curved pipes. Zhang et al. [4] used Galerkin method to study Oldroyd-B fluid flow in circular curved pipes. They obtained the flow field in high Dean and Weissenberg numbers. Phan-Thien and Zheng [5] obtained self-similar solution for the flow of Oldroyd-B fluid between two curved plates. They presented numerical and approximate analytical solutions for self-similar equations and showed that these two solutions have a good agreement with each other at Reynolds number below 1000. The study by Fan et al. [6] has used a finite volume method to find a solution for fully developed creeping and inertial flow of Oldroyd-B and $UCMN_2$ fluids in a curved pipe. They showed that the first normal stress difference amplifies the intensity of secondary flows and decreases the pressure drop value. They expressed that this phenomenon is due to the effect of axial normal stress called hoop stress. They used order of magnitude technique to show how centrifugal forces are balanced with radial pressure gradient and hoop stress in the core region of the flow. Also, they proved that the effect of negative second normal stress difference neutralize the effect of first normal stress difference, and the increment in the second normal stress difference results in the decrement of the secondary flows intensity and flow resistance ratio. Their results have a good agreement with experimental results [7-9]. Chen et al. [10] used the numerical and perturbation method for solving fully developed flow of Oldroyd-B fluid in a rotating circular curved pipe. As they have shown, the dynamic pressure distribution in cross-section of pipe is balanced with Coriolis force shifting the maximum axial velocity from the centre to outer side of curvature. They also investigated the effects of co and counter-rotation and showed that in counter-rotation case, weaker Taylor vortices are visible. There is a critical situation in counter-rotation condition in which the secondary flow intensity is the lowest. They affirmed that the critical counter-rotation number is four times of the Weissenberg number in curvature ratio. However, very few works have been done on non-Newtonian fluid flows in curved duct with square cross-section. For example, Zhang et al. [11] worked on Oldroyd-B fluid in a rotating curved duct. Helin et al. [12] and Boutabaa et al. [13] studied PTT and Oldroyd-B fluids in curved duct and investigated the effects of Deborah number on the formation of Dean vortices.

Besides numerical investigations, analytical solutions especially via perturbation method are used to study the flow of non-Newtonian fluids in circular curved pipes. The first research about this kind of fluids was done by Thomas and Walters [14]. They studied fully developed flow of Oldroyd-B fluid in a curved pipe by the perturbation technique and showed that the intensity of secondary flows has direct relation with elastic properties of Oldroyd-B fluid. Sarin [15, 16] and Robertson and Muller [17] have carried out similar works on the Oldroyd-B fluids. Sarin confirmed that the secondary flows intensity is related to the fourth order of Deborah number. He also explained the relation between the maximum value of axial velocity and the central position of vortices on the Debroah number. Robertson and Muller [17] studied fully developed creeping and inertial flow of Oldroyd-B fluid in a curved pipe and solved it analytically using perturbation method. They showed that for creeping flow, the flow rate in a curved pipe is slightly larger than in a straight pipe at low Weissenberg numbers. By increasing the Weissenberg number and decreasing the viscosity of Newtonian solvent in inertial flow, an obvious decrement in the resistance ratio is observable. Imeto et al. [18, 19] investigated the power law and White-Metzner fluid flow inside different shapes of curved circular pipes such as sinusoid, parabolic and hyperbolic. They obtained that the sooner adaptation with curvature occurs when the elastic property of fluids increases. The study by Das [20] has considered Bingham fluid flow in curved pipes. By analytical study on the viscoelastic fluid flow, Zhang et al. [21] and Shen et al. [22] have paid attention to heat transfer phenomenon using Oldroyd-B model as the constitutive equations.

Some researchers studied the second order fluid flow in curved pipes via perturbation method. Sharma and Prakash [23] studied the first normal stress difference (considered zero value for second normal stress difference) and found that the intensity of the secondary flows will increase with the increment of first normal stress difference. Bowen et al. [24] studied creeping flow of upper convected Maxwell (UCM) and the second order fluid (SOF) in curved pipe to calculate the higher order terms of axial velocity and its effect on the flow rate in a curved pipe. They showed that in the creeping flow of viscoelastic fluid flow inside the curved pipes, the flow rate ratio can be more than one comparing with the straight pipes for similar pressure gradient. They affirmed that for the creeping flow of the second order and UCM fluids, the drag reduction and enhancement are observed at large Weissenberg number respectively. Jitchote and Robertson [25] studied the inertial flow of the second order fluid in a curved pipe and considered negative second normal stress difference. They affirmed that it is not possible to find a unique solution for this problem except for some special cases. Also, they found secondary flows in their research using first order term of perturbation series. They have not considered the higher order terms of the axial velocity and stream function for their perturbation solution and subsequently no flow rate has been reported.

In this paper, an analytical solution for fully developed second order fluid flow in a curved circular pipe is presented using a perturbation method to solve the governing equations. Fig. 1 shows the geometry of curved pipe which is used in current research. Here, the toroidal coordinate system $(\tilde{r},\phi,\tilde{s})$ is used for studying the flow. According to the picture, r_0 is the radius of the pipe and the R is the pitch radius of curvature. Due to the complication of the solution, the MAPLE software has been used in representing the expressions. According to the result of Jitchote and Robertson [25], the solution of this problem is available analytically in absence of second normal stress difference. Here, unlike the research of Jitchote and Robertson [25], the solution is carried out using the high order terms of perturbation series. It is important to emphasize that it is not possible to find a relation for flow rate based on the first order terms of perturbation series. By calculating second order terms, an analytical relation for the flow rate of the second order fluid in a curved pipe is obtained and based on this solution, the effect of retardation time on the drag reduction of flow in a curved pipe is investigated and compared with the effect of relaxation time that reported in research of Robertson and Muller [17]. Also, in the presence of the high order terms, the axial velocity, secondary flows and stress components are studied in detail.

2. MATHEMATICAL MODELING

2.1. Non-dimensional Parameter

The non-dimensional parameters of the current study are:

$$s=\frac{\tilde{s}}{r_0},\qquad r=\frac{\tilde{r}}{r_0},\qquad u=\frac{\tilde{\underline{v}}.e_r}{W_0},\qquad v=\frac{\tilde{\underline{v}}.e_\phi}{W_0},\qquad w=\frac{\tilde{\underline{v}}.e_s}{W_0},\qquad \tau=\frac{r_0}{\eta W_0}\tilde{\tau}$$
$$\gamma=\tilde{\gamma}\frac{r_0}{W_0},\qquad \overset{\nabla}{\gamma}=\overset{\nabla}{\tilde{\gamma}}\left(\frac{r_0}{W_0}\right)^2,\qquad We=\frac{\lambda W_0}{r_0},\qquad \mathrm{Re}=\frac{\rho W_0 r_0}{\eta},\qquad \delta=\frac{r_0}{R} \tag{1}$$

Where, r and s are the toroidal coordinates, r_0 the radius of pipe, W_0 the reference velocity, $\tilde{\underline{v}}$ the velocity vector, $\tilde{\tau}$ the stress, λ the time constant of fluid, η the viscosity, $\tilde{\gamma}$ and $\overset{\nabla}{\tilde{\gamma}}$ the first and the second contravariant convected derivative of the shear rate tensor, Re the Reynolds number, δ the curvature ratio, R the curvature radius of the curved pipe, and We the Weissenberg number. Also, λ introduce the time constant of fluid. Note that in the upper convected Maxwell fluid and the second order fluid, λ is defined based on the relaxation and retardation time of material respectively.

2.2. Constitutive Equation

The second order fluid flow considered here is similar to the Criminale-Eriksen-Filbey (CEF) viscoelastic model except that the rheological functions in the second order fluid are independent of shear rate. This constitutive equation is suitable for modeling of steady shear flow of viscoelastic materials and it is current for industrial calculations [26]. Stress tensor for incompressible SOF can be written as [27]:

$$\tilde{\tau} = \tilde{\eta}\tilde{\gamma}-\frac{1}{2}\tilde{\Psi}_1\overset{\nabla}{\tilde{\gamma}}+\tilde{\Psi}_2\left(\tilde{\gamma}.\tilde{\gamma}\right) \tag{2}$$

Where, $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are the first and the second normal stress difference coefficient. In steady state condition, the first and the second order of contravariant convected derivative of the shear rate tensors are defined as below [27]:

$$\tilde{\gamma} = \nabla\tilde{V}+\nabla\tilde{V}^T \tag{3-1}$$

$$\overset{\nabla}{\tilde{\gamma}} = \tilde{V}.\nabla\tilde{\gamma}-\left[\left(\nabla\tilde{V}\right)^{\mathrm{T}}\cdot\tilde{\gamma}+\tilde{\gamma}\cdot\left(\nabla\tilde{V}\right)\right] \tag{3-2}$$

Substituting $\tilde{\Psi}_1 = 0$ into Eq. (2), Reiner-Rivlin model is obtained and both $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ set equal to zero, the Newtonian fluid model is achieved. In the second order fluid, the relaxation time is zero and the retardation time (λ) is directly proportional to the first normal stress difference [27]:

$$\lambda = \frac{\tilde{\Psi}_1}{2\tilde{\eta}} \tag{4}$$

As shown by Jitchote *et al.* [25], the singularity in the perturbation solution appears when the $\tilde{\Psi}_2$ is included in Eq. (2). Therefore, the effect of the second normal stress difference is neglected in the current work. By using relations (1), (2), and (4), the dimensionless constitutive equation of the second order fluid is obtained:

$$\tau = \gamma - We\overset{\nabla}{\gamma} \tag{5}$$

The detailed expression of the above relation in toroidal coordinate system is presented in appendix section of current paper.

2.3. Governing Equations

In the fully developed flow, the derivatives of all the parameters of the flow respect to the angle of curvature path (ζ) except static pressure are zero. According to Fig. 1, in the fully developed situation, the pressure gradient can be expressed as:

$$\frac{\partial \tilde{P}}{\partial \zeta} = cte < 0 \tag{6}$$

By considering G as the absolute value of the axial pressure drop, it can be written:

$$\frac{\partial \tilde{P}}{\partial \tilde{s}} = \frac{1}{\tilde{R}}\frac{\partial P}{\partial \zeta} = -G \tag{7}$$

In relation (7), G is the constant value indicating the axial pressure drop. To non-dimensionalize the governing equations, the maximum velocity of the fully developed Newtonian fluid flow in a straight circular pipe with same pressure gradient, viscosity and hydraulic diameter is considered as the reference velocity [17]:

$$W_0 = \frac{G r_0^2}{4\eta} \tag{8}$$

The non-dimensional axial pressure gradient of Newtonian fluid flow in a straight pipe is as follows [17]:

$$\frac{\partial p}{\partial s} = -4 \tag{9}$$

It is important to knowing that the reference velocity is defined based on the equality of the pressure gradient in curved pipe with a Newtonian fluid flow in straight duct. Therefore, the relation (9) can be generalized to the pressure gradient of curved pipe in pitch direction.

The continuity equation and momentum equations in r, ϕ and s directions for the steady fully developed flow in toroidal coordinate system are as follows:

$$\frac{\partial(u r B)}{\partial r} + \frac{\partial(v B)}{\partial \phi} = 0 \tag{10-1}$$

$$\mathrm{Re}\left[u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} - \frac{v^2}{r} - \delta\frac{w^2\cos\phi}{B}\right] = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{1}{r}\left(\tau_{rr} - \tau_{\phi\phi}\right) + \frac{\delta}{B}\left(\tau_{rr}\cos\phi - \tau_{r\phi}\sin\phi - \tau_{ss}\cos\phi\right) \tag{10-2}$$

$$\mathrm{Re}\left[u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \phi}+\frac{u\,v}{r}+\delta\frac{w^2\sin\phi}{B}\right] = -\frac{1}{r}\frac{\partial p}{\partial \phi}+\frac{\partial \tau_{\phi r}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\phi\phi}}{\partial \phi}+\frac{2}{r}\tau_{r\phi}$$
$$+\frac{\delta}{B}\left(\tau_{r\phi}\cos\phi-\tau_{\phi\phi}\sin\phi-\tau_{ss}\sin\phi\right) \qquad (10\text{-}3)$$

$$\mathrm{Re}\left[u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \phi}+\delta\frac{w}{B}\left(u\cos\phi-v\sin\phi\right)\right] = -\frac{1}{B}\frac{\partial p}{\partial s}+\frac{\partial \tau_{sr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\phi s}}{\partial \phi}+\frac{1}{r}\tau_{sr}$$
$$+\frac{2\delta}{B}\left(\tau_{rs}\cos\phi-\tau_{\phi s}\sin\phi\right) \qquad (10\text{-}4)$$

Where, B is curvature radius of each point expressed as

$$B = 1+\delta\, r\cos(\phi) \qquad (11)$$

Also in the toroidal coordinate system, the relations between the lateral velocity components and the stream function of secondary flows are:

$$u = -\frac{1}{r\,B}\frac{\partial \psi}{\partial \phi}, \qquad v=-\frac{1}{B}\frac{\partial \psi}{\partial r} \qquad (12)$$

Where, ψ is the stream function. By substituting relation (12) into Eq. (10-1), the continuity equation is satisfied. Also, the momentum equation in main flow direction (Eq. (10-4)) is expressed as

$$\mathrm{Re}\left[\frac{1}{r\,B}\left(\frac{\partial \psi}{\partial r}\frac{\partial w}{\partial \phi}-\frac{\partial \psi}{\partial \phi}\frac{\partial w}{\partial r}\right)-\delta\frac{w}{B^2}\left(\frac{1}{r}\frac{\partial \psi}{\partial \phi}\cos\phi+\frac{\partial \psi}{\partial r}\sin\phi\right)\right]=$$
$$\frac{4}{B}+\frac{\partial \tau_{rs}}{\partial r}+\frac{1}{r}+\frac{\partial \tau_{\phi s}}{\partial \phi}+\frac{1}{r}\tau_{rs}+\frac{2\delta}{B}\left(\tau_{rs}\cos\phi-\tau_{\phi s}\sin\phi\right), \qquad (13)$$

By deducing the ϕ derivative of Eq. (10-2) and r derivative of equation (10-3) from each other, the equation in terms of stream function is obtained:

$$\mathrm{Re}\left\{\frac{1}{r\,B^2}\left(\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \phi}\nabla^2\psi-\frac{\partial \psi}{\partial \phi}\frac{\partial}{\partial r}\nabla^2\psi\right)+\frac{\delta}{B}\left[2w\left(\frac{\partial w}{\partial r}\sin\phi+\frac{1}{r}\frac{\partial w}{\partial \phi}\cos\phi\right)\right.\right.$$
$$+\frac{1}{B^2}\left(\sin\phi\left(\frac{3}{r}\left(\frac{\partial \psi}{\partial r}\right)^2+\frac{3}{r^2}\frac{\partial^2\psi}{\partial \phi^2}\frac{\partial \psi}{\partial r}-\frac{1}{r^2}\frac{\partial \psi}{\partial \phi}\frac{\partial^2\psi}{\partial r\,\partial \phi}+\frac{1}{r^3}\left(\frac{\partial \psi}{\partial \phi}\right)^2+2\frac{\partial \psi}{\partial r}\frac{\partial^2\psi}{\partial r^2}\right)\right.$$
$$\left.\left.+\cos\phi\left(\frac{3}{r}\frac{\partial^2\psi}{\partial r^2}\frac{\partial \psi}{\partial \phi}-\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial^2\psi}{\partial r\,\partial \phi}+\frac{3}{r^2}\frac{\partial \psi}{\partial r}\frac{\partial \psi}{\partial \phi}+\frac{2}{r^3}\frac{\partial \psi}{\partial \phi}\frac{\partial^2\psi}{\partial \phi^2}\right)\right)\right]$$
$$\left.+\delta^2\frac{3}{B^4}\left[\frac{\sin 2\phi}{2}\left(\frac{1}{r^2}\left(\frac{\partial \psi}{\partial \phi}\right)^2-\left(\frac{\partial \psi}{\partial r}\right)^2\right)-\cos 2\phi\frac{1}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial \psi}{\partial \phi}\right)\right]\right\}=$$
$$\left[-\frac{1}{r}\frac{\partial^2\tau_{rr}}{\partial r\partial \phi}-\frac{1}{r^2}\frac{\partial \tau_{rr}}{\partial \phi}+\frac{1}{r}\frac{\partial^2\tau_{\phi\phi}}{\partial r\partial \phi}+\frac{1}{r^2}\frac{\partial \tau_{\phi\phi}}{\partial \phi}+\frac{\partial^2\tau_{r\phi}}{\partial r^2}+\frac{3}{r}\frac{\partial \tau_{r\phi}}{\partial r}-\frac{1}{r^2}\frac{\partial^2\tau_{r\phi}}{\partial \phi^2}\right]$$

$$+\frac{\delta}{B}\left[\begin{array}{l}\sin\phi\left(\frac{1}{r}\frac{\partial\tau_{r\phi}}{\partial\phi}-\frac{1}{r}\left(\tau_{\phi\phi}-\tau_{rr}\right)+\frac{\partial\tau_{ss}}{\partial r}-\frac{\partial\tau_{\phi\phi}}{\partial r}\right)\\+\cos\phi\left(\frac{1}{r}\frac{\partial\tau_{ss}}{\partial\phi}-\frac{1}{r}\frac{\partial\tau_{rr}}{\partial\phi}+\frac{2}{r}\tau_{r\phi}-\frac{\partial\tau_{r\phi}}{\partial r}\right)\end{array}\right] \tag{14}$$

Therefore, the governing equations are reduced into Eqs. (13) and (14) where ∇^2 denotes the Laplacian operator defined as:

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} \tag{15}$$

Also, the no slip condition is valid as boundary condition at the walls:

$$w = \psi=\frac{\partial\psi}{\partial r}=0 \text{ at } r = 1 \tag{16}$$

3. PERTURBATION SOLUTION

In this section, the perturbation method is used to obtain the solution of w and ψ for the fully developed flow of second order fluid inside a curved pipe. The forms of perturbation series for flow field parameters are as follows

$$w = \sum_{n=0}^{\infty}\delta^n w^{(n)}(r,\phi),\ \psi=\sum_{n=1}^{\infty}\delta^n\psi^{(n)}(r,\phi),\ \tau=\sum_{n=0}^{\infty}\delta^n\tau^{(n)}(r,\phi) \tag{17}$$

Where, δ is the curvature ratio and is considered as perturbation parameter. Due to lack of secondary flows in the stationary straight pipe, the zero order term of the stream function which introducing the secondary flows in straight pipe is zero ($\psi^{(0)} = 0$).

3.1. Solution to Order δ^0

The zero order stress components are necessary to find the first term of the axial velocity in relation (17). Arranging the coefficients of δ^0 in Eq. (13) yields below expression:

$$\frac{\partial\tau_{rs}^{(0)}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\phi s}^{(0)}}{\partial\phi}+\frac{\tau_{rs}^{(0)}}{r} = -4 \tag{18}$$

Using relation (17) and the constitutive equation of the second order fluid presented in Appendix, it can be expressed:

$$\tau_{rs}^{(0)} = \frac{\partial w^{(0)}}{\partial r},\qquad \tau_{\phi s}^{(0)}=\frac{1}{r}\frac{\partial w^{(0)}}{\partial\phi} \tag{19}$$

By substituting relation (19) into Eq. (18) yields, the characteristic equation of zero order equation is obtained:

$$\nabla^2 w^{(0)} = -4 \tag{20}$$

The solution of $w^{(0)}$ which satisfies the boundary condition (relation (16)) is:

$$w^{(0)} = 1 - r^2 \tag{21}$$

This result is exactly similar to the axial velocity of Newtonian fluids in the straight pipe.

3.2. Solution to Order δ^1

By substituting the terms δ^1 of order in the stress components and rearranging the same order terms of Eq. (14), the characteristic equation of $\psi^{(1)}$ is obtained. Due to large size of the characteristic equation that related to the higher order terms, these equations are not represented in this paper. The general solution of the characteristic equation of $\psi^{(1)}$ is:

$$\psi^{(1)} = g_1(r)\sin(\phi) \tag{22}$$

Where,

$$g_1(r) = \frac{1}{288} r\, w^{(0)^2} \left(\mathrm{Re}(r^2 - 4) - 24 We\right) \tag{23}$$

After rearranging the terms of order δ^1 in Eq. (14), the characteristic equation of $w^{(1)}$ is obtained. The close form of the solution of this equation is as follows:

$$w^{(1)} = f_1(r)\cos(\phi) \tag{24}$$

Where,

$$f_1(r) = -\frac{1}{11520} r\, w^{(0)} \begin{pmatrix} 8640 + \mathrm{Re}^2 (r^6 - 9r^4 + 21r^2 - 19) + 1920(r^2 - 1)We^2 \\ + \mathrm{Re}(-120r^4 + 520r^2 - 440)We \end{pmatrix} \tag{25}$$

The similar relations were obtained previously by Jitchote and Robertson [25] for the second order fluid flow in a curved pipe, but the higher order terms are not considered in their research. Calculation of the first order terms is only suitable for the exhibition of secondary flows in the curved pipes not for the accurate expression for velocity field. It is obvious that the integral of relation (25) over the cross-section of the pipe is equal to zero. Therefore, using the first order approximation of the axial velocity is not sufficient for calculation of flow rate in the curved pipe in comparison with the straight pipe. Therefore, the calculation of the higher order terms leads to obtaining more accurate expression for the secondary flows and axial velocity. Thus, unlike the Jitchote and Robertson's research [25], the current study goes beyond the first order term. Note that the derivation of these expressions is more difficult and leads to more complicated solutions.

3.3. Solution to order δ^2

The higher order terms in the series solution can be obtained using the same method employed for obtaining the first order terms of the axial velocity and stream function. The characteristic equations are obtained by substituting the stress components of the second order fluid with order δ^0 to δ^2 into the governing equations (Eqs. (13) and (14)) and rearranging the second order terms. The solutions of these equations which satisfy the boundary conditions are:

$$\psi^{(2)} = g_2(r)\sin(2\phi) \tag{26-1}$$

$$w^{(2)} = f_{20}(r) + f_{22}(r)\cos(2\phi) \tag{26-2}$$

$g_2(r)$, $f_{20}(r)$, and $f_{22}(r)$ are:

$$g_2(r) = -\frac{1}{464486400} r^2 w^{(0)\,2} \left(\begin{array}{l} \mathrm{Re}(564480 r^2 - 1290240) + \\ \mathrm{Re}^3 (5r^8 - 134 r^6 + 777 r^4 - 2792 r^2 + 4979) + \\ (5160960 - 1935360 r^2) We^3 + \\ \mathrm{Re}(248640 r^4 - 1330560 r^2 + 1659840) We^2 + \\ \left(\mathrm{Re}^2 \left(-4224 r^6 + 39712 r^4 - 124672 r^2 + 172944\right) - 26611200\right) We \end{array} \right) \tag{27-1}$$

$$f_{20}(r) = w^{(0)} \left(\begin{array}{l} -\frac{3}{32} + \frac{11}{32} r^2 + \mathrm{Re}^2 \left(\frac{7}{230400} r^8 - \frac{17}{57600} r^6 + \frac{11}{19200} r^4 - \frac{43}{230400} r^2 - \frac{37}{57600}\right) \\ + \mathrm{Re}^4 \left(\frac{1}{106168320} r^{14} - \frac{121}{743178240} r^{12} + \frac{803}{743178240} r^{10} - \frac{4523}{1238630400} r^8 \right. \\ \left. + \frac{26261}{3715891200} r^6 - \frac{29179}{3715891200} r^4 + \frac{17161}{3715891200} r^2 - \frac{1373}{1238630400}\right) \\ + \left(\frac{1}{36} r^6 + \frac{1}{108}(-5 r^4 + r^2 + 1)\right) We^4 \\ + \left(\mathrm{Re}\left(-\frac{11}{2880} r^8 + \frac{3}{160} r^6 - \frac{13}{540} r^4 + \frac{109}{17280} r^2 + \frac{49}{17280}\right)\right) We^3 \\ + \left(\begin{array}{l} \frac{1}{4} r^4 - \frac{19}{48} r^2 + \frac{7}{48} \\ + \mathrm{Re}^2 \left(\begin{array}{l} \frac{71}{414720} r^{10} - \frac{577}{414720} r^8 + \frac{3211}{829440} r^6 - \frac{401}{92160} r^4 \\ + \frac{431}{276480} r^2 + \frac{13}{92160} \end{array} \right) \end{array} \right) We^2 \\ + \left(\begin{array}{l} \mathrm{Re}^3 \left(\begin{array}{l} -\frac{13}{5806080} r^{12} + \frac{979}{34836480} r^{10} - \frac{11129}{87091200} r^8 \\ + \frac{194993}{696729600} r^6 - \frac{218147}{696729600} r^4 + \frac{107563}{696729600} r^2 \\ - \frac{13397}{696729600} \end{array} \right) \\ + \mathrm{Re}\left(-\frac{83}{4608} r^6 + \frac{341}{4608} r^4 - \frac{367}{4608} r^2 + \frac{31}{1536}\right) \end{array} \right) We \end{array} \right) \tag{27-2}$$

$$\frac{r^2 w^{(0)}}{117050572800}\left(\begin{array}{l}\left(\begin{array}{l}36578304000+\mathrm{Re}^4\,(160r^{12}-2801r^{10}+19123r^8-70547r^6+174649r^4\\-240206r^2+145690)+\mathrm{Re}^2\,(1693440r^6-125314560r^4+259519680r^2\\-196015680)\end{array}\right)\\+\left(5202247680r^4-11054776320r^2+5419008000\right)We^4\\+\mathrm{Re}\left(-392878080r^6+2343720960r^4-3894912000r^2+1835688960\right)We^3\\+\left(\begin{array}{l}39016857600r^2-32514048000\\+\mathrm{Re}^2\left(\begin{array}{l}9918720r^8-92534400r^6+303730560r^4-441806400r^2\\+218796480\end{array}\right)\end{array}\right)We^2\\+\left(\begin{array}{l}\mathrm{Re}^3\left(\begin{array}{l}-71316r^{10}+960876r^8-4772124r^6+12373956r^4\\-17566164r^2+10024812\end{array}\right)\\+\mathrm{Re}\left(-1280240640r^4+5222568960r^2-3434296320\right)\end{array}\right)We\end{array}\right) \tag{27-3}$$

4. SOLUTION OF FLOW RATE

In this section, an analytical solution for the flow rate of the second order fluid flow in curved pipe is presented. The flow rate through a pipe can be obtained as:

$$Q = \int_0^{2\pi}\int_0^1 w\,r\,dr\,d\phi \tag{28}$$

Where the three terms of the perturbation expansion for w are:

$$w = w^{(0)}(r)+\delta\,f_1(r)\cos\phi+\delta^2\left(f_{20}(r)+f_{22}(r)\cos 2\phi\right). \tag{29}$$

By substituting relations (21), (25) and (27) into relation (29), and substituting the result into relation (28), the following relation for the flow rate is obtained:

$$Q_c = Q_s\left(1+\delta^2\left(\begin{array}{l}\frac{1}{48}-\frac{11}{17280}\mathrm{Re}^2-\frac{1541}{4180377600}\mathrm{Re}^4+\left(\frac{1}{1290240}\mathrm{Re}^3+\frac{1}{240}\mathrm{Re}\right)We\\+\left(\frac{1}{18}+\frac{691}{2903040}\mathrm{Re}^2\right)We^2+\left(\frac{11}{4320}\mathrm{Re}\right)We^3+\left(\frac{1}{135}\right)We^4\end{array}\right)\right)$$

Where, Q_s is the non-dimensional flow rate in the straight pipe under the same axial pressure gradient and equal to $\pi/2$. Dimensional magnitude of this flow rate is

$$\tilde{Q}_s = \frac{\pi r_o^2}{8\eta}G \tag{31}$$

Relation (30) is valid for the small curvature ratio. If We in relation (30) is set equal to zero, the solution is reduced to:

$$Q_c = Q_s\left(1+\delta^2\left(\frac{1}{48}-\frac{11}{17280}\mathrm{Re}^2-\frac{1541}{4180377600}\mathrm{Re}^4\right)\right) \quad (32)$$

The above relation is obtained by Topakoglu [3] for Newtonian fluid flow in a curved pipe. The flow rate of the creeping flow of the second order fluid in a curved pipe is obtained by considering the zero value for Reynolds number in relation (30):

$$Q_c = Q_s\left(1+\delta^2\left(\frac{1}{48}+\frac{1}{18}We^2+\frac{1}{135}We^4\right)\right) \quad (33)$$

This relation was presented by Bowen et al. [24] previously. Therefore, it is evident that relation (30) presented in the current research for the flow rate of inertial flow of the second order fluid in a curved pipe covers the solution of both Topakoglu [3] and Bowen et al. [24].

5. RESULTS AND DISCUSSION

In this section, based on the perturbation solution that presented in sections 3 and 4, the second order fluid flow in a curved pipe is studied. In the flow of UCM and the second order fluids under constant pressure gradient with the same viscosity and hydraulic diameter, the Weissenberg number depends on the relaxation and the retardation time respectively. The effect of these times constant on the viscoelastic flow in the curved pipe is investigated using the results of the second order fluid flow obtained from the current research and the UCM fluid presented in Robertson and Muller's [17] work.

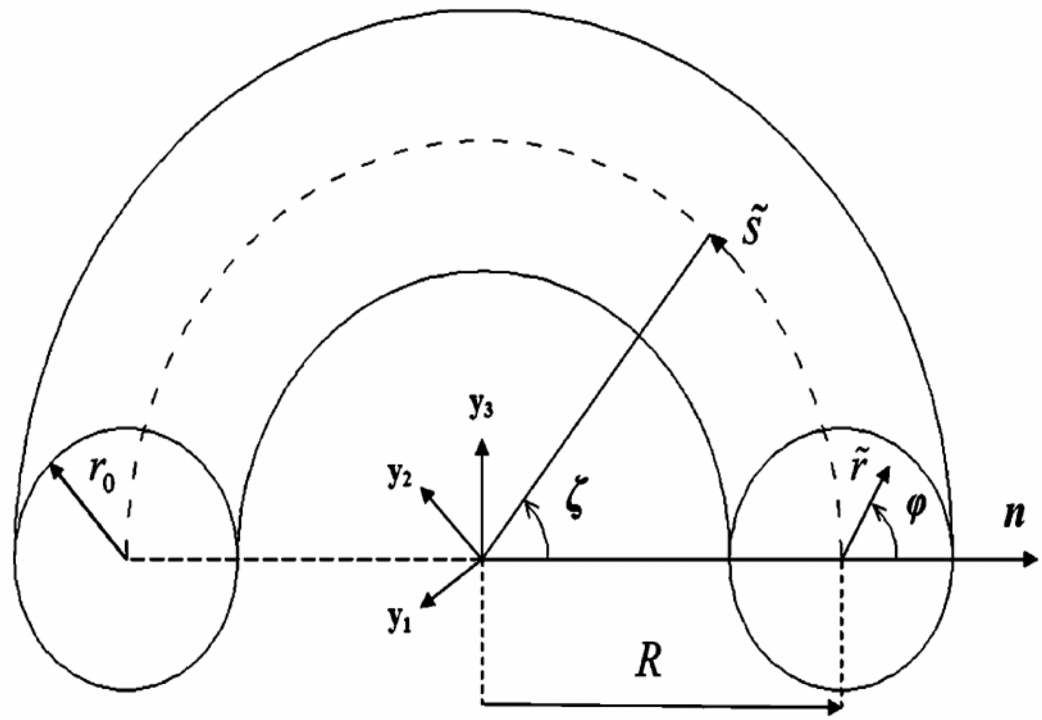


Figure. 1: Geometry of the Problem

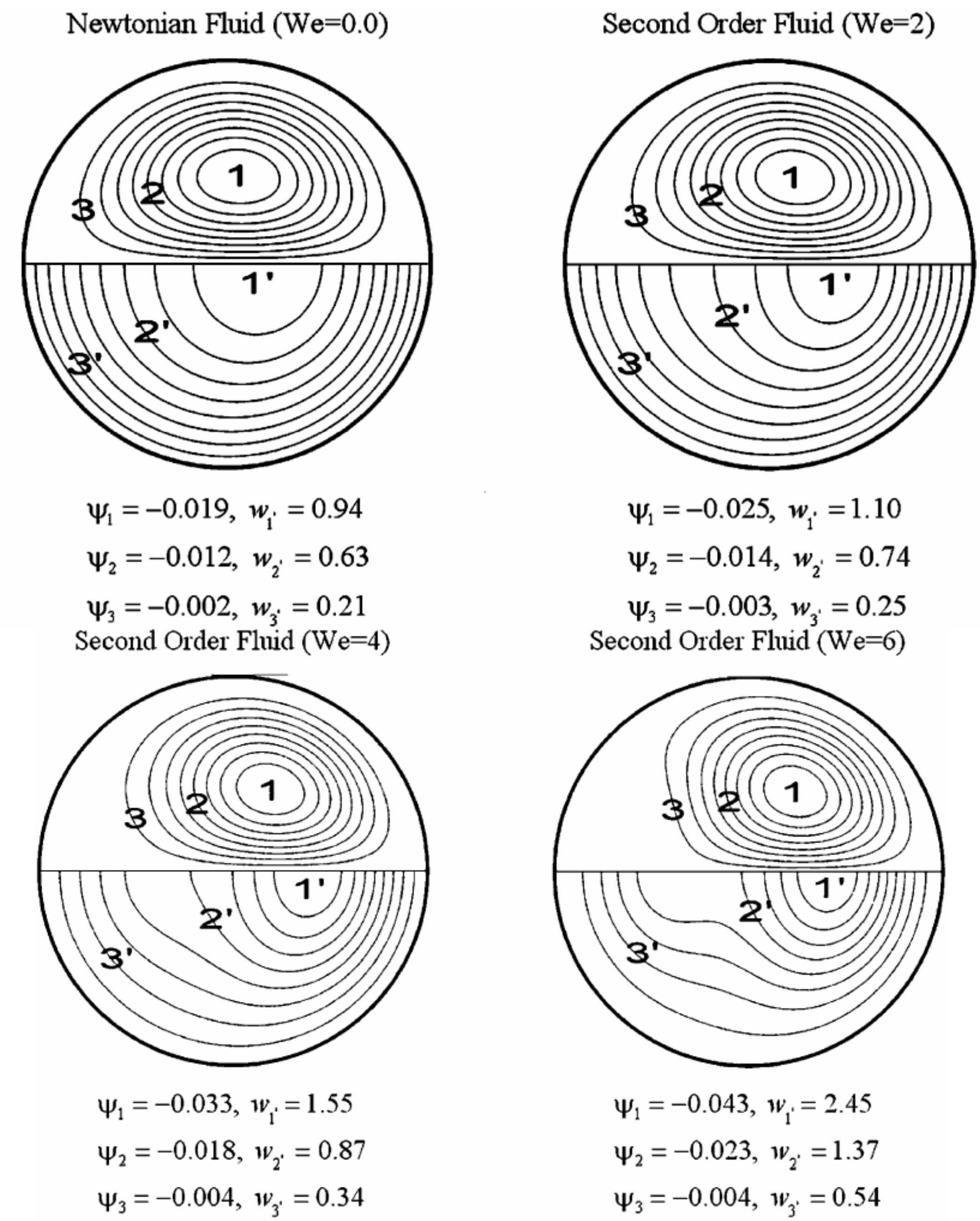


Figure. 2: Stream Function of Secondary Flows (Upper Half) and Axial Velocity (Lower Half) of Second Order Fluid Flow at different Values of Weissenberg Number (Re = 50 and $\delta = 0.1$)

In Fig. 2, the secondary flows in upper half and the axial velocity in lower half of the second order fluid flow at different Weissenberg numbers have been shown for Re=50 and d $\delta = 0.1$. In Newtonian fluid, the effect of centrifugal force generates a pair of secondary flows called Taylor vortices. According to Fig. 2, by increasing Weissenberg number of the second order fluid, the maximum of axial velocity and secondary flows intensity are increased and the position of axial velocity shifts to the outer curved pipe region.

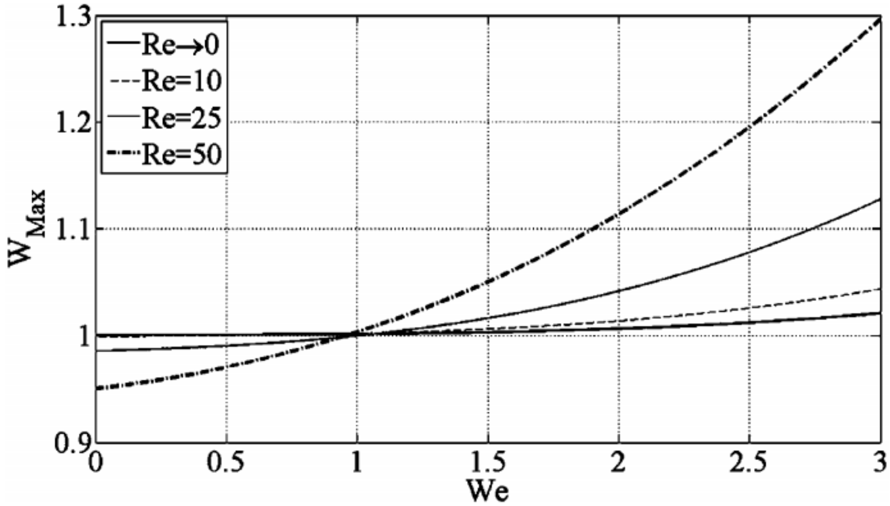


Figure. 3: Effect of Weissenberg Number on Maximum Value of Axial Velocity Profile of Second Order Fluid Flow in Curved Pipe

Fig. 3 illustrates the effect of Weissenberg number on the maximum main flow velocity. According to this figure, in creeping ($\mathrm{Re} \simeq 0$) and inertial flow, the increasing of Weissenberg number enhances the maximum of axial velocity. Also, the effect of Weissenberg number becomes stronger at large Reynolds number. This effect is due to the second order dependency of non-Newtonian part of the second order fluid to the shear rate.

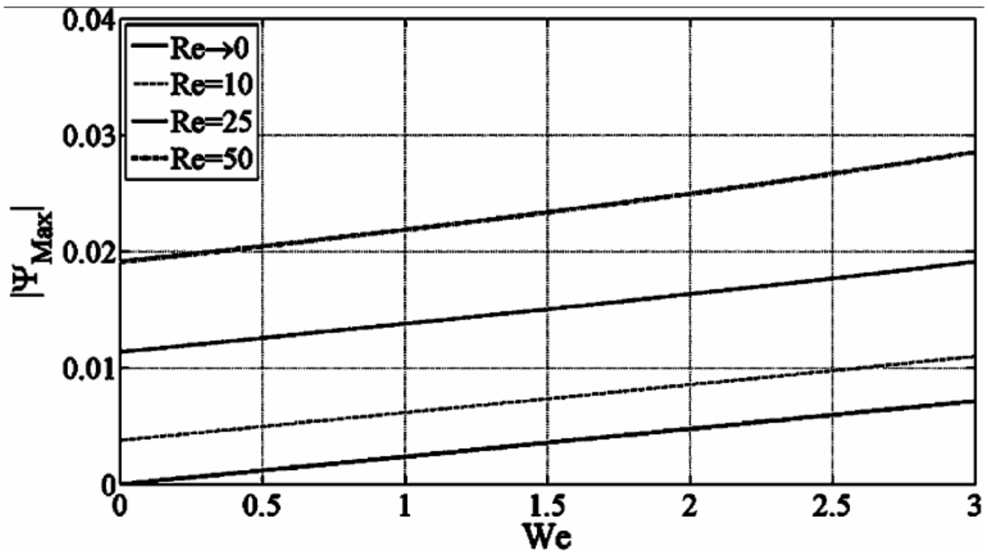


Figure. 4: Effect of Weissenberg Number on Maximum Value of Stream Function of Second Order Fluid Flow in Curved Pipe

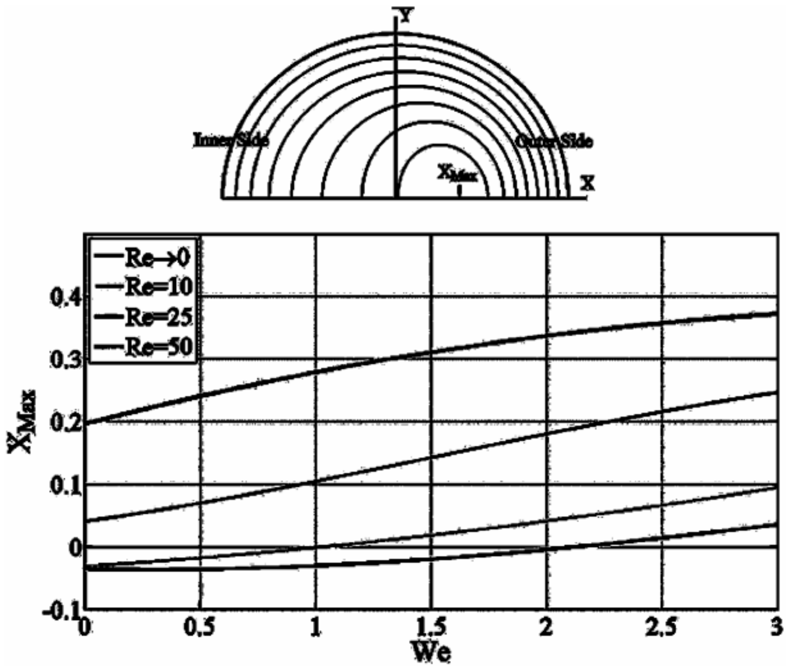
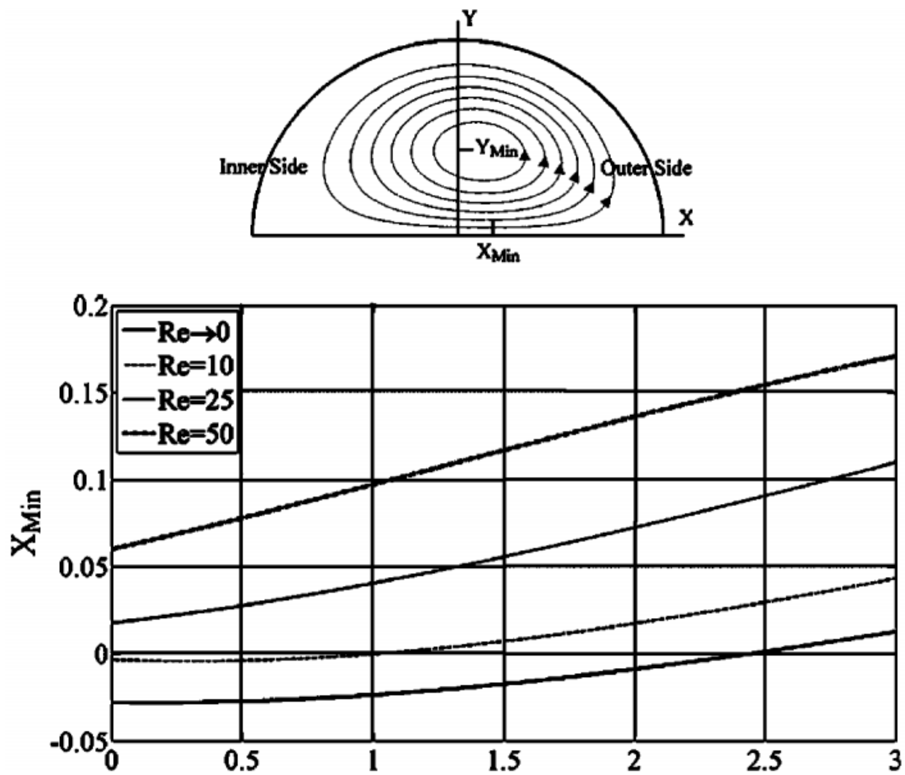


Figure. 5: Effect of Weissenberg Number on Position of Maximum Value of Axial Velocity



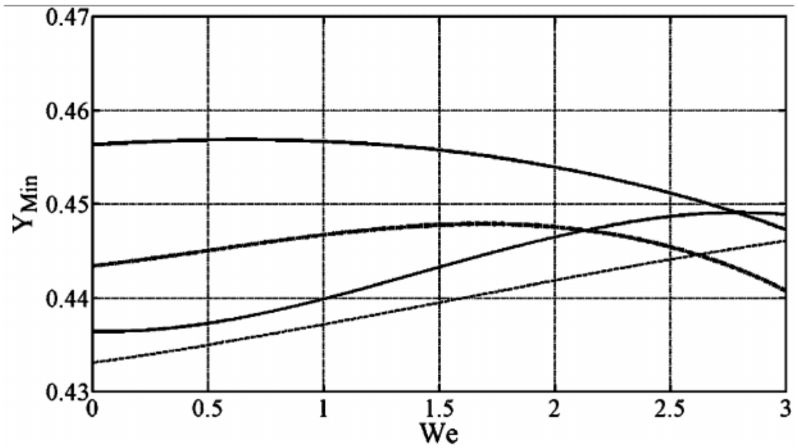


Figure. 6: Effect of Weissenberg Number on Position of Secondary Flow Center

Similarly, Fig. 4 shows the effect of Weissenberg number on the maximum value of stream function. As is evident, the intensity of the secondary flow in creeping Newtonian fluid ($\mathrm{Re} \simeq 0$ and We = 0.0) is equal to zero. In this case, the effects of the centrifugal force and the fluid elasticity are trivial due to lack of secondary flow. But by increasing of Weissenberg number, the secondary flow intensifies similar to the inertial flow. The effect of Weissenberg number on the position of maximum axial velocity (X_{max}) is shown in Fig. 5. Regarding to this figure, the position of maximum axial velocity stays on the symmetric line of the flow. It should be noticed that at Reynolds number less than 10 and small Weissenberg number, the position of maximum axial velocity tends to the inner side of curved pipe (X_{max} is negative). In this case, the centrifugal force and the elastic force are infinitesimal resulting in very weak secondary flow.

Therefore, in the absence of secondary flows and due to the larger value of the axial pressure gradient near the inner side in comparison to outer side of the curved pipe, the distribution of the axial velocity gets closer to the inner side. By increasing Reynolds number and Weissenberg number (increasing of centrifugal and elastic forces), the secondary flow intensity increases and the position of axial velocity shifts to the outer side of curved pipe. Fig. 6 shows the effect of Weissenberg number on the longitudinal (X_{Min}) and the lateral positions (Y_{Min}) of secondary flows centre. According to this figure, increasing of Weissenberg number raises the X_{Min}. It is showing that the center of secondary flows shifts to the outer side of curvature. The effect of Weissenberg number on the lateral position of vortices is trivial, by increasing Weissenberg number first the Y_{Min} increases then it starts decreasing.

According to the research of Robertson and Muller [17], the relation of flow rate for UCM fluid flow in a curved pipe is as follow:

$$Q_c = Q_s\left\{1+\frac{\delta^2}{48}\left[\begin{array}{l}1-\mathrm{Re}^2\left(\frac{11}{360}+\mathrm{Re}^2\frac{1541}{87091200}\right)+\frac{8}{3}We^2\left(1-\frac{1}{15}We^2\right)+\\ \frac{1}{26880}We\,\mathrm{Re}\left(\mathrm{Re}^2+5376\right)-\frac{101}{60480}We^2\,\mathrm{Re}^2-\frac{2}{45}We^3\,\mathrm{Re}\end{array}\right]\right\} \quad (34)$$

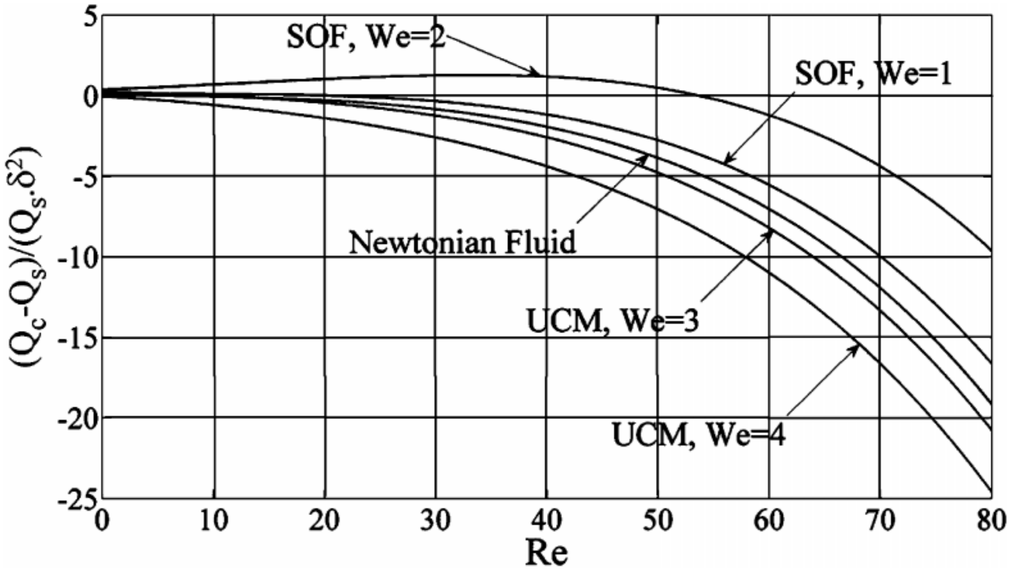


Figure. 7: Flow Rate difference Ratio of Newtonian, Upper Convected Maxwell and Second Order Fluid Versus Reynolds Number

Fig. 7 shows the flow rate difference ratio of Newtonian, UCM and the second order fluid versus Reynolds number at different values of Weissenberg number. According to the perturbation solution, the flow rate difference ratio for these fluids is in the second order of curvature ratio. It should be noticed that the second order dependency of flow rate ratio is only valid in low Dean Number and due to the restrictions of the perturbation method; it is impossible to extend it to the high Dean numbers. According to the figure, as the Reynolds number increases, the flow rate difference ratio of Newtonian fluid decreases. For UCM fluid at large Weissenberg number (large relaxation time), the flow rate is lower than the Newtonian fluid. Unlike the UCM fluid, by increasing Weissenberg number (increasing retardation time) of the second order fluid, the flow rate ratio becomes larger than the Newtonian fluid. In the other hand, by increasing the retardation time of fluid, the drag reduction is caused in viscoelastic flow in curved pipes while the effect of relaxation time is reveres at $We > 2.1$. It should be mentioned that Bowen et al. [24] observed similar effect for the drag reduction/enhancement for creeping flow of the UCM and the second order fluids but in the current research, this effect is proved for the inertial flows analytically. The effect of the retardation time on the secondary flows and on the flow rate of the second order fluid flow in a curved pipe have been studied here. According to Eq. (4), retardation time of second order fluid has a direct relation with the first normal stress difference. In the core region, the order of axial velocity is larger than the order of secondary flows. Fan et al. [6] obtained force balance relation in the core region of viscoelastic fluid flow by using this specification and neglecting the second normal stress difference:

$$\mathrm{Re}\frac{w^2}{B} \approx \frac{\partial P}{\partial n}+\frac{\tau_{ss}}{B} \qquad (35)$$

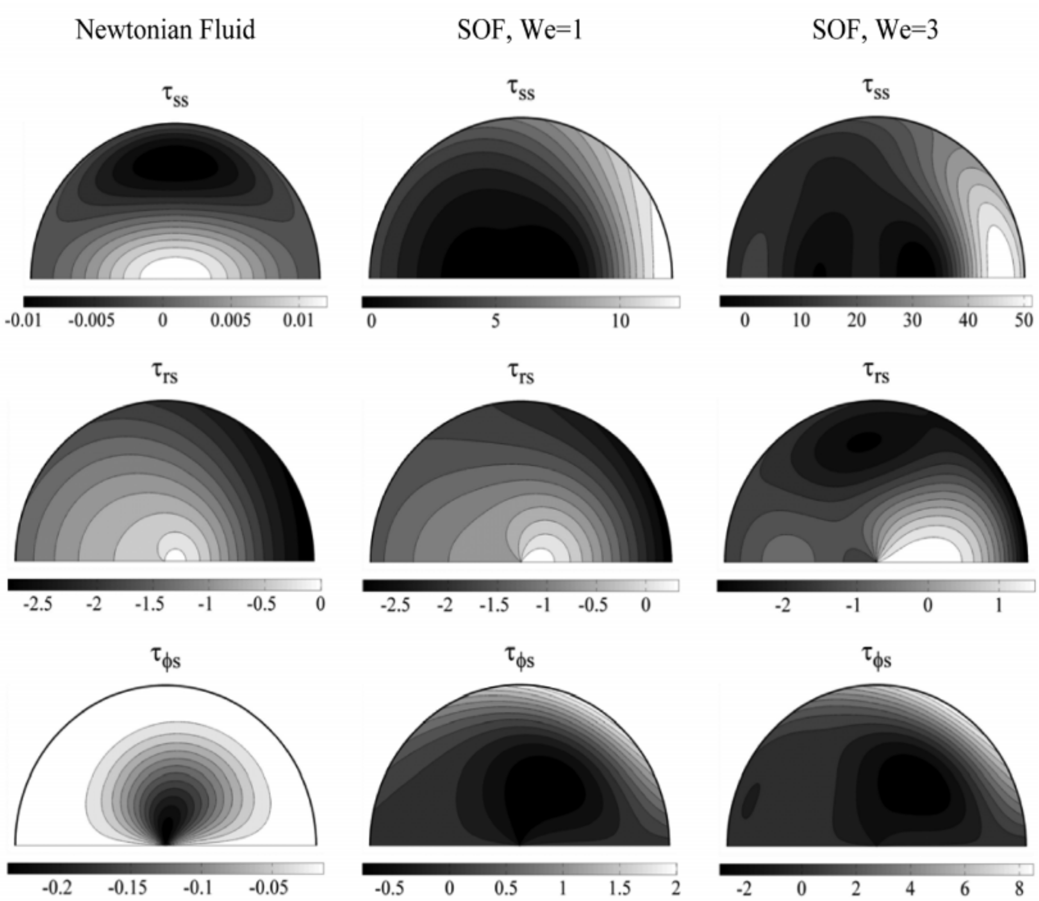


Figure. 8: Contours of τ_{ss}, τ_{rs} and $\tau_{\varphi s}$ at Re = 50 and δ = 0.1

According to Eq. (35), the effect of centrifugal force is balanced by the effect of pressure gradient in the direction of the curvature radius and hoop stress effect (τ_{ss}) in the core region of flow. Fig. 8 shows the hoop stress (τ_{ss}) distribution for the Newtonian fluid flow and the second order fluid flow in two Weissenberg numbers of 1 and 3. Here, the curvature ratio is equal to 0.1 and the Reynolds number is equal to 50. Mashelkar and Devarajan [28] showed that in the core region of the inertial flow of the Newtonian fluid in the curved duct, the pressure gradient is balanced by the effect of centrifugal force. In other hand, in Newtonian fluid, the hoop stress does not play main role in the secondary flow generation. According to Fig. 8, the amount of hoop stress in the Newtonian fluid is very small, but in the second order fluid flow the amount of hoop stress is enhanced noticeably by increasing the Weissenberg number. Therefore, the maximum amount of the hoop stress in the second order fluid flow at We = 3 is 5000 times larger than the Newtonian fluid flow. This result has a good accordance with the relation for the hoop stress that has been obtained based on the order of magnitude technique by Fan et al. [6]. Regarding to their studies, the hoop stress is in ε order for the Newtonian fluid but largely dependent on the first normal stress difference. Near the outer side of the curved pipe and in the vicinity of it, the amount of axial velocity and the effect of centrifugal force is small. According to Fig. 8, the amount of the hoop stress of the second order fluid flow is very large in this region. Therefore, the relation (35) for balance force is not valid in this region. For keeping the balance, the momentum mechanism acts and secondary flows in the direction of Taylor vortices are generated. Therefore by increasing the Weissenberg number, the intensity of the resultant secondary flow increases. Fig. 8 shows the stress components of τ_{rs} and $\tau_{\phi s}$ of the second order fluid flow. These two stress components are effective in the axial flow equation and regarding to their distribution, Weissenberg number effect on increasing flow rate of second order fluid flow can be affirmed. According to this figure, by increasing the Weissenberg number, unlike τ_{rs}, the distribution of $\tau_{\phi s}$ changes noticeably and the maximum of absolute amount of this stress component at We = 3 is 100 times larger than the Newtonian fluid flow. Thus, it is clear that in the second order fluid flow, the large variation in $\tau_{\phi s}$ increases the axial velocity and flow rate in comparison with the Newtonian fluid flow.

6. CONCLUSION

In this paper, the inertial flow of the second order fluid flow in a circular curved pipe has been studied using perturbation method to present an analytical solution for the flow field. Under the specific pressure gradient, Weissenberg number of the second order and the UCM fluid flows are the criterion of the retardation time and relaxation time respectively. Therefore, by comparing the results of the current research with the results of previous works for UCM fluid, the effects of these two time constants of materials on the viscoelastic flow in curved pipes are investigated.

Most important results of this research can be outlined as below:

- Increasing the Weissenberg number of second order fluid amplifies the secondary flow intensity. Increasing the retardation time shifts the centre of Taylor vortices towards the outer region of curved pipe and its effect on the changes of the centre position of these vortices respective to the lateral side is trivial.
- In the core region of the second order fluid flow, the centrifugal force is balanced by the pressure gradient in the curvature direction and by the hoop stress (τ_{ss}). Increasing the retardation time of the second order fluid generates the strength hoop stress near the outer wall where the value of axial velocity and centrifugal force is small and the momentum mechanism controls the balance and generates the strong secondary flows in the direction of Taylor vortices.
- Increasing the relaxation time and retardation time shifts the position of maximum axial velocity towards the outer curved region of the pipe.
- Increasing the retardation time of the second order fluid is accompanied by the drag reduction in the flow inside the curved pipes. In the UCM fluid, the drag reduction behavior is observed in the small amount of the relaxation time while the drag enhancement behavior is observed in the large amount of relaxation time (We > 2.1).

- Increasing the retardation time of the second order fluid strengthens the $\tau_{\phi s}$ which increases the flow rate in the curved pipes.

Appendix: Stress Components of the Second Order Fluid

Stress components of the second order fluid in toroidal coordinate system can be expressed in the form of $\tau = \gamma - We\overset{\nabla}{\gamma}$, where:

$$\gamma_{rr} = 2\frac{\partial u}{\partial r} \qquad \gamma_{r\phi} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial \phi} - v\right)$$

$$\gamma_{rs} = \frac{\partial w}{\partial r} - \frac{\delta}{B} w\cos\phi \qquad \gamma_{\phi\phi} = \frac{2}{r}\left(u + \frac{\partial v}{\partial \phi}\right)$$

$$\gamma_{\phi s} = \frac{1}{r}\frac{\partial w}{\partial \phi} + \frac{\delta}{B} w\sin\phi \qquad \gamma_{ss} = \frac{2\delta}{B}\left(u\cos\phi - v\sin\phi\right) \tag{A1}$$

And

$$\overset{\nabla}{\gamma}_{rr} = u\frac{\partial \gamma_{rr}}{\partial r} + \frac{v}{r}\frac{\partial \gamma_{rr}}{\partial \phi} - 2\frac{\partial u}{\partial r}\gamma_{rr} - \frac{2}{r}\frac{\partial u}{\partial \phi}\gamma_{r\phi},$$

$$\overset{\nabla}{\gamma}_{r\phi} = u\frac{\partial \gamma_{r\phi}}{\partial r} + \frac{v}{r}\left(\frac{\partial \gamma_{r\phi}}{\partial \phi} + \gamma_{rr}\right) - \frac{\partial u}{\partial r}\gamma_{r\phi} - \frac{1}{r}\frac{\partial u}{\partial \phi}\gamma_{\phi\phi} - \frac{\partial v}{\partial r}\gamma_{rr} - \frac{1}{r}\left(u + \frac{\partial v}{\partial \phi}\right)\gamma_{r\phi},$$

$$\overset{\nabla}{\gamma}_{rs} = u\frac{\partial \gamma_{rs}}{\partial r} + \frac{v}{r}\frac{\partial \gamma_{rs}}{\partial \phi} - \frac{\partial u}{\partial r}\gamma_{rs} - \frac{1}{r}\frac{\partial u}{\partial \phi}\gamma_{\phi s} - \frac{\partial w}{\partial r}\gamma_{rr} - \frac{1}{r}\frac{\partial w}{\partial \phi}\gamma_{r\phi}$$
$$+ \frac{\delta}{B}(-u\cos\phi + v\sin\phi)\gamma_{rs} + w\gamma_{rr}\cos\phi - w\gamma_{r\phi}\sin\phi,$$

$$\overset{\nabla}{\gamma}_{\phi\phi} = u\frac{\partial \gamma_{\phi\phi}}{\partial r} + \frac{v}{r}\left(\frac{\partial \gamma_{\phi\phi}}{\partial \phi} + 2\gamma_{r\phi}\right) - 2\frac{\partial v}{\partial r}\gamma_{r\phi} - \frac{2}{r}\left(u + \frac{\partial v}{\partial \phi}\right)\gamma_{\phi\phi},$$

$$\overset{\nabla}{\gamma}_{\phi s} = u\frac{\partial \gamma_{\phi s}}{\partial r} + \frac{v}{r}\left(\gamma_{rs} + \frac{\partial \gamma_{\phi s}}{\partial \phi}\right) - \frac{\partial v}{\partial r}\gamma_{rs} - \frac{1}{r}\left(u + \frac{\partial v}{\partial \phi}\right)\gamma_{\phi s} - \frac{\partial w}{\partial r}\gamma_{r\phi} - \frac{1}{r}\frac{\partial w}{\partial \phi}\gamma_{\phi\phi}$$
$$+ \frac{\delta}{B}(-u\cos\phi + v\sin\phi)\gamma_{\phi s} + w\gamma_{r\phi}\cos\phi - w\gamma_{\phi\phi}\sin\phi,$$

$$\overset{\nabla}{\gamma}_{ss} = u\frac{\partial \gamma_{ss}}{\partial r} + \frac{v}{r}\frac{\partial \gamma_{ss}}{\partial \phi} - 2\frac{\partial w}{\partial \phi}\gamma_{\phi s} + 2\frac{\delta}{B}(-u\cos\phi + v\sin\phi)\gamma_{ss}$$
$$+ w\gamma_{rs}\cos\phi - w\gamma_{\phi s}\sin\phi, \tag{A2}$$

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