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ON WAVES GENERATED DUE TO A LINE SOURCE IN FRONT OF A VERTICAL WALL WITH A GAP

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ABSTRACT: A mixed boundary value problem arising in the problem of generation of waves due to a line source in front of a vertical wall with a gap is studied here by using a dual integral equations formulation. The dual integral equations are reduced to a singular integral equation with logarithmic kernel in the interval (a, b), $a > b > 0$. Using the boundedness behaviour of the function satisfying the singular integral equation at the end points of the interval (a, b) the amplitude of waves at infinity is obtained analytically in a very simple manner.

1. INTRODUCTION

The solution of mixed boundary value problems using various mathematical techniques arising in the theory of scattering and radiation of water waves by a vertical barrier is well known in the literature.A number of researchers developed various mathematical techniques to solve this class of problems. Recently, Chakrabarti, Manam and Banerjea [1] used a very simple method to solve completely the mixed boundary value problem(BVP) arising in the theory of scattering of water waves by a vertical wall with a gap. They reduced the boundary value problem to dual integral equations which in turn are reduced to a singular integral equation with logarithmic kernel in the domain (a, b), $a > b > 0$. The forcing function of this integral equation involve two unknowns constants. The function satisfying this integral equation is bounded at both end points of the domain (a, b) and consequently, the solution of this singular integral equation exists if and only if two solvability criteria are satisfied. From these two solvability conditions the two unknown constants in the forcing function are determined and hence the solution of mixed boundary value problem is obtained. In the present paper we have studied the problem of generation of water waves due to presence of a line source in front of a vertical wall with a gap. This problem was considered earlier by Banerjea and Kar [2] who obtained the wave amplitude at infinity by an application of Green's theorem without actually solving the mixed boundary value problem. In the present paper we used the idea of Chakrabarti etal [1] to reduce the problem to a singular integral equation with logarithmic kernel in the domain (a, b). Using the solvability conditions the corresponding mixed boundary value problem is solved in closed form. The known results are recovered by considering the limiting case.

2. STATEMENT AND FORMULATION OF THE PROBLEM

The motion is described by the velocity potential $\mathrm{Re}\{\Phi(x, y) \exp(-i\sigma t)$, where Φ satisfies the following BVP:

$$\nabla^2\Phi = 0 \qquad \text{in the fluid region except at } (\xi, \eta), \tag{2.1}$$

$$K\Phi + \Phi_y = 0 \quad \text{on} \quad y = 0, \tag{2.2}$$

where $K = \frac{\sigma^2}{g}$, g being accelaration of gravity, σ being circular frequency.

$$\Phi_x = 0, \text{ for } x = 0, y \in B, \quad B = (0, a) + (b,\infty), \tag{2.3}$$

$$\Phi \sim \ln \rho \text{ as } \rho \to 0 \text{ where, } \rho = \{(x - \xi)^2 + (y - \eta)^2\}^{1/2}, \tag{2.4}$$

$$r^{1/2}\nabla\Phi \text{ is bounded as } r \to 0,\ r = \{(x)^2 + (y - c)^2\}^{1/2},\ c = a \text{ or } b, \tag{2.5}$$

$$\nabla\Phi \to 0 \text{ as } y \to \infty, \tag{2.6}$$

$$\Phi \sim \begin{cases} B_+ \exp(-Ky + iKx) & \text{as } x \to \infty \\ B_- \exp(-Ky - iKx) & \text{as } x \to -\infty \end{cases} \tag{2.7}$$

where B_+ and B_- are amplitudes of radiated waves at infinity on either side of the wall.

Let,

$$\Phi = G + \phi, \tag{2.8}$$

where , $G(x, y, \xi, \eta)$ is the velocity potential due to presence of a line source at (ξ, η) in absence of a barrier and G satisfies (2.1), (2.2), (2.4) and it behaves as outgiong wave as $|x - \xi| \to \infty$. The form of G is given by(cf[3])

$$G(x, y, \xi, \eta) = -2\int_0^\infty \frac{L(k,\eta)L(k,y)\exp(-k|\xi-\eta|}{k(k^2+K^2)}\,dk - 2\pi ie\exp\left(-K(y+\eta)+iK|x-\xi|\right) \tag{2.9}$$

where,

$$L(k, y) = k\cos ky - K\sin ky.$$

Thus ϕ is the correction to G and it satisfies

$$\nabla^2\phi = 0, \qquad y > 0. \tag{2.10}$$

$$K\phi + \phi_y = 0, \qquad \text{on } y = 0. \tag{2.11}$$

$$\phi_x(0, y) = f(y) = -G_x(o, y, \xi, \eta), \qquad x = 0, y \in B. \tag{2.12}$$

$$\phi(0^-, y) = \phi(0^+, y), \qquad y \in (a, b). \tag{2.13}$$

$$\nabla\phi \to 0 \qquad y \to \infty. \tag{2.14}$$

$$r^{1/2}\nabla\phi \quad \text{is bounded as} \quad r \to 0. \tag{2.15}$$

$$\phi(x,y) \sim \begin{cases} C_1 \ \ \exp(-Ky+iKx) & \text{as } x \to \infty \\ C_2 \ \ \exp(-Ky-iKx) & \text{as } x \to -\infty \end{cases}. \tag{2.16}$$

where C_1, C_2 are two unknown complex constants to be determined.

3. THE METHOD OF SOLUTION

By Havelock's expansion of water wave potential a suitable representation of ϕ satisfying (2.10), (2.11), (2.14) and (2.16) is given by

$$\phi(x, y) \sim \begin{cases} C_1 \ \ \exp(-Ky+iKx) & +\int_0^\infty A(k)L(k,y)\exp(-kx)\,dk, x > 0 \\ C_2 \ \ \exp(-Ky-iKx) & +\int_0^\infty B(k)L(k,y)\exp(kx)\,dk, x < 0. \end{cases} \tag{3.1}$$

By Havelock's expansion theorem

$$C_1 = -C_2,$$
$$A(k) = -B(k). \tag{3.2}$$

Using (3.1) in (2.12) and (2.13) we have

$$\int_0^\infty k\,A(k)L(k, y)\,dk = iKC_1\exp(-Ky) - f(y), \qquad y \in B. \tag{3.3}$$

Also,

$$\int_0^\infty A(k)L(k, y)\,dk = -C_1\exp(-Ky). \qquad y \in (a, b). \tag{3.4}$$

The dual integral equations (3.3) and (3.4) can be put in the alternative form as

$$\int_0^\infty kA(k)\sin ky\,dk = \begin{cases} iC_1\sinh Ky - \exp(Ky)\int_0^y f(t)\exp(-Kt)dt + D_2\exp(Ky), 0 < y < a \\ -\frac{iC_1}{2}\exp(-Ky) - \exp(Ky)\int_\infty^y f(t)\exp(-Kt)dt + D_3\exp(Ky), b < y < \infty \end{cases} \tag{3.5}$$

and

$$\int_0^\infty A(k)\sin ky\,dk = D_1\exp(Ky) + \frac{C_1}{2K}\exp(-Ky), \qquad y \in (a, b) \tag{3.6}$$

where D_1, D_2 and D_3 are arbitrary constants. In order to accommodate the origin as well as the point at infinity along the y-axis, the arbitrary constants D_2 and D_3 in (3.5) becomes zero.

Then the dual integral equations (3.5) – (3.6) can be rewritten as

$$\int_0^\infty kA(k)\sin ky\,dk = \begin{cases} iC_1\sinh Ky - \exp(Ky)\int_0^y f(t)\exp(-Kt)\,dt, 0<y<a \\ -\frac{iC_1}{2}\exp(-Ky) - \exp(Ky)\int_\infty^y f(t)\exp(-Kt)\,dt, b<y<\infty \end{cases} \tag{3.7}$$

and

$$\int_0^\infty A(k)\sin ky\,dk = D_1\exp(Ky) + \frac{C_1}{2K}\exp(-Ky), \qquad y \in (a, b) \tag{3.8}$$

Now, let,

$$\int_0^\infty k\,A(k)\sin ky\,dk = g(y), \qquad y \in (a, b) \tag{3.9}$$

where $g(y)$ is an unknown function to be determined. Utilizing the relations (3.7) and (3.9), we obtain, by using Fourier Sine Transform,

$$A(k) = \frac{2}{\pi k}\int_0^\infty M(t)\sin kt\,dt, \tag{3.10}$$

where,

$$M(t) = \begin{cases} iC_1\sinh Kt - \exp(Kt)\int_0^t f(u)\exp(-Ku)\,du, & 0<t<a, \\ -\frac{iC_1}{2}\exp(-Kt) - \exp(Kt)\int_\infty^t f(u)\exp(-Ku)\,du, & b<t<\infty \\ g(t) & a<t<b \end{cases}$$

Substituting A(k) into the equation (3.8) and simplifying we get

$$\frac{1}{\pi}\int_a^b g(t)\ln\left|\frac{y+t}{y-t}\right|dt = h(y) \qquad \text{for } y \in (a,b) \tag{3.11}$$

where,

$$h(y) = D_1\exp(Ky) + \frac{C_1}{2K}\exp(-Ky) - \frac{1}{\pi}\int_B M(t)\ln\left|\frac{y+t}{y-t}\right|dt\,.$$

Thus $h(y)$ contains two unknowns D_1 and C_1. Next, the integral equation (3.11) will be solved completely and D_1 and C_1 will also be determined.

4. DETERMINING THE COMNSTANTS

In order to solve the integral equation (3.11) completely,we must known the behaviour of the unknown function $g(y)$ at the end points $y = a$ and $y = b$ which can be determined as follows.

Let

$$\phi_x(0, y) = s(y), \qquad y \in G. \tag{3.12}$$

Using (3.1) we get,

$$\left(\frac{d}{dy}-K\right)\int_0^\infty kA(k)\sin ky\,dk = iKC_1\exp(-Ky)-s(y),\qquad y\in(a,b).$$

This can be written alternatively as

$$\int_0^\infty kA(k)\sin ky\,dk = -\frac{i}{2}C_1\exp(-Ky)+D_4\exp(Ky)-\exp(ky)\int s(y)\exp(-Ky)\,dy,\ y\in(a,b).$$

Where D_4 is an arbitrary constant. Comparing with (3.9) we have after simplification

$$s(y) = iC_1K\exp(-Ky)+Kg(y)-\frac{\partial g}{\partial y},\ y\in(a,b).$$

Noting (2.15) and (3.12) observe that

$$g(y) \sim Oj\,|y-t|^{1/2})\ ay\to t,$$

where $t = a^+$ and b^-.

It can be shown (cf[1]) that the solution of the integral equation (3.11) which is bounded at both end points is given by,

$$g(u) = \frac{2}{\pi}((u^2-a^2)(b^2-u^2))^{1/2}\int_a^b \frac{th'(t)}{\left((t^2-a^2)(b^2-t^2)\right)^{1/2}(u^2-t^2)}dt\ \ \mathrm{u}\in(a,b) \tag{3.13}$$

provided that

$$1)\int_a^b \frac{th'(t)}{\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}dt = 0 \tag{3.14a}$$

$$2)\ C+2\int_a^b \frac{th'(t)(t^2-a^2)^{1/2}}{(b^2-t^2)^{1/2}}dt = 0\,, \tag{3.14b}$$

where C is given by

$$C = 2\frac{a\pi - I_1}{I_2}\int_a^b \frac{h(x)}{(x^2-a^2)^{1/2}(b^2-x^2)^{1/2}}dx + 2\int_a^b \frac{h(x)(x^2-a^2)^{1/2}}{(b^2-x^2)^{1/2}}dx$$

with

$$I_1 = \int_a^b \frac{\ln\left|\frac{a+x}{a-x}\right|(x^2-a^2)^{1/2}}{(b^2-x^2)^{1/2}}dx$$

$$I_2 = \int_a^b \frac{\ln\left|\frac{a+x}{a-x}\right|}{\left((x^2-a^2)(b^2-x^2)\right)^{1/2}}dx$$

Utilizing the function $h(y)$ given by (3.11), in the relations (3.14a) and (3.14b) we obtain a linear system of equations in D_1 and C_1,

$$r_1D_1 + r_2C_1 = b_1 \tag{3.15}$$

$$r_3 D_1 + r_4 C_1 = d_1 \tag{3.16}$$

where,

$$r_1 = K\int_a^b \frac{t\exp(Kt)}{\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}\,dt\,,$$

$$r_2 = -\frac{1}{2}\int_a^b \frac{t\exp(-Kt)}{\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}\,dt - \frac{i}{2}\int_{-a}^a \frac{y\exp(-Ky)}{\left((y^2-a^2)(b^2-y^2)\right)^{1/2}}dy - \frac{i}{2}\int_b^\infty \frac{y\exp(-Ky)}{\left((y^2-a^2)(b^2-y^2)\right)^{1/2}}\,dy,$$

$$r_3 = \int_a^b \frac{\exp(Kt)\left((a\pi - I_1) + I_2(t^2-a^2)\right)}{I_2\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}\,dt + K\int_a^b \frac{t\exp(Kt)(t^2-a^2)^{1/2}}{(b^2-t^2)^{1/2}}\,dt$$

$$r_4 = \frac{1}{2K}\int_a^b \frac{\exp(-Kt)\left((a\pi - I_1) + I_2(t^2-a^2)\right)}{I_2\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}\,dt + \frac{i}{2\pi}\int_a^b \frac{\left((a\pi - I_I) + I_2(t^2-a^2)\right)}{I_2\left(\left((t^2-a^2)(b^2-t^2)\right)^{1/2}\right)}\,dt$$

$$\left[\int_{-a}^a \exp(-Ky)\ln\left|\frac{t+y}{t-y}\right|dy\right]dt - \frac{i}{2\pi}\int_a^b \frac{\left((a\pi - I_1) + I_2(t^2-a^2)\right)}{I_2\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}$$

$$\left[\int_b^\infty \exp(-Ky)\ln\left|\frac{t+y}{t-y}\right|dy\right]dt - \frac{1}{2}\int_a^b \frac{t\exp(-Kt)(t^2-a^2)^{1/2}}{(b^2-t^2)^{1/2}}\,dt$$

$$+\frac{i}{2}\int_{-a}^{a_t} \exp(-Kt)\left[\frac{(a^2-t^2)^{1/2}}{(b^2-t^2)^{1/2}} - 1\right]dt + \frac{i}{2}\int_b^\infty t\exp(-Kt)\left[\frac{(t^2-a^2)^{1/2}}{(b^2-t^2)^{1/2}} - 1\right]dt,$$

$$b_1 = -\pi\exp(k\eta - iK\xi)\left[\int_{-a}^a \frac{y\exp(-Ky)}{\left((y^2-a^2)(b^2-y^2)\right)^{1/2}}\,dy - \int_b^\infty \frac{y\exp(-Ky)}{\left((y^2-a^2)(b^2-y^2)\right)^{1/2}}\,dy\right]$$

$$-2\int_0^\infty \frac{L(k,\eta)\exp(k\xi)}{k^2+K^2}\int_a^b \frac{y\cos ky}{\left((y^2-a^2)(b^2-y^2)\right)^{1/2}}\,dy\,dk,$$

$$d_1 = \pi\exp(-k\eta - iK\xi)\int_{-a}^a y\exp(-Ky)\left[\frac{(a^2-y^2)^{1/2}}{(b^2-y^2)^{1/2}} - 1\right]dy + 2\int_0^\infty \frac{L(k,\eta)\exp(k\xi)}{k^2+K^2}\int_0^\infty y\sin ky$$

$$\left[1 - \frac{(a^2-y^2)^{1/2}}{(b^2-y^2)^{1/2}}\right]dy\,dk + \pi\exp(-K\eta - iK\xi)\int_b^\infty y\exp(-Ky)\left[\frac{(y^2-a^2)^{1/2}}{(b^2-y^2)^{1/2}} - 1\right]dy$$

$$+2\int_0^\infty \frac{L(k,\eta)\exp(k\xi)}{k^2+K^2}\int_b^\infty y\sin ky\left[1 - \frac{(y^2-a^2)^{1/2}}{(b^2-y^2)^{1/2}}\right]dy\,dk + \exp(-k\eta - iK\xi)\int_a^b \frac{(t^2-a^2)^{1/2}}{(b^2-t^2)^{1/2}}$$

$$\left[\int_b^\infty \exp(-Ky)\ln\left|\frac{t+y}{t-y}\right|dy - \int_{-a}^a \exp(-Ky)\ln\left|\frac{t+y}{t-y}\right|dy\right]dt + \frac{2}{\pi}\int_0^\infty \frac{L(k,\eta)\exp(k\xi)}{k^2+K^2}$$

$$\int_a^b \frac{(t^2-a^2)^{1/2}}{(b^2-t^2)^{1/2}}\left[\int_b^\infty in\left|\frac{t+y}{t-y}\right|\sin ky\, dy - \int_0^a \ln\left|\frac{t+y}{t-y}\right|\sin ky\, dy\right]dt\, dk + \frac{a\pi - I_1}{I_2} exp(-K\eta - iK\xi)$$

$$\int_a^b \frac{1}{\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}\left[\int_{-a}^a \exp(-Ky)\ln\left|\frac{t+y}{t-y}\right|dy - \int_b^\infty \exp(-Ky)\ln\left|\frac{t+y}{t-y}\right|dy\right]dt$$

$$+\frac{2}{\pi}\int_0^\infty \frac{L(k,\eta)\exp(k\xi)}{k^2+K^2}\int_a^b \frac{1}{\left((t^2-a^2)(b^2-t^2)\right)^{1/2}}\left[\int_b^\infty \ln\left|\frac{t+y}{t-y}\right|\sin ky\, dy - \int_0^a \ln\left|\frac{t+y}{t-y}\right|\sin ky\, dy\right]dt.$$

Solving (3.15) and (3.16) we get,

$$D_1 = \frac{r_4 b_1 - r_2 d_1}{r_4 r_1 - r_2 r_3} \tag{3.17}$$

$$C_1 = \frac{r_3 b_1 - r_1 d_1}{r_2 r_3 - r_4 r_1}. \tag{3.18}$$

The final form of the solution $\phi(x, y)$ can be obtained by using the relations (3.2), (3.10), (3.13), (3.17) and (3.18) in the relation (3.1). Thus knowing ϕ, Φ can be completely determined from (2.8) after using (2.9).

5. SOLUTION FOR OTHER CASES

Case 1: $a \to 0$ and b (> 0) fixed.

This limiting case represents the fully submerged barrier and it can be shown that the integral equation (3.11) reduces to a special singular integral equation and is given by

$$\frac{1}{\pi}\int_0^b g(t)\ln\left|\frac{y+t}{y-t}\right|dt = h(y), \qquad y \in (0, b) \tag{4.1}$$

where,

$$h(y) = -\frac{C_1}{K}\sinh Ky - \frac{1}{\pi}\int_b^\infty \left[-\frac{iC_1}{2}\exp(-Kt) - \exp(Kt)\int_\infty^t f(u)\exp(-Ku)du\right]\ln\left|\frac{y+t}{y-t}\right|dt.$$

Since

$$h(0) = 0,$$

it is easily observe that the two conditions of solvability (3.14a) and (3.14 b) are reduces to one condition in this case and is given by

$$\int_0^b \frac{h'(t)}{(b^2-t^2)^{1/2}}dt = 0. \tag{4.2}$$

Utilizing the above condition we find that

$$C_1 = \frac{2}{\Delta_1}\int_b^\infty \frac{h_1(y)\exp(Ky)}{(y^2-b^2)^{1/2}}dy \tag{4.3}$$

where,

$$\Delta_1 = \pi\, I_0(Kb) - iK_0(Kb),$$

$$h_1(y) = \int_{\infty}^{y} f(t)\exp(-Kt)dt. \tag{4.4}$$

The result in (4.3)and (4.4) coincide with [5].

Case 2: a (> 0) fixed and $b \to \infty$.

This case represents a barrier which is partially immersed to a depth 'a' below the mean free surface. In this case (3.11) becomes

$$\frac{1}{\pi}\int_a^{\infty} g(t)\ln\left|\frac{y+t}{y-t}\right| dt = h(y), \qquad y \in (a,\infty) \tag{4.5}$$

where,

$$h(y) = \frac{C_1}{2K}\exp(-Ky) - \frac{1}{\pi}\int_0^a \left[iC_1 \sinh Kt - \exp(Kt)\int_0 f(u)\exp(-Ku)du \right] \ln\left|\frac{y+t}{y-t}\right| dt\,.$$

Transforming the above integral equation (4.5) into an integral equation of the form (4.1) and following the case-1,we obtain the solvability criterion in this case as

$$\int_a^{\infty} \frac{th'(t)}{(t^2-a^2)^{1/2}} dt = 0. \tag{4.6}$$

Using the above condition we obtain,

$$C_1 = -\frac{2i}{a\Delta_2}\int_0^a \frac{yh_2(y)\exp(Ky)}{(a^2-y^2)^{1/2}} dy, \tag{4.7}$$

where,

$$\Delta_2 = \pi\, I_1(Ka) + iK_1(Ka), \tag{4.8}$$

$$h_2(y) = \int_0^{y} f(u)\exp(-Ku)\, du. \tag{4.9}$$

The result in (4.7), (4.8) and (4.9) coincide with [4].

6. CONCLUSION

The present analysis provides an efficient method to solve the class of water wave scattering and radiation problems involving vertical barrier under the assumption of linearised theory.

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