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FOURTH GRADE FLUID FLOW IN FLAT POROUS CHANNEL

M. Ebrahimpour, M.Gorji & G. Domairry*
Department of Mechanical Engineering, Babol University of Technology, Babol, Iran

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Abstract: *In this paper steady flow of the fourth grade fluid inside porous channel with monotonous injection and suction rates has been analytically investigated with two flat porous plates at $y = b$ and $y = b$ intervals through HAM method, and the effect of variations in each parameter on velocity profile has been shown. The results indicate that velocity profile will decrease if R, K, β or γ increases.*

Keywords: *Homotopy Analysis Method (HAM); Fourth Grade Fluid; Porous Channel*

1. INTRODUCTION

Nowadays in different industries such as chemical industries, food industries, etc, application of non-Newtonian fluids is more than Newtonian fluids, since the behavior of these fluids is more complicated in comparison with Newtonian fluids. Thus numerous models have been presented to predict their behavior. One of the most important non-Newtonian fluids is differential type fluids which has been taken into consideration by numerous scientists such as Rajagopal [1] and Siddigui et al. [2] so that several articles have been presented about flow of these fluids in various geometrical situations by these scientists.

The simplest kind of non-Newtonian fluids, differential type of fluid, is second grade fluid. Numerous researches have been done on flow of this fluid inside porous channel, convergent or divergent channel, on porous plate. Another kind of differential fluids which shows a more complete behavior is the third grade fluid. This fluid has been analyzed inside the porous channel by Berman [3], Hayat et al. [4], and on the porous plate by Ayub et al. [5]. But, the most complete differential fluid is the fourth grade fluid whose flow on flat plate has been analyzed by Sajid [6] et al, but no research has been reported about flow of this fluid inside the porous channel. Flow of a third grade fluid through a porous flat channel has been investigated by Donald Ariel [7, 8] he has also studied flow of a second grade fluid in porous channel [9]. fourth grade fluid has investigated by V. Marinca et al [10] and T.Hayat et al [11].

In this paper, steady flow of the fourth grade fluid, inside the porous channel consisting of two parallel plates at $y = b$ and $y = -b$ intervals, has been studied. It is considered that the flow

* Corresponding Author: *amirganga111@yahoo.com*

is under steady pressure gradient and the fluid enters injection channels with the speed that it is sucked through the other wall. Equation for the flow of this fluid inside the porous channel is of fifth grade, while the number of boundary conditions proportionate to no slip condition is only two. The method to solve this equation is HAM method which was presented by Liao [12].

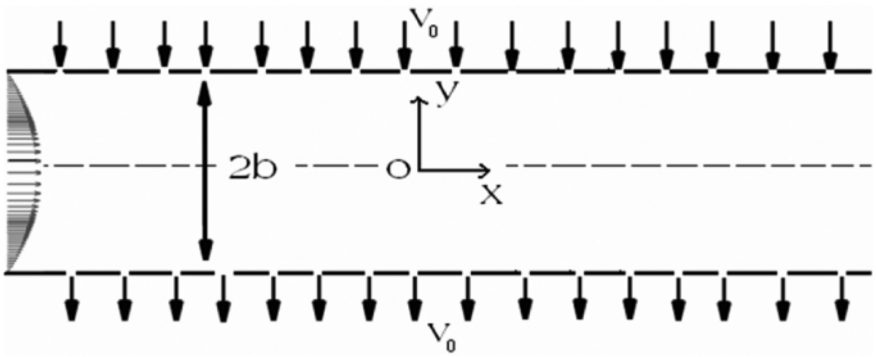

Figure.1: The Physical Geometry of the Problem Under Discussion

2. CONSTITUTIVE RELATIONS

Differential type fluids are one of the most important subclass of non-Newtonian fluids. We consider the fourth grade fluid as a subclass of differential type fluids.

The constitutive law for this fluid is given by:

$$\begin{aligned}\tau = & -pI + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2 (A_2 A_1 + A_1 A_2) + \beta_3 (tr\, A^2{}_1) A_1 \\ & + \gamma_1 A_4 + \gamma_2 (A_3 A_1 + A_1 A_3) + \gamma_3 A^2{}_2 + \gamma_4 (A_2 A^2{}_1 + A^2{}_1 A_2) \\ & + \gamma_5 (tr\, A_2) A_2 + \gamma_6 (tr\, A_2) A^2{}_1 + (\gamma_7\, tr\, A_3 + \gamma_8\, tr(A_2 A_1)) A_1\end{aligned} \tag{1}$$

Where T is the Cauchy stress tensor, P is pressure, μ is the viscosity and α_1, α_2, $\beta_i(i = 1 - 3)$, $\gamma_i(i = 1 - 8)$ are material constant, A_i $(i = 1 - 4)$ are Rivlin- Erickson tensor which are defined as:

$$A_1 = L + L^T \tag{2}$$

$$A_2 = \frac{D}{Dt} A_1 + L^T \cdot A_1 + A_1 \cdot L \tag{3}$$

$$A_n = \frac{D}{Dt} A_{n-1} + L^T\ A_{n-1} + A_{n-1}\ L \qquad (n \geq 2) \tag{4}$$

$$L = V(L_{IJ} = v_{J,I}) \tag{5}$$

where V is a velocity field, ∇ is the gradient operator, $\frac{D}{Dt}$ is defined as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (V.\nabla) \tag{6}$$

Where $\frac{\partial}{\partial t}$ is the derivation with respect to time.

3. MATHEMATICAL FORMULATION

Consider the steady flow of fourth grade fluid in a flat porous channel which is made of two porous flat plates. The coordinate axis is placed in the middle of channel and two plates are assumed to be at $y = b$ and $y = -b$. There is a cross flow which is injected to the channel through upper wall and is sucked off through the lower wall with the equal velocity.

The flow is assumed to be under the constant pressure gradient.

Let $V = (u(x, y), v(x, y))$ be the velocity field and the governing equations are as follows:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z} \quad (7)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z} \quad (8)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad (9)$$

Where $\tau_{xx}, \tau_{xy}, \tau_{xz}, \tau_{yy}, \tau_{yz}, \tau_{zz}$ are the components of the stress tensor, which will be defined as follows after substituting Eqs. (2) – (5) into Eq. (1):

$$\tau_{xx} = -p+\alpha_2\left(\frac{du}{dy}\right)^2+2\beta_2 V_0\frac{du}{dy}\frac{d^2u}{dy^2}+2\gamma_2 V_0^2\frac{du}{dy}\frac{d^3u}{dy^{32}}+\gamma_3 V_0^2\left(\frac{d^2u}{dy^2}\right)^2+2\gamma_6\left(\frac{du}{dy}\right)^4 \quad (10)$$

$$\tau_{xy} = \mu\frac{du}{dy}+\alpha_1 V_0\frac{d^2u}{dy^2}+\beta_1 V_0^2\frac{d^3u}{dy^3}+2(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^3+\gamma_1 V_0^3\frac{d^4u}{dy^4}$$
$$+(6\gamma_6+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)V_0\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} \quad (11)$$

$$\tau_{yy} = -p+(2\alpha_1+\alpha_2)\left(\frac{du}{dy}\right)^2+2V_0(3\beta_1+\beta_2)\frac{du}{dy}\frac{d^2u}{dy^2}+6\gamma_1 V_0^2\frac{d}{dy}\left(\frac{du}{dy}\frac{d^2u}{dy^2}\right)$$
$$+2(\gamma_1+\gamma_2)V_0\frac{d^3u}{dy^3}\frac{du}{dy}+\gamma_3 V_0^2\left(\frac{d^2u}{dy^2}\right)^2+2(2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6)\left(\frac{du}{dy}\right)^4 \quad (12)$$

$$\tau_{zz} = -p \quad (13)$$

$$\tau_{xz} = \tau_{zy} = 0 \quad (14)$$

$$\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy} \quad (15)$$

Note: because of the same injection and suction velocity we have:

$$v = V_0 \quad (16)$$

where V_0 is the velocity of cross flow.

By substituting Eqs. (10) – (12) and Eq. (15) into Eq. (7), we obtain:

$$\rho V_0 \frac{du}{dy} = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} + \alpha_1 V_0 \frac{d^3 u}{dy^3} + \beta_1 V_0^2 \frac{d^4 u}{dy^4} + 6(\beta_2 + \beta_3)\left(\frac{d^2 u}{dy^2}\right)\left(\frac{du}{dy}\right)^2$$
$$+ \gamma_1 V_0^3 \frac{d^5 u}{dy^5} + (6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8) V_0 \frac{d}{dy}\left(\frac{du}{dy}\frac{d^2 u}{dy^2}\right) \quad (17)$$

The boundary condition subjected to the equation 16 is:

$$u(b) = u(-b) = 0 \quad (18)$$

Now we can use the following non dimensional variables:

$$U = -\frac{\mu u}{b^2}\left(\frac{\partial p}{\partial x}\right)^{-1}, \ \eta = \frac{y}{b} \quad (19)$$

Substituting Eq. (19) in to Eq. (17) gives us:

$$\frac{d^2 U}{d\eta^2} - R\frac{dU}{d\eta} + RK\frac{d^3 U}{d\eta^3} + \beta_1 \frac{d^4 U}{d\eta^4} + \beta\left(\frac{dU}{d\eta}\right)^2 \frac{d^2 U}{d\eta^2} + \gamma_1 \frac{d^5 U}{d\eta^5}$$
$$+ \gamma\left(2\frac{dU}{d\eta}\frac{d^2 U}{d\eta^2} + \left(\frac{dU}{d\eta}\right)^2 \frac{d^3 U}{d\eta^3}\right) = -1 \quad (20)$$

And for simplicity, we can write it as the form:

$$U'' - RU' + RKU''' + \beta_1 U^{(4)} + \beta (U')^2 U'' + \gamma_1 U^{(5)} + \gamma\left(2U'U''^2 + U'^2 U'''\right) = -1 \quad (21)$$

where:

$$R = \frac{\rho V_0 b}{\mu}, \ K = \frac{\alpha_1}{\rho b^2}, \ \beta = \frac{6(\beta_2 + \beta_3) b^2}{\mu^3}\left(\frac{\partial p}{\partial x}\right)^2, \ \beta_1 = \frac{\beta_1 V_0^2}{b^2 \mu}, \ \gamma_1 = \frac{\gamma_1 V_0^3}{b^3 \mu}$$
$$\gamma = \frac{2bV_0}{\mu^3}\left(\frac{\partial p}{\partial x}\right)^2 (3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \quad (22)$$

And boundary condition subjected to equation above is:

$$U(-1) = U(1) = 0 \quad (23)$$

4. APPLYING HOMOTOPY ANALYSIS METHOD

In this section we use HAM to solve equation 21 subjected to boundary condition (23).

We select initial guess in the form:

$$U_0(\eta) = 0 \quad (24)$$

And auxiliary linear part as:

$$L[U] = U_{\eta\eta}(\eta) - R\, U_{\eta}(\eta) \quad (25)$$

which has a property as:

$$L\left[C_1 + C_2 e^{R\eta}\right] = 0 \tag{26}$$

where $C_i(i = 1, 2)$ are constants. $p \in [0, 1]$ denotes the embedding parameter and $\hbar$ indicates non-zero auxiliary parameters. Then we construct the following equation.

Zeroth-order deformation equation:

$$(1-P)L_1\left[U(\eta;p) - U_0(\eta)\right] = p\hbar N_1\left[U(\eta;p)\right] \tag{27}$$

$$U(-1,p) = 0,\ U(1,\ p) = 0 \tag{28}$$

$$\begin{aligned} N\left[\phi(\tau;p)\right] =& U_{\eta\eta}(\eta,p) - RU_{\eta}(\eta,p) + RKU_{\eta\eta\eta}(\eta,p) + \beta_1 U_{\eta\eta\eta\eta}(\eta,p) \\ &+ \beta\,(U_{\eta}(\eta,p))^2\, U_{\eta\eta}(\eta,p) + \gamma_1 U_{\eta\eta\eta\eta\eta}(\eta,p) \\ &+ \gamma\Big(2U_{\eta}(\eta,p)U_{\eta\eta}(\eta,p) + \big(U_{\eta}(\eta,p)\big)^2 U_{\eta\eta\eta}(\eta,p)\Big) + 1 \end{aligned} \tag{29}$$

For $p = 0, 1$ we have:

$$U(\eta,0) = U_0(\eta)\ ,\ U(\eta,1) = U(\eta) \tag{30}$$

When p increase from 0 to 1 then $U(\eta, p)$ vary from $U_0(\eta)$ to $U(\eta)$. By Taylor's theorem and using Eq. (30) we can write:

$$U(\eta;p) = U_0(\eta) + \sum_{m-1}^{\infty} U_m(\eta)\, p^m \quad U_m(\eta) = \frac{1}{m!}\frac{\partial^m U(\eta;p)}{\partial p^m} \tag{31}$$

Therefore we have through Eq. (31):

$$U(\eta) = U_0(\eta) + \sum_{m-1}^{\infty} U_m(\eta) \tag{32}$$

mth-order deformation equation:

$$L\left[U_m(\eta) - \chi_m U_{m-1}(\eta)\right] = \hbar R_m^U(\eta) \tag{33}$$

$$U_m(-1) = U_m(1) = 0 \tag{34}$$

$$\begin{aligned} R_m(\eta) =& U''_{m-1}(\eta) - RU'_{m-1}(\eta) + RKU'''_{m-1}(\eta) + \beta_1 U^{(4)}_{m-1}(\eta) + \gamma_1 U^{(5)}_{m-1}(\eta) + \\ &\sum_{k=0}^{m-1} U'_{m-1-k}(\eta) \sum_{l=0}^{k}\left[\beta\, U'_{k-l}(\eta)\ U''_l(\eta) + \gamma\left\{U'_{k-l}(\eta)U'''_l + 2U''_{k-l}U''_l\right\}\right] \end{aligned} \tag{35}$$

where:

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m \ge 1 \end{cases} \tag{36}$$

And now by m times derivation of the zeroth order deformation equation (27) with respect to p and solve them we obtain:

$$U_1(\eta) = \frac{\left(xe^{-R} - xe^{R} + 2e^{Rx} - e^{-R} - e^{R}\right)}{R\left(e^{-R} - e^{R}\right)} \tag{37}$$

Because of the large size of $U_2(\eta)$, $U_3(\eta)$, $U_4(\eta)$ we show the $U_2(\eta)$ in appendix and the others schematically as follows for the various parameters of problem.

5. CONVERGENCE OF THE HAM SOLUTION

The convergence and rate of approximation for the HAM solution strongly depends on the value of auxiliary parameter $\hbar$. By means of so-called $\hbar$-curves, it is easy to find out so-called valid regions of auxiliary parameters to gain a convergent solution series. According to figs.2-4, the convergence ranges for $U''(\eta)$ is variable for different values of parameters.

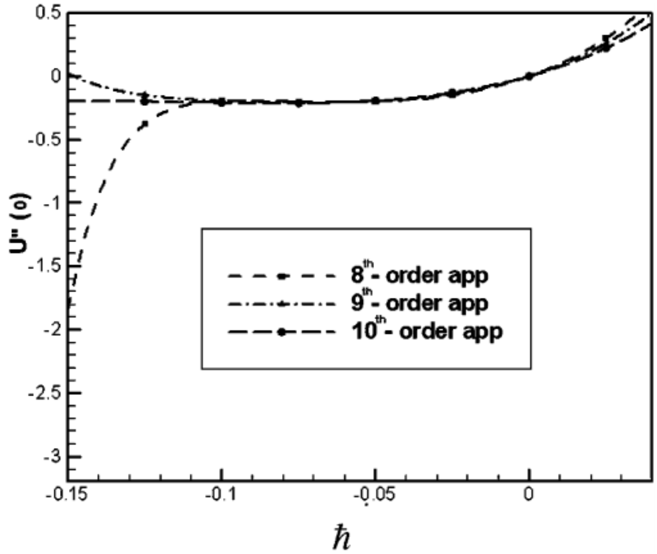

Figure. 2: The $\hbar$ Validity of for $R=3$, $\beta_1=0.2$, $\beta=5$, $\gamma_1=0.1$, $\gamma=2$, $K=0.5$ and 8th-9th-10th order Approximation

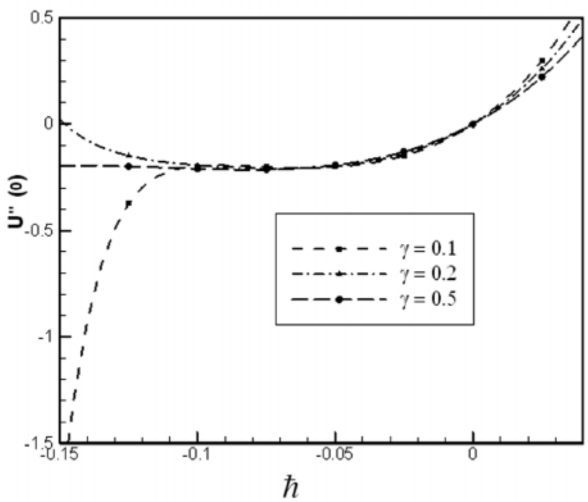

Figure. 3: The $\hbar$ Validity of for $R=3$, $\beta_1=0.2$, $\beta=5$, $\gamma_1=0.1$, $\gamma=2$, $K=0.5, \gamma=1,2,4$

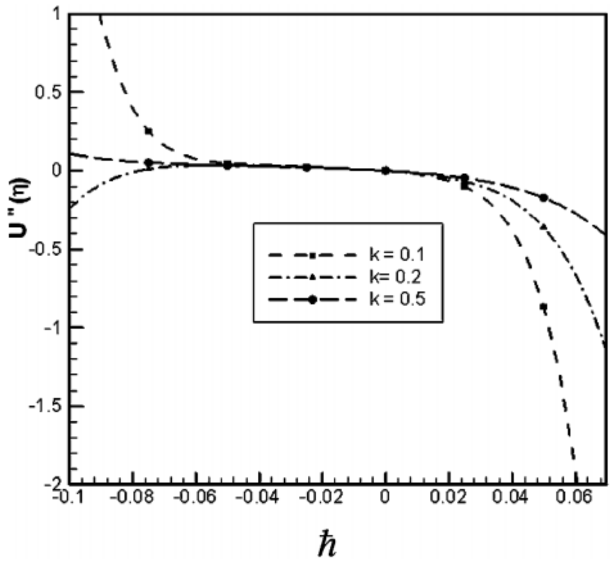

Figure. 4: The $\hbar$ validity of for $R = 3$, $\beta_1 = 0.2$, $\beta = 5$, $\gamma_1 = 0.1$, $\gamma = 2$, $K = 0.1, 0.2, 0.5$

6. RESULTS AND DISCUSSION

As it can be seen in table 1, the results of HAM solution for $\beta_1 = 0, \gamma_1 = 0, \gamma = 0$ (Third grade fluid) are in very good agreement with the those of exact solution [12]. According to table 1, among R, K, β by increasing one and while the other two remain fxed, velocity profile will decrease. The graphs of the velocity variation are plotted with respect to the η for the fixed $\beta_1 = 0.1$, $\beta = 5$, $\gamma_1 = 0.5$, $K = 0.1$ and different value of R, γ. Fig 5 shows the velocity field in $R = 1$ and various γ, as it can be seen, velocity profile decreases whereas boundary layer thickness increases. Fig 6 displays variation of $U(\eta)$ at $R = 2$ and different γ. Figs 7 and 8 shows velocity profile for various γ at $R = 3$ and $R = 5$ respectively.

Above figures are showing well that by assuming of β_1, β, γ_1, K to be constant, the velocity profile decreases by increasing R, cross flow Reynolds number and γ as a fourth grade fluid parameter. Another interesting point that should be noted is that the effect of γ variation on the velocity profile decreases by increasing R, cross flow Reynolds number.

Also the results show that HAM is a powerful mathematical tool for solving nonlinear partial differential equations and systems of nonlinear partial differential equations that have wide applications in engineering.

Table 1
The Value of $U(1/2)$ at the Center Line of Porous Channel for Various R, Cross- flow Reynold's Number K, the Visco-elastic Fluid Parameter and β Third Grade Fluid Parameter and $\beta_1 = 0, \gamma_1 = 0, \gamma = 0$

R	*K*	*β*	*U(1/2)* *REF [12]*	$\hbar$	*HAM* *solution*
1	0.1	1	0.109334	-0.329	0.109333
		2	0.106976	-0.311	0.106978

Table 1 Contd...

Table 1 Contd...

		5	0.100651	-0.308	0.100647
	0.2	1	0.102101	-0.291	0.102103
		2	0.099691	-0.289	0.099691
	0.5	1	0.087626	-0.281	0.087626
		2	0.085471	-0.278	0.085452
2	0.1	1	0.089912	-0.261	0.089925
		2	0.088692	-0.252	0.088692
		5	0.085708	-0.275	0.085729
	0.2	1	0.078179	-0.262	0.078181
		2	0.077288	-0.257	0.077291
		5	0.075064	-0.264	0.075069
	0.5	1	0.060643	-0.231	0.060646
		2	0.060100	-0.229	0.060120
		5	0.058690	-0.240	0.058681
5	0.1	1	0.052089	-0.0147	0.052089
		2	0.051974	-0.0141	0.051974
		5	0.051638	-0.0137	0.051638
		10	0.051102	-0.0139	0.051102
		20	0.050119	-0.0140	0.050119
	0.2	1	0.041948	-0.035	0.041945
		2	0.041886	-0.033	0.041829
		5	0.041703	-0.037	0.041737
		10	0.041403	-0.031	0.041449
		20	0.040852	-0.039	0.040849

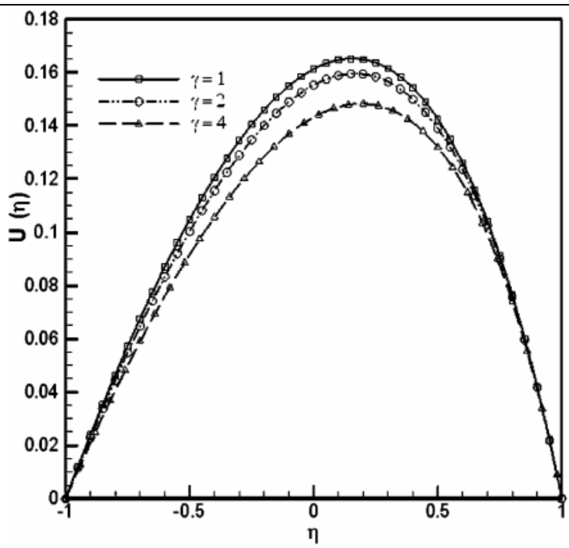

Figure. 5: The 8th-order Approximation of $U(\eta)$ for Various Value of γ at $\hbar = -0.085$ $\beta_1 = 0.1, \beta = 5$, $\gamma_1 = 0.5$, $K = 0.1$, $\gamma = 1, 2, 4$ $R = 1$

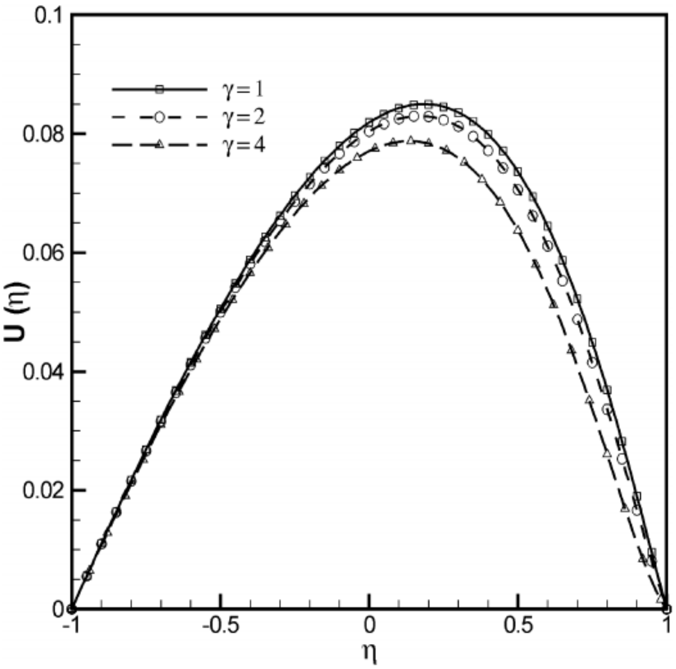

Figure. 6: The 8th-order Approximation of $U(\eta)$ for various value of γ at $\hbar = -0.085$ $\beta_1 = 0.1, \beta = 5$, $\gamma_1 = 0.5$, $K = 0.1$, $\gamma = 1,2,4$ $R = 2$

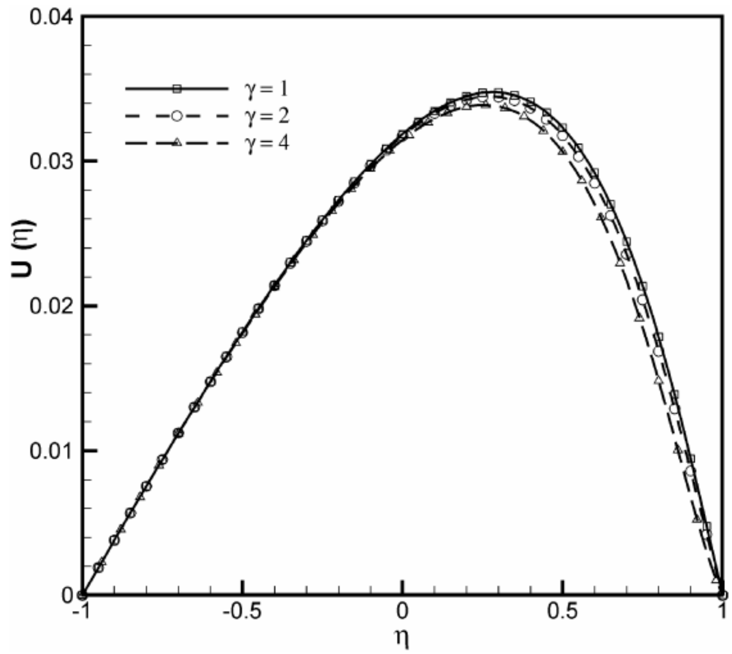

Figure. 7: The 8th-order Approximation of $U(\eta)$ for various value of γ at $\hbar = -0.085$ $\beta_1 = 0.1, \beta = 5$, $\gamma_1 = 0.5$, $K = 0.1$, $\gamma = 1,2,4$ $R = 3$

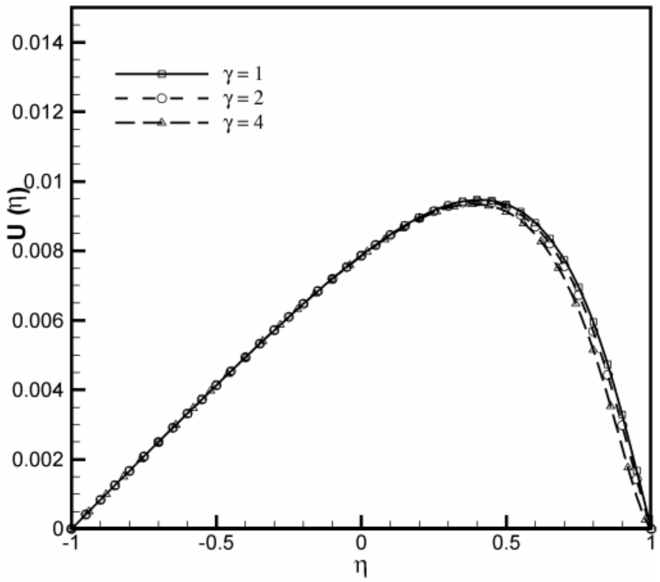

Figure. 8: The 8th-order Approximation of $U(\eta)$ for Various Value of γ at $\hbar = -0.085$ $\beta_1 = 0.1, \beta = 5$, $\gamma_1 = 0.5$, $K = 0.1$, $\gamma = 1, 2, 4$ $R = 5$

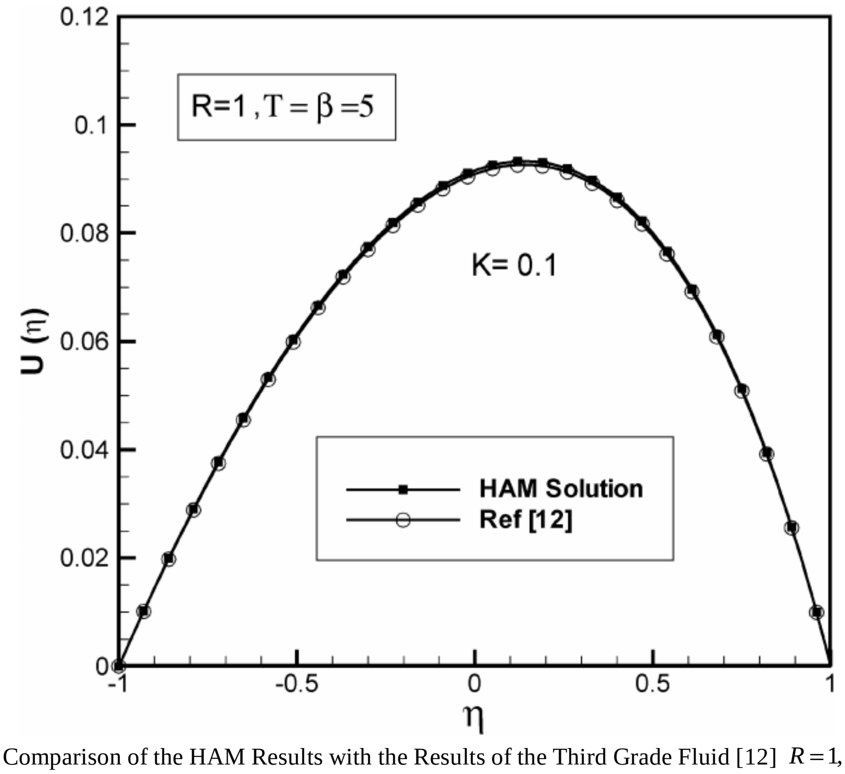

Figure. 9: Comparison of the HAM Results with the Results of the Third Grade Fluid [12] $R=1$, $T=\beta=5$, $\gamma_1 = 0$, $\beta_1 = 0$, $\gamma = 0$, $K = 0.1$

7. CONCLUSION

The effect of R, cross flow Reynolds and γ, fourth grade fluid parameter variation on fourth grade fluid flow in a porous channel have been studied. Analytical solution of the governing equation has also been obtained by using a homotopy analysis method (HAM). The particular conclusions drawn from this study can be listed as follows:

1. The velocity profile decreases and boundary layer thickness increases due to an increase in cross flow Reynolds R and fourth grade fluid parameter γ at $\beta_1 = 0.1$, $\beta = 5$, $\gamma_1 = 0.5$, $K = 0.1$.
2. The effect of γ on velocity profile decreases when R increases.
3. The value of velocity for the fourth grade fluid flow in a porous channel is less than that for the third grade fluid flow in the same condition.

8. APPENDIX

$$
\begin{aligned}
U_2(\eta) = \;& (h(6h\gamma_1 R^4 e^{R(\eta+1)}\eta + 6hR^3Ke^{R(\eta+1)}\eta \;\; 2h + 4he^{4R} \;\; 2e^{R(\eta\;\;1)} + 4\eta \\
& 8hR^3Ke^{2R} + 2e^{R(\eta+3)}h\beta_1R^3 + 2e^{R(\eta+3)}hKR^3 \;\; 2e^{R(\eta+5)}hKR^3 \\
& +2h\beta_1R^3\eta e^{R(\eta+5)} \;\; 6e^{R(\eta+3)} + 4h\eta + 2e^{R(\eta+1)}hR^3K \\
& +2e^{R(\eta+1)}hR^4\gamma_1 + 2e^{R(\eta+1)}hR^3\beta_1 \;\; 6e^{2R}\eta \;\; 6e^{R(\eta+3)}hR^3K\eta \\
& 6e^{6R} + 4hR^3K + 2hR^3Ke^{R(\eta+5)}\eta \;\; 6he^{2R}\eta \;\; 6e^{R(\eta+3)}hR^4\gamma_1\eta \\
& 2e^{R(\eta\;\;1)}hR^3K \;\; 2e^{R(\eta+5)}h\beta_1R^3 \;\; 2e^{R(\eta-1)}h\beta_1R^3 \\
& +2e^{R(\eta+3)}hR^4\gamma_1 + 4e^{4R}hR^4\gamma_1 + 4e^{4R}h\beta_1R^3 \;\; he^{6R} \;\; \eta e^{6R} \\
& 6e^{R(\eta+3)}h \;\; 2h\beta_1R^3e^{R(\eta\;\;1)}\eta + 2hR^4\gamma_1e^{R(\eta+5)}\eta + 2he^{4R} \\
& 2e^{R(\eta-1)} + 6e^{R(\eta+1)} + 4hR^4\gamma_1 + 6e^{R(\eta+1)}h\beta_1R\eta + 4\eta e^{4R} \\
& he^{6R}\eta \;\; 2hR^3Ke^{R(\eta-1)}\eta \;\; 8h\beta_1R^3e^{2R} + 2he^{R(\eta+5)} \\
& 2hR^4\gamma_1e^{R(\eta-1)} \;\; 2hR^4\gamma_1e^{R(\eta+5)} + 4h\beta_1R^3 + he^{\;\;2R} \\
& +6he^{R(\eta+1)} \;\; 8hR^4\gamma_1e^{2R} \;\; \eta e^{2R} + he^{\;\;2R} + 2e^{R(\eta+5)} \\
& 2hR^4\gamma_1e^{R(\eta-1)}\eta \;\; 2))
\end{aligned}
$$

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