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APPLICATION OF THE (G'/G)-EXPANSION METHOD FOR TWO NONLINEAR EVOLUTION EQUATIONS

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Abstract: *In this paper, the (G'/G)-expansion method is applied to seek traveling wave solutions of two nonlinear evolution equations. This traveling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. It is shown that the proposed method is direct, effective and can be used for many other nonlinear evolution equations in mathematical physics.*

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1. INTRODUCTION

The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In the past several decades, new exact solutions have helped to find new phenomena. A variety of powerful methods, such as inverse scattering method [1], the tanh-sech method [2-4], extended tanh method [5-7], sine-cosine method [8, 9], homogeneous balance method [10], Exp-function method [11, 12], improved tanh-function method[13] and modified tanh-coth method [14, 15] have been used to develop nonlinear dispersive and dissipative problems. Recently, Wang et al. [16] proposed the (G'/G)-expansion method and showed that it is powerful for finding analytic solutions of PDEs. Next, Bekir [17] applied the method to some nonlinear evolution equations gaining traveling wave solutions. Later, Zhang et al. [18] further extended the method to solve an evolution equation with variable coefficients. In this paper we apply the (G'/G)-expansion method to the Kuramoto-Sivashinsky (KS) and The Benjamin-Bona-Mahony (BBM) equations.

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The KS equation describes the fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium [19]. This equation was examined as a prototypical example of spatiotemporal chaos in one space dimension [20]. Moreover, this equation was originally derived in the context of plasma instabilities, flame front propagation, and phase turbulence in reaction diffusion system [20]. The BBM equation, was first proposed in 1972 by Benjamin et al [21]. This equation is an alternative to the Kortewegde Vries (KdV) equation, and describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. The BBM equation is not only convenient for shallow water waves but also for hydromagnetic and acoustic waves, and therefore it has some advantages compared to the KdV equation.

2. DESCRIPTION OF THE (G'/G)-EXPANSION METHOD

We suppose that the given nonlinear partial differential equation for $u(x, t)$ is in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{1}$$

Where P is a polynomial in its arguments. The essence of the (G'/G)-expansion method can be presented in the following steps:

Step 1: Seek traveling wave solutions of Eq. (1) by taking $u(x,t) = U(\xi), \xi = x - ct$, and transform Eq. (1) to the ordinary differential equation

$$Q(U, U', U'', ...) = 0, \tag{2}$$

Where prime denotes the derivative with respect to ξ.

Step 2: If possible, integrate Eq. (2) term by term one or more times. If possible this yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3: Introduce the solution $U(\xi)$ of Eq. (2) in the finite series form

$$U(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{3}$$

Where a_i are real constants with $a_N \neq 0$ to be determined, N is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4}$$

Where λ and μ are real constants to be determined.

Step 4: Determine N. This, usually, can be accomplished by balancing the linear term (s) of highest order with the highest order nonlinear term(s) in Eq. (2).

Step 5: Substituting (3) and (4) into Eq. (2) yields an algebraic equation involving powers of (G'/G). Equating the coefficients of each power of (G'/G) to zero gives a system of algebraic equations for a_i, λ, μ and c. Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminate $\Delta = \lambda^2 - 4\mu$ the solutions of Eq. (4) are well known to us. Then, as a final step, we can obtain exact solutions of the given Eq. (1).

3. APPLICATIONS

In this section, we will apply the (G'/G)-expansion method on two of the well-known non-linear evolution equations, namely, the (KS) and (BBM) equations.

3.1. The Kuramoto-Sivashinsky Equation

We consider the KS equation

$$u_t + auu_x + bu_{xx} + ku_{xxxx} = 0, \tag{5}$$

where a, b and k are nonzero real constants. Make the transformation $u(x,t) = U(\xi), \xi = x - ct$, where c is constant. Integrating once with respect to ξ, Eq. (5) becomes

$$-cU + \frac{a}{2}U^2 + bU' + kU''' = 0, \tag{6}$$

where prime denote the derivative with respect to ξ. Now, we make an ansatz (3) for the solution of Eq. (6). Balancing the highest order derivative term U''' with the highest power nonlinear term U^2, yields the leading order $N = 3$. Therefore, we can write the solution of Eq. (6) in the form

$$U(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2 + a_3(\frac{G'}{G})^3, \ a_3 \neq 0 \tag{7}$$

By using (4) and (7) we derive that

$$U^2(\xi) = a_3^2(\frac{G'}{G})^6 + 2a_2a_3(\frac{G'}{G})^5 + (2a_1a_3 + a_2^2)(\frac{G'}{G})^4 + (2a_3a_0 + 2a_1a_2)(\frac{G'}{G})^3$$
$$+(a_1^2 + 2a_2a_0)(\frac{G'}{G})^2 + 2a_1a_0(\frac{G'}{G}) + a_0^2, \tag{8}$$

$$U'(\xi) = -3a_3(\frac{G'}{G})^4 - (2a_2 + 3a_3\lambda)(\frac{G'}{G})^3 - (a_1 + 2a_2\lambda + 3a_3\mu)(\frac{G'}{G})^2$$
$$-(a_1\lambda + 2a_2\mu)(\frac{G'}{G}) - a_1\mu, \tag{9}$$

$$U'''(\xi) = -60a_3(\frac{G'}{G})^6 - (24a_2 + 144a_3\lambda)(\frac{G'}{G})^5 + \ldots$$
$$-(a_1\lambda^2\mu + 6a_2\mu^2\lambda + 2_{a_1}\mu_2 + 6a_3\mu^3), \tag{10}$$

Substituting (7)-(10) into (6), setting coefficients of $(\frac{G'}{G})^i (i = 0,1,\ldots,6)$ to zero, we obtain the following under-determined system of algebraic equations for $a_0, a_1, a_2, a_3, c, \lambda$ and μ:

$$(\frac{G'}{G})^0 : -ca_0 + \frac{1}{2}aa_0^2 - ka_1\lambda^2\mu - 6ka_2\mu^2\lambda - ba_1\mu - 2ka_1\mu^2 - 6ka_3\mu^3 = 0,$$

$$(\frac{G'}{G})^1 : -ca_1 + aa_1a_0 - ba_1\lambda - 2ba_2\mu - ka_1\lambda^3 - 16ka_2\mu^2 - 8ka_1\lambda\mu - 14ka_2\lambda^2\mu - 36ka_3\lambda\mu^2 = 0,$$

$$(\frac{G'}{G})^2 : -ca_2 + \frac{1}{2}aa_1^{\,2} - ba_1 + aa_2a_0 - 2ba_2\lambda - 3ba_3\mu - 7ka_1\lambda^2 - 8ka_1\mu - 8ka_2\lambda^3$$
$$-52ka_2\lambda\mu - 60ka_3\mu^2 - 57ka_3\lambda^2\mu = 0,$$
$$(\frac{G'}{G})^3 : -ca_3 + 2ba_2 + aa_3a_0 + aa_1a_2 - 3ba_3\lambda - 38ka_2\lambda^2 - 40ka_2\mu - 27ka_3\lambda^3 - 12ka_1\lambda$$
$$-168ka_3\lambda\mu = 0,$$
$$(\frac{G'}{G})^4 : \frac{1}{2}aa_2^{\,2} - 3ba_3 - 6ka_1 + aa_1a_3 - 111ka_3\lambda^2 - 114ka_3\mu - 54ka_2\lambda = 0,$$
$$(\frac{G'}{G})^5 : -24ka_2 + aa_2a_3 - 114ka_3\lambda = 0,$$
$$(\frac{G'}{G})^6 : \frac{1}{2}aa_3^{\,2} - 60ka_3 = 0,$$

Solving this system by Maple, gives
first solution set:

$$a_0 = \pm\frac{30b}{19a}\sqrt{\frac{-b}{19k}}, \quad a_1 = \frac{90b}{19a}, \quad a_2 = 0, \quad a_3 = \frac{120k}{a}$$
$$c = \pm\frac{30b}{19}\sqrt{\frac{-b}{19k}}, \quad \lambda = 0, \quad \mu = \frac{b}{76k}, \quad \frac{b}{k} < 0, \tag{11}$$

second solution set:

$$a_0 = \pm\frac{30b}{19a}\sqrt{\frac{11}{19k}}, \quad a_1 = \frac{-270b}{19a}, \quad a_2 = 0, \quad a_3 = \frac{120k}{a},$$
$$c = \pm\frac{30b}{19}\sqrt{\frac{11b}{19k}}, \quad \lambda = 0, \quad \mu = \frac{-11b}{76k}, \quad \frac{b}{k} > 0, \tag{12}$$

where λ and μ are arbitrary constants. Substituting Eq. (11) and Eq. (12) into Eq. (7) yields

$$u(\xi) = \pm\frac{30b}{19a}\sqrt{\frac{-b}{19k}} + \frac{90b}{19a}(\frac{G'}{G}) + \frac{120k}{a}(\frac{G'}{G})^3, \tag{13}$$

where $\xi = x - (\pm\frac{30b}{19}\sqrt{\frac{-b}{19k}})t$, and

$$u(\xi) = \pm\frac{30b}{19a}\sqrt{\frac{11b}{19k}} - \frac{270b}{19a}(\frac{G'}{G}) + \frac{120k}{a}(\frac{G'}{G})^3, \tag{14}$$

where $\xi = x - (\pm\frac{30b}{19}\sqrt{\frac{11b}{19k}})t$.

Substituting general solutions of Eq. (4) into (13) and (14) we have three types of traveling wave solutions of the KS equation as follows:

When $\lambda^2-4\mu>0$,

$$u_1(\xi) = \pm\frac{30b}{19a}\sqrt{\frac{-b}{19k}}-\frac{45b\lambda}{19a}-\frac{15k\lambda^3}{a}$$
$$+(\frac{45b}{19a}+\frac{45k\lambda^2}{a})\sqrt{\lambda^2-4\mu}(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})$$
$$+(\frac{180k\lambda\mu}{a}-\frac{45k\lambda^3}{a})(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$
$$+(\frac{15k\lambda^2}{a}-\frac{60k\mu}{a})\sqrt{\lambda^2-4\mu}(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^3, \quad (15)$$

where $\xi=x-(\pm\frac{30b}{19}\sqrt{\frac{-b}{19k}})t$.

$$u_2(\xi) = \pm\frac{30b}{19a}\sqrt{\frac{11b}{19k}}+\frac{135b\lambda}{19a}-\frac{15k\lambda^3}{a}$$
$$+(\frac{45k\lambda^2}{a}-\frac{135b}{19a})\sqrt{\lambda^2-4\mu}(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})$$
$$+(\frac{180k\lambda\mu}{a}-\frac{45k\lambda^3}{a})(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$
$$+(\frac{15k\lambda^2}{a}-\frac{60k\mu}{a})\sqrt{\lambda^2-4\mu}(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^3, \quad (16)$$

where, $\xi=x-(\pm\frac{30b}{19}\sqrt{\frac{11b}{19k}})t$.

When $\lambda^2-4\mu<0$,

$$u_3(\xi) = \pm\frac{30b}{19a}\sqrt{\frac{-b}{19k}}-\frac{45b\lambda}{19a}-\frac{15k\lambda^3}{a}$$
$$+(\frac{45b}{19a}+\frac{45k\lambda^2}{a})\sqrt{4\mu-\lambda^2}(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})$$
$$+(\frac{180k\lambda\mu}{a}-\frac{45k\lambda^3}{a})(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$
$$+(\frac{45k\lambda^2}{a}-\frac{60k\mu}{a})\sqrt{4\mu-\lambda^2}(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^3, \quad (17)$$

where $\xi = x - (\pm \frac{30b}{19}\sqrt{\frac{-b}{19k}})t$.

$$u_4(\xi) = \pm \frac{30b}{19a}\sqrt{\frac{11b}{19k}} + \frac{45b\lambda}{a} - \frac{15k\lambda^3}{a}$$
$$+(\frac{45k\lambda^2}{a} - \frac{135b}{19a})\sqrt{4\mu-\lambda^2}\,(\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})$$
$$+(\frac{180k\lambda\mu}{a} - \frac{45k\lambda^3}{a})(\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$
$$+(\frac{15k\lambda^2}{a} - \frac{60k\mu}{a})\sqrt{4\mu-\lambda^2}\,(\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^3, \quad (18)$$

where $\xi = x - (\pm \frac{30b}{19}\sqrt{\frac{11b}{19k}})t$.

When $\lambda^2 - 4\mu = 0$,

$$u_5(\xi) = -\frac{15}{361}\frac{K(\xi)}{a(c_1+c_2\xi)^3}, \quad (19)$$

Where

$$K(\xi) = \mp 2b\sqrt{\frac{-19b}{k}}c_1 \mp 6b\sqrt{\frac{19b}{k}}c_1{}^2c_2\xi \mp 6b\sqrt{\frac{-19b}{k}}c_1c_2{}^2\xi^2 \mp 2b\sqrt{\frac{-19b}{k}}c_2{}^3\xi^3 + 57b\lambda c_1{}^3$$
$$+171b\lambda c_1{}^2c_2\xi + 171b\lambda c_1c_2{}^2\xi^2 - 114bc_2c_1{}^2 - 228bc_2{}^2c_1\xi - 114bc_2{}^3\xi^2 + 57bc_2{}^3\lambda\xi^3$$
$$+361k\lambda^3c_1{}^3 - 2166k\lambda^2c_1{}^2c_2 + 1083k\lambda^3c_1{}^2c_2\xi + 4332k\lambda c_1c_2{}^2 - 4332k\lambda^2c_1c_2{}^2\xi$$
$$+1083k\lambda^3c_1c_2{}^2\xi^2 - 2888kc_2{}^3 + 4332kc_2{}^3\lambda\xi - 2166kc_2{}^3\lambda^2\xi^2 + 361kc_2{}^3\lambda^3\xi^3,$$

where $\xi = x - (\pm \frac{30b}{19}\sqrt{\frac{-b}{19k}})t$. And

$$u_6(\xi) = -\frac{15}{361}\frac{L(\xi)}{a(c_1+c_2\xi)^3}, \quad (20)$$

Where

$$L(\xi) = \mp 2b\sqrt{\frac{209b}{k}}c_1{}^3 \mp 6b\sqrt{\frac{209b}{k}}c_1{}^2c_2\xi \mp 6b\sqrt{\frac{209b}{k}}c_1c_2{}^2\xi^2 \mp 2b\sqrt{\frac{209b}{k}}c_2{}^3\xi^3 - 171b\lambda c_1{}^3$$
$$-513b\lambda c_1{}^2c_2\xi - 513b\lambda c_1c_2{}^2\xi^2 + 342bc_2c_1{}^2 + 684bc_2{}^2c_1\xi + 342bc_2{}^3\xi^2 - 171bc_2{}^3\lambda\xi^3$$
$$+361k\lambda^3c_1{}^3 - 2166k\lambda^2c_1{}^2c_2 + 1083k\lambda^3c_1{}^2c_2\xi + 4332k\lambda c_1c_2{}^2 - 4332k\lambda^2c_1c_2{}^2\xi$$
$$+1083k\lambda^3c_1c_2{}^2\xi^2 - 2888kc_2{}^3 + 4332kc_2{}^3\lambda\xi - 2166kc_2{}^3\lambda^2\xi^2 + 361kc_2{}^3\lambda^3\xi^3,$$

Where $\xi = x - (\pm\frac{30b}{19}\sqrt{\frac{11b}{19k}})t$, and c_1 and c_2 are arbitrary constants.

Comparing our results with Wazzan [15], Peng [22] and Wazwaz's results [23] show that our results are more general.

In particular, if, $c_1 \neq 0$, $c_2 = 0$, $\lambda = 0$, $\mu < 0$ then u_1 becomes

$$u_1(\xi) = \pm\frac{30b}{19a}\sqrt{\frac{-b}{19k}} + \frac{45b}{19a}\sqrt{\frac{-b}{19k}}\tanh(\frac{1}{2}\sqrt{\frac{-b}{19k}}\xi) - \frac{15b}{19a}\sqrt{\frac{-b}{19k}}\tanh^3(\frac{1}{2}\sqrt{\frac{-b}{19k}}\xi), \tag{21}$$

where $\xi = x - (\pm\frac{30b}{19}\sqrt{\frac{-b}{19k}}t)$, and u_2 becomes

$$u_2(\xi) = \pm\frac{30b}{19a}\sqrt{\frac{11b}{19k}} - \frac{135b}{19a}\sqrt{\frac{11b}{19k}}\tanh(\frac{1}{2}\sqrt{\frac{11b}{19k}}\xi) + \frac{165b}{19a}\sqrt{\frac{11b}{19k}}\tanh^3(\frac{1}{2}\sqrt{\frac{11b}{19k}}\xi), \tag{22}$$

where $\xi = x - (\pm\frac{30b}{19}\sqrt{\frac{11b}{19k}})t$,

Which are the solutions of the KS equation. The solutions (21) and (22) are the same as Eq. (16) and Eq. (19) respectively [23]. Therefore the solutions in [23] are only a special case of the our solutions.

3.2. The Benjamin-Bona-Mahony Equation (BBM)

We consider the BBM equation

$$u_t + au_x + uu_x + bu_{xxt} = 0, \tag{23}$$

Where a and b are nonzero real constants. Making the transformation $u(x,t) = U(\xi)$, $\xi = x - ct$ and integrating once with respect to ξ, Eq. (23) becomes

$$(a-c)U + \frac{1}{2}U^2 - bcU'' = 0, \tag{24}$$

Where prime denote the derivative with respect to ξ. Balancing the highest derivative term U'' with the highest power nonlinear term U^2 gives the leading order $N = 2$. Therefore, we can write the solution of Eq. (24) in the form

$$U(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2, \; a_2 \neq 0. \tag{25}$$

By using (4) and substituting (25) into Eq. (24) and setting the coefficients of $(\frac{G'}{G})^i \, (i = 0,1,\ldots,6)$ to zero, we obtain the following under-determined system of algebraic equations for $a_0, a_1, a_2, c, \lambda$ and μ:

$$(\frac{G'}{G})^0 : aa_0 - ca_0 + \frac{1}{2}a_0^2 - bca_1\lambda\mu - 2bca_2\mu^2 = 0,$$

$$(\frac{G'}{G})^1 : aa_1 - ca_1 + a_1a_0 - bca_1\lambda^2 - 2bca_1\mu - 6bca_2\lambda\mu = 0,$$

$$(\frac{G'}{G})^2 : aa_2 - ca_2 + a_2a_0 + \frac{1}{2}a_1^2 - 3bca_1\lambda - 4bca_2\lambda^2 - 8bca_2\mu = 0,$$

$$(\frac{G'}{G})^3 : a_1a_2 - 2bca_1 - 10bca_2\lambda = 0,$$

$$(\frac{G'}{G})^4 : \frac{1}{2}a_2^2 - 6bca_2 = 0.$$

Solving this system by Maple, gives first solution set:

$$a_0 = \frac{12ba\mu}{1+b(\lambda^2-4\mu)},\quad a_1 = \frac{12ba\lambda}{1+b(\lambda^2-4\mu)},\quad a_2 = \frac{12ba}{1+b(\lambda^2-4\mu)},\quad c = \frac{a}{1+b(\lambda^2-4\mu)}, \quad (26)$$

Second solution set:

$$a_0 = \frac{-2ba(\lambda^2+2\mu)}{-1+b(\lambda^2-4\mu)},\ a_1 = \frac{-12ba\lambda}{-1+b(\lambda^2-4\mu)},\ a_2 = \frac{-12ba}{-1+b(\lambda^2-4\mu)},\ c = \frac{-a}{-1+b(\lambda^2-4\mu)}, \quad (27)$$

Where λ and μ are arbitrary constants. Substituting Eq. (26) and Eq. (27) into Eq. (25) yields

$$u(\xi) = \frac{12ba\mu}{1+b(\lambda^2-4\mu)} + \frac{12ba\lambda}{1+b(\lambda^2-4\mu)}(\frac{G'}{G}) + \frac{12ba}{1+b(\lambda^2-4\mu)}(\frac{G'}{G})^2, \quad (28)$$

where $\xi = x - \frac{a}{1+b(\lambda^2-4\mu)}t$. And

$$u(\xi) = -\frac{2ba(\lambda^2+2\mu)}{-1+b(\lambda^2-4\mu)} - \frac{12ba\lambda}{-1+b(\lambda^2-4\mu)}(\frac{G'}{G}) - \frac{12ba}{-1+b(\lambda^2-4\mu)}(\frac{G'}{G})^2, \quad (29)$$

where $\xi = x + \frac{a}{-1+b(\lambda^2-4\mu)}t$.

Substituting general solutions of Eq. (4) into (28) and (29) we have three types of traveling wave solutions of the BBM equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{3ba(\lambda^2-4\mu)}{1+b(\lambda^2-4\mu)} + \frac{3ba(\lambda^2-4\mu)}{1+b(\lambda^2-4\mu)}(\frac{c_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2, \quad (30)$$

where $\xi = x - (\frac{a}{1+b(\lambda^2-4\mu)})t.$

$$u_2(\xi) = \frac{ba(\lambda^2-4\mu)}{-1+b(\lambda^2-4\mu)} - \frac{3ba(\lambda^2-4\mu)}{-1+b(\lambda^2-4\mu)}(\frac{c_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2, \quad (31)$$

where $\xi = x + (\frac{a}{-1+b(\lambda^2-4\mu)})t.$

When $\lambda^2 - 4\mu < 0$,

$$u_3(\xi) = -\frac{3ba(4\mu-\lambda^2)}{1+b(4\mu-\lambda^2)} + \frac{3ba(4\mu-\lambda^2)}{1+b(4\mu-\lambda^2)}\left(\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2, \quad (32)$$

where $\xi = x - (\frac{a}{1+b(\lambda^2-4\mu)})t$.

$$u_4(\xi) = -\frac{ba(4\mu-\lambda^2)}{-1+b(4\mu-\lambda^2)} - \frac{3ba(4\mu-\lambda^2)}{-1+b(4\mu-\lambda^2)}\left(\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2, \quad (33)$$

where $\xi = x + (\frac{a}{-1+b(\lambda^2-4\mu)})t$.

When $\lambda^2 - 4\mu = 0$,

$$u_5(\xi) = \frac{12bac_2^2}{(c_1+c_2\xi)^2}, \quad (34)$$

where $\xi = x - (\frac{a}{1+b(\lambda^2-4\mu)})t$. And

$$u_6(\xi) = \frac{12bac_2^2}{(c_1+c_2\xi)^2}, \quad (35)$$

where $\xi = x + (\frac{a}{-1+b(\lambda^2-4\mu)})t$ and c_1 and c_2 are arbitrary constants.

In particular, if $c_1 \neq 0$, $c_2 = 0$, $\lambda > 0$, $\mu = 0$, then u_1 becomes

$$u_1(\xi) = \frac{-3ba\lambda^2}{1+b\lambda^2} + \frac{3ba\lambda^2}{1+b\lambda^2}(\tanh^2\frac{\lambda\xi}{2}), \quad (36)$$

and u_2 becomes

$$u_2(\xi) = \frac{ba\lambda^2}{-1+b\lambda^2} - \frac{3ba\lambda^2}{-1+b\lambda^2}(\tanh^2\frac{\lambda\xi}{2}). \quad (37)$$

Which are the solitary wave solutions of the BBM equation.

Comparing our results with Bekir's results [14] and results of [24] show that our results are more general. For instance for $a=1$, $b=-1$, $c=\frac{-1}{3}$ and $\lambda = 2$ the solution (36) is the same as in ([24], p. 71) and for $a=1$, $b=-1$, $c=\frac{1}{5}$ and $\lambda = 2$ the solution (37) is the same as in ([24], p. 71).

4. CONCLUSIONS

In this paper, the use of the (G'/G)-expansion method is intyoduced by applying it to two nonlinear equations to illustrate the validity and advantages of the method. The exact traveling wave solutions being determined in this study are more general, and it is not difficult to arrive at some known analytic solutions for certain choices of the parameters c_1 and c_2. Compared with the methods used in [14-15, 22-24], one can see that the (G'/G) -expansion method is not only simple and straightforward, but also avoids tedious calculations. This verifies that the method can be used for many other nonlinear evolution equations.

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