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# AN INTEGRO–DIFFERENTIAL EQUATION RELATED TO A ROBOT WITH INTERNAL SAFETY DEVICE

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***ABSTRACT***: We introduce a robot–safety device system composed of a robot with internal (built–in) safety device. The system is characterized by a safety shut–down rule and by the natural feature of standby. In order to obtain the point availability of the twin–system, we introduce a stochastic process endowed with a time–dependent potential satisfying a (non-standard) integro–differential equation. The explicit solution procedure requires the (new) notion of virtual lifetime - versus - effective lifetime of the robot. The analysis of the long–run availability requires the introduction of a signed measure.

As a particular example, we consider the family of the Weibull–Gnedenko distributions to model the robot's failure and repair process. Finally, we display a computer–plotted graph of the point availability by a direct numerical solution of the equation. Our numerical approach, based on a grid generation technique, is strongly motivated by the complexity of the exact solution.

***Mathematics Subject Classification:*** 60K10

***Keywords:*** Robot, safety device, virtual lifetime, effective lifetime, availability, integro–differential equation, functional equation.

## 1. INTRODUCTION

Up–to–date robots are often connected with a safety device, e.g. [10]. Such a device prevents possible damage, caused by a robot failure, in the neighbouring environment. The usual "bugbears" are software failures, e.g. [5], common–cause failures, e.g. [3] and physical failures, e.g. [1]. Moreover, the random behaviour of the entire system (robot, safety unit, repair facility) requires some additional measures to ensure the safety of man–machine interactions [5]. In the previous Literature, we have considered a robot with (external) safety device, called the **T**–system, e.g. [9], [13]. The **T**–system is characterized by the natural feature of standby for the safety device and by an admissible risky state [12]. As a variant, we introduce a robot with internal (built–in) safety unit, henceforth called the **S**–system, subjected to the following safety shut–down rule : "Any repair of the failed safety device requires a shut–down of the operative robot". On the other hand, the safety unit need not to operate if the robot is in repair. Consequently, upon failure of the robot (safety device) the operative safety device (robot) is put in standby until the repair of the robot (safety device) has been completed. The **S**–system is attended by two different repairmen. Repairman $R_S$ is skilled in repairing the safety unit, whereas repairman R is assumed to be an expert in repairing the robot. Any repair is supposed to be perfect and general.

Apart from a statistical generalization of the **T**–system with regard to the previous (restrictive) assumption of a constant failure rate of the robot, we also introduce the notion of virtual lifetime - versus - effective lifetime of the robot (S–system). In order to obtain the point availability and the long–run availability of the S–system, we introduce a stochastic process endowed with a time–dependent potential satisfying a (non–standard) integro–differential equation. The explicit solution procedure requires the distribution of the robot's virtual lifetime and the introduction of a signed measure.

As a particular example, we consider the family of Weibull–Gnedenko distributions [6] to model the robot's failure and repair process. Finally, we display a computer–plotted graph of the point availability by a direct numerical solution of the equation. Our numerical approach, based on a grid generation technique, e.g. [8], is strongly motivated by the complexity of the exact solution.

## 2. FORMULATION

Consider the **S**–system satisfying the following conditions.

- The operative safety device has a constant failure rate $\lambda_s$ and a general repair time distribution $R_s(\cdot)$, $R_s(0) = 0$. Let $f_s$ be the random variable corresponding to the failure rate $\lambda_s$. Clearly, $f_s$ is exponentially distributed with mean $\lambda_s^{-1}$. The repair time is denoted by $r_s$.
- In order to define the virtual lifetime of the robot, we first consider a robot without a safety device, starting to operate at some time origin $t = 0$. The lifetime of the robot is denoted by $f$ with general distribution $F(\cdot)$, $F(0) = 0$. Clearly, $f$ is the time measured from $t = 0$ onwards until the robot fails. Next, we consider the $\boldsymbol{S}$–system. Let

$$v_f := \begin{cases} f + \sum_{i=1}^{n_f} r_{s,i} & , \text{ if } n_f > 0, \\ f & , \text{ if } n_f = 0, \end{cases}$$

  where $r_{s,i}$; $i = 1, 2, \ldots$ denotes the $i$–th repair time of the safety unit and $n_f$ the number of $\lambda_s$–failures *during* $f$. The random variable $v_f$ is called the *virtual* lifetime of the robot. Clearly, $v_f$ reduces to $f$ if $n_f = 0$. Therefore, we call $f$ the *effective* lifetime of the robot. The repair time of the robot is denoted by $r$ with general distribution $R(\cdot)$, $R(0) = 0$. Finally, let $F_v(\cdot) := \mathbf{P}\{v_f \leq \cdot\}$.
- The variables $f, f_s, r, r_s$ are supposed to be statistically independent with finite mean and any repair is perfect.
- Finally, we assume that both the robot and the safety unit are free from standby failures (the so–called "cold" standby mode).

In order to describe the random behaviour of the **S**–system, we introduce a stochastic process $\{N_t, t \geq 0\}$, $N_0 = 0$ **P**–a.s., with state space $\{A, B, C\} \subset [0, \infty)$ characterized by the following mutually exclusive events :

$\{N_t = A\}$ : "The robot and the safety device are both operative at time $t$."

$\{N_t = B\}$ : "The safety unit is under progressive repair and the robot is in standby at time $t$."

$\{N_t = C\}$ : "The robot is under progressive repair and the safety device is in standby at time $t$."

State $A$ is called the safe state and state $B$ is called the shut–down state. Note that the event : "The robot and the safety device are both under repair at time $t$" is a **P**–null set! Let

$$\wp(t) := \mathbf{P}\{N_t = A\}, \; t \geq 0.$$

We recall that the **S**–system is only available (functioning) in state $A$. Therefore, the point availability of the **S**–system is given by $\wp(\cdot)$. Let

$$\wp(\infty) := \lim_{t\to\infty} \wp(t),$$

provided that the precious limit exists. $\wp(\infty)$ is called the *long–run* availabilityof the **S**–system. Observe that

$$\wp(\infty) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \wp(t)dt.$$

**Notations**

- Let $C$ be the complex $\omega$–plane, $\mathbf{C}^+ := \{\omega \in \mathbf{C} : \text{Im } \omega > 0\}$.
- The indicator (function) of the event $\{N_t = A\}$ is denoted by $\mathbf{1}\{N_t = A\}$.
- The Borel algebra on $[0, \infty)$ is denoted by $\mathcal{B}([0, \infty))$.

- The Laplace–transform of any function $\alpha(\cdot)$, locally integrable and bounded on $[0, \infty)$ is denoted by the corresponding character marked with an asterisk. For instance,

$$\alpha^*(z) := \int_0^\infty e^{-zt}\alpha(t)dt, \qquad z > 0.$$

- If $\alpha(\cdot)$ is a right–continous function of bounded variation on $[0, \infty)$, then we define

$$\alpha^\vee(z) := \int_{0-}^\infty e^{-zt} d\alpha(t), \qquad z \geq 0,$$

where

$$\int_{0-}^\infty e^{-zt} d\alpha(t) := \alpha(0) + \int_0^\infty e^{-zt} d\alpha(t)$$

Note that the product rule, e.g. [2, Appendix] implies that

$$\alpha^{\wedge}(z) = z\alpha^*(z), \qquad z > 0.$$

## 3. PRELIMINARY PROPERTIES

For direct reference, we state the following properties. Let

$$p_s(t) := \int_{0-}^t e^{-\lambda_s(t-u)} d\sum_{n=0}^{\infty} \varphi_s^{n\text{Ú}}(u)$$

where

$$\varphi_s(u) := \int_0^u (1 - e^{-\lambda_s(u-v)}) dR_s(v).$$

$\varphi_s^{n*}(\cdot)$ denotes the $n$–fold convolution of $\varphi_s(\cdot) \equiv \varphi_s^{1\text{Ú}}(\cdot)$. For $n = 0$, $\varphi_s^{0\text{Ú}}(u)$ represents the Heaviside unit–step function with the unit–jump at $u = 0$.

**Property 3.1** [11], [13]

- $p_s(0) = 1$; $0 < p_s(t) \leq 1$, $p_s(\infty) = (1 + \lambda_s \mathbf{E} r_s)^{-1}$.
- $p_s(\cdot)$ is Lebesgue–absolutely continuous on $(0, \infty)$ and of bounded variation on $[0, \infty)$.
- 

$$p_s^*(s) = \frac{1}{1 + \lambda_s \mathbf{E} r_s \gamma(z)}, z \geq 0, \tag{1}$$

where

$$\gamma(z) := \begin{cases} \frac{1 - \mathbf{E} e^{-zr_s}}{z \mathbf{E} r_s} & \text{, if } z \neq 0. \\ 1 & \text{, if } z = 0. \end{cases}$$

Note that

$$\gamma(z) = \frac{1}{\mathbf{E} r_s} \int_0^\infty e^{-zu}(1 - R_s(u))du.$$

**Remarks 3.1**

- Observe that the function $p_s(t)$, $t \geq 0$ induces a finite signed measure on $\mathcal{B}([0, \infty))$ denoted by $\mu_s(\cdot)$. Clearly, by property 3.1

$$\int_{[0,\infty]} \mu_s(dx) = \frac{1}{1 + \lambda_s \mathbf{E} r_s}. \tag{2}$$

- It is fairly obvious that the state probability $p_s(\cdot)$ can be generalized for arbitrary distributions by means of Renewal Theory, e.g. [7].

Involving the functions $F_v(\cdot)$ and $R(\cdot)$, let

$$\varphi(u) := \int_0^u F_v(u - w)\, dR(w)$$

and

$$p_R(t) := \int_{0-}^t (1 - F_v(t-u))\, d\sum_{n=0}^{\infty} \varphi^{n\text{Ú}}(u). \tag{3}$$

The following properties are stated without proof.

**Property 3.2**

- $p_R(0) = 1$, $0 < p_R(t) \leq 1$.
- If $f$ is non–lattice, then

$$p_R(\infty) = \frac{\mathbf{E} v_f}{\mathbf{E} r + \mathbf{E} v_f}. \tag{4}$$

-

$$p^*_R(z) = \frac{1}{z} \frac{1 - \mathbf{E} e^{-z v_f}}{1 - \mathbf{E} e^{-zr} \mathbf{E} e^{-z v_f}}, \qquad z > 0 \tag{5}$$

**Remarks 3.2:** Consider the decomposition of Eq. (3), i.e.

$$p_R(t) = \sum_{n=0}^{\infty} \varphi^{n\text{Ú}}(t) - \int_{0-}^t F_v(t-u)\, d\sum_{n=0}^{\infty} \varphi^{n\text{Ú}}(u).$$

Clearly, $p_R(\cdot)$ is a difference of two right–continuous increasing functions. Hence, $p_R(\cdot)$ is of bounded variation on any compact of $[0, \infty)$. However, since $p_R(\infty)$ exists, we may conclude that $p_R(\cdot)$ is a right–continuous function of bounded variation on $[0, \infty)$.

Consequently, $p_R(\cdot)$ is uniquely determined by $p^*_R(\cdot)$. Note that our remark is crucial to determine the point availability $\wp(\cdot)$ by inversion of the corresponding Laplace–transform $\wp^*(\cdot)$. See Chapter 5 for further details.

**Theorem 3.1**

$$F_v(t) = \int_0^t \sum_{k=0}^{\infty} R_s^{k\text{Ú}}(t-u) e^{-\lambda_s u} \frac{(\lambda_s u)^k}{k!} dF(u).$$

$$\mathbf{E} v_f = \mathbf{E} f (1 + \lambda_s \mathbf{E} r_s).$$

**Proof**

- By the law of total probability,

$$\mathbf{P}\{v_f \le t\} = \int_0^{\infty} \mathbf{P}\{v_f \le t \,|\, f = u\} dF(u)$$

$$= \int_0^t \mathbf{P}\left\{\sum_{i=1}^{n_u} r_{s,i} \le t - u\right\} dF(u)$$

$$= \int_0^t \sum_{k=0}^{\infty} R_s^{k\acute{U}}(t-u)\mathbf{P}\{n_u = k\} dF(u).$$

But $\{n_u, u \ge 0\}$ is a homogeneous Poisson process with parameter $\lambda_s$, i.e.

$$\mathbf{P}\{n_u = k\} = e^{-\lambda_s u}\frac{(\lambda_s u)^k}{k!}; k = 0, 1, \ldots$$

Hence,

$$\mathbf{P}\{v_f \le t\} = \int_0^t \sum_{k=0}^{\infty} R_s^{k\acute{U}}(t-u) e^{-\lambda_s u}\frac{(\lambda_s u)^k}{k!} dF(u).$$

- The Laplace–Stieltjes convolution theorem entails that

(6)

Thus, $\mathbf{E}v_f$ follows from the relation

$$\mathbf{E}v_f = -\frac{\partial}{\partial z}\mathbf{E}e^{-zv_f}\bigg|_{z=0}.$$

## 4. INTEGRO–DIFFERENTIAL EQUATION

It should be noted that state $A$ is *non–regenerative* for the process $\{N_t, t \ge 0\}$. In order to obtain a Markov characterization of state $A$, let $X_t$ be the remaining (effective) lifetime of the robot being operative at time $t$.

We are going to assume (without loss of generality, see forthcoming Remarks 5.1.) that $F(\cdot)$ is Lebesgue–absolutely continuous on $[0, \infty)$ with Radon–Nikodym derivative $\mathcal{F}(\cdot)$ of bounded variation on $[0, \infty)$. For $t > 0$, $x > 0$, let

$$\wp(t, x)dx := \mathbf{P}\{N_t = A, X_t \in dx\}.$$

Note that

$$\wp(t) = \int_0^{\infty} d_x \mathbf{P}(N_t = A, X_t \le x\} = \int_0^{\infty} \wp(t, x)\, dx.$$

Observing the **S**–system in some time interval $(t, t + \Delta)$, $\Delta \downarrow 0$ and grouping terms of $o(\Delta)$, $\Delta \downarrow 0$ reveals that

$$\wp(t + \Delta, x - \Delta) = \wp(t, x)(1 - \lambda_s \Delta) +$$

$$\int_0^t \wp(t - u, x) dR_s(u)\mathbf{P}\{f_s \le x + \Delta \,|\, f_s > x\} +$$

$$\int_0^t \wp(t - u, 0) dR(u)\mathcal{F}(x)\,\Delta + o(\Delta).$$

Clearly,

$$\mathbf{P}\{f_s \le x+\Delta \mid f_s > x\} = \frac{\mathbf{P}\{x < f_s \le x+\Delta\}}{\mathbf{P}\{f_s > x\}}.$$

On the other hand,

$$\frac{\mathbf{P}\{x < f_s \le x+\Delta\}}{\mathbf{P}\{f_s > x\}} = 1 - e^{-\lambda_s \Delta} = \lambda_s \Delta + o(\Delta).$$

Hence,

$$\wp(t+\Delta, x-\Delta) = \wp(t, x)(1-\lambda_s \Delta) + \lambda_s \Delta \int_0^t \wp(t-u, x)\,dR_s(u) +$$

$$\int_0^t \wp(t-u, 0)\,dR(u)\mathcal{F}(x)\,\Delta + o(\Delta)\,.$$

Invoking the definition of the directional derivative

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\wp(t,x) := \lim_{\Delta \downarrow 0} \frac{\wp(t+\Delta, x-\Delta) - \wp(t,x)}{\Delta}$$

entails that $\wp(t, x)$, $t > 0$, $x > 0$ satisfies the integro–differential equation

$$\left(\lambda_s + \frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\wp(t,x) = \lambda_s \int_0^t \wp(t-u,x)\,dR_s(u) + \int_0^t \wp(t-u,0)\,dR(u)\mathsf{F}(x) \tag{7}$$

with initial condition

$$\wp(0, x) = \mathcal{F}(x),\ x \ge 0.$$

## 5. FUNCTIONAL EQUATION

Note that Eq.(7) is well–adapted to a Laplace–Fourier transformation. In fact, the function $\wp(t, \cdot)$ is locally integrable and bounded on $[0, \infty)$. Hence, the Laplace–transform $\cdot\,\wp^*(z, \cdot)$ exists for $z > 0$. Moreover, the integrability of the functions $\wp^*(\cdot, x)$ and $\mathcal{F}(x)$ on $[0, \infty)$ also implies the integrability of $d\wp(\cdot, x)/dx$.

Consequently, $\wp(\cdot, x)$ vanishes if $x$ tends to infinity. Applying a Laplace–Fourier transformation to Eq.(7) and taking the initial condition into account, yields the functional equation

$$(z + \lambda_s(1 - \mathbf{E}e^{-zr_s}) + i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t} 1\{N_t = A\})\,dt$$

$$+\wp^*(z, 0)(1 - \mathbf{E}e^{-zr}\mathbf{E}e^{i\omega f}) = \mathbf{E}e^{i\omega f} \tag{8}$$

valid for $z > 0$, $\mathrm{Im} \ge 0$. In order to determine $\wp^*(z, 0)$, we first remark that

$$z + \lambda_s(1 - \mathbf{E}e^{-zr_s}),\ z > 0$$

is a positive, increasing, concave function with range $(0, \infty)$.

Consequently, the equation

$$i\omega + z + \lambda_s(1 - \mathbf{E}e^{-zr_s}) = 0$$

holds for any pair $(z, \omega)$ : $\mathrm{Re}\ \omega = 0$ and $\mathrm{Im}\ \omega = z + \lambda_s(1 - \mathbf{E}e^{-zr_s})$. But the function

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t = A\})\,dt, z > 0$$

is analytic in $\mathbf{C}^+$. Consequently, $\wp^*(z, 0)$ is uniquely determined by the equation

$$\mathbf{E}e^{i\omega f} = \wp^*(z,0)(1 - \mathbf{E}e^{-zr}\,\mathbf{E}e^{i\omega f})$$

at the point $\omega = i(z + \lambda_s(1 - \mathbf{E}e^{-zr_s}))$. Taking Eq.(6) into account, reveals that

$$\wp^*(z,0) = \frac{\mathbf{E}e^{-zv_f}}{1-\mathbf{E}e^{-zv_f}\,\mathbf{E}e^{-zr}}, \qquad z > 0.$$

Inserting $\omega = 0$ into Eq.(8) yields

$$\wp^*(z) = \frac{1-\wp^*(z,0)(1-\mathbf{E}e^{-zr})}{z+\lambda_s(1-\mathbf{E}e^{-zr_s})}.$$

Whence,

$$\wp^*(z) = \frac{1-\mathbf{E}e^{-zv_f}}{1-\mathbf{E}e^{-zv_f}\,\mathbf{E}e^{-zr}}\,\frac{1}{z+\lambda_s(1-\mathbf{E}e^{-zr_s})}.$$

By Eqs.(1), (5) we have

$$\wp^*(z) = p_R^*(z)p_s^{\vee}(z).$$

Finally, taking Remark 3.2 into account, we obtain by inversion

$$\wp(t) = \int_{[0,\infty)} p_R(t-x)\mathbf{1}\{0 \le x < t\}\mu_s(dx).$$

Next, we deal with the long–run availability $\wp(\infty)$ of the **S**–system. It is plain that the existence of $p_R(\infty)$ implies the existence of $\wp(\infty)$. Applying the bounded convergence theorem, e.g. [4, page 84], entails that

$$\wp(\infty) = p_R(\infty)\int_{[0,\infty)} \mu_s(dx).$$

Hence, by Eqs.(2),(4) and Theorem 3.1

$$\wp(\infty) = \frac{\mathbf{E}f}{\mathbf{E}f(1+\lambda_s\mathbf{E}r_s)+\mathbf{E}r}.$$

**Remarks 5.1** It is clear that the function $p_R(\cdot)$ also exists as a Lebesgue–Stieltjes integral for *general F*. Consequently, our initial assumption concerning the existence of $\mathcal{F}$ is totally superfluous to ensure the existence of $\wp(\cdot)$. We summarize the following result.

**Theorem 5.1** Let $F$, $R$, $R_s$ be general distributions with finite mean. Suppose that $f$ is *non–lattice*, then

$$\wp(t) = \int_{[0,t)} p_R(t-x)\mu_s(dx),$$

$$\wp(\infty) = \frac{\mathbf{E}f}{\mathbf{E}f(1+\lambda_s\mathbf{E}r_s)+\mathbf{E}r}.$$

## 6. NUMERICAL APPROACH

A particular but important family $\Im$ of current probability distributions with *non–rational* Laplace–Stieltjes transforms, such as the Weibull–Gnedenko distribution

$$W_\beta(x) := 1 - e^{-x^\beta}, \ \beta > 0$$

is fairly suitable to model failure processes (for instance, metal fatigue). See [6, page 70, references]. However, since $W_\beta(\cdot)$, $\beta > 1$ has an increasing hazard rate $\beta x^{\beta-1}$ we may infer that $W_\beta(\cdot)$ is also suitable to model repair times!

Unfortunately, if at least $\mathcal{F} \in \mathfrak{I}$, an explicit evaluation of $\wp(\cdot)$ in terms of a *finite* sum of linear combinations of algebraic and/or transcendental functions is totally excluded. Therefore, a *numerical* approach of our equation is completely motivated by the *complexity* of the exact solution.

## 7. NUMERICAL SCHEME

In view of our forthcoming application, we may assume that $F$, $R$ and $R_s$ have density functions of bounded variation denoted by $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{R}_s$. We recall that $\wp(\cdot, x)$ vanishes at infinity. Consequently, in order to construct an appropriate numerical procedure, we convert the region $(0, \infty) \times (0, \infty)$ into the truncated region $(0, T) \times (0, L)$, for some $T$, $L > 0$. This truncation requires the boundary condition

$$\wp(t, L) = 0.$$

The initial condition

$$\wp(0, x) = \mathcal{F}(x)$$

completes the initial boundary-value problem. Let $\wp_i^n := \wp(t_n, x_i)$, $\mathcal{R}^n := \mathcal{R}(t_n)$, $\mathcal{R}_s^n := \mathcal{R}_s(t_n)$, $\mathcal{F}_i := \mathcal{F}(x_i)$, where $x_i := i\Delta x$, $t_n := n\Delta t$, $i = 0, \dots, N_x + 1$, $n = 0, \dots, N_t$; $\Delta x := L/(N_x + 1)$ denotes the spatial step, $\Delta t := T/Nt$ the time step and $N_x + 1 \times N_t$ the size of the corresponding finite-difference grid. The approximation of the boundary condition at $x = L$ is then given by

$$\wp_{N_x+1}^n = 0\,,$$

whereas the initial condition is given by

$$\wp_i^0 = \mathcal{F}(x_i).$$

We propose to compute the approximation of the unknown function by means of the first order finite-difference scheme

$$\lambda_s \wp_i^{n+1} + \frac{\wp_i^{n+1} - \wp_1^n}{\Delta t} - \frac{\wp_{i+1}^{n+1} - \wp_i^{n+1}}{\Delta x} = \lambda_s I_i^{n+1} + I^{n+1}\mathsf{F}_i,$$

where

$$I_i^{n+1} := \wp_i^0 \mathcal{R}_s^{n+1} \Delta t/2 + \sum_{j=1}^{n} \mathcal{R}_s^{n+1-j} \wp_i^j \Delta t + \wp_i^{n+1} \mathcal{R}_s^0 \Delta t/2$$

and

$$I_i^{n+1} := \wp_i^0 \mathcal{R}^{n+1} \Delta t/2 + \sum_{j=1}^{n} \mathcal{R}^{n+1-j} \wp_0^j \Delta t + \wp_0^{n+1} \mathcal{R}^0 \Delta t/2$$

are approximations of the integrals by the trapezoidal rule. The calculations are running backwards in the space coordinate $i = N_x, N_x - 1, \dots, 0$ and forward in the time coordinate $n = 0, \dots, N_t$. We employ the following iterations. At the first iteration ($k = 1$), $\wp_0^{n+1}$ is replaced by $\wp_0^n$. The calculation of $\wp_0^{n+1,1}$ is now straightforward since $I^{n+1}$ can be evaluated explicitly. Next, $\wp_0^{n+1}$ is replaced by $\wp_0^{n+1,1}$ to obtain $\wp_i^{n+1,2}$. The process continues max $\left|\wp_i^{k+1,n+1} - \wp_i^{k,n+1}\right| < \epsilon$, where $\epsilon$ is the prescribed accuracy. Finally, we calculate $\wp^n$ by

$$\wp^n = \wp_0^n \Delta x/2 + \sum_{i=1}^{N_{x-1}} \wp_i^n \Delta x + \wp_{N_x}^n \Delta x/2.$$

Let $\wp_\Delta$ denote a numerical solution related to a step $\Delta$. The convergence of the numerical scheme is then evaluated by the standard error estimates max $|\wp_{\Delta x} - \wp_{\Delta x/2}|$ and max $|\wp_{\Delta t} - \wp_{\Delta t/2}|$. However, the estimates are only accurate if $L$ is large enough. Therefore, we estimate the suitability of $L$ by comparing $\wp^n$ for large $n$ with the *exact* asymptotic value

$$\wp(\infty) = \frac{\mathbf{E}_f}{\mathbf{E}r + \mathbf{E}f(1 + \lambda_s \mathbf{E}r_s)}.$$

## 8. APPLICATION

As an example, we consider the particular case $\lambda_s = 0.5$, $F(\cdot) = W_2(\cdot)$, $R_s(\cdot) = W_1(\cdot)$, $R(\cdot) = W_3(\cdot)$. Figure 1 displays the graph of $\wp(t)$, $\wp(\infty) = 0.474$. The graph clearly indicates that the availability of the **S**–system is not worse than $\inf_{t \geq 0} \wp(t) = 0.32$. Figure 2 displays the graph of $\wp(t, x)$.

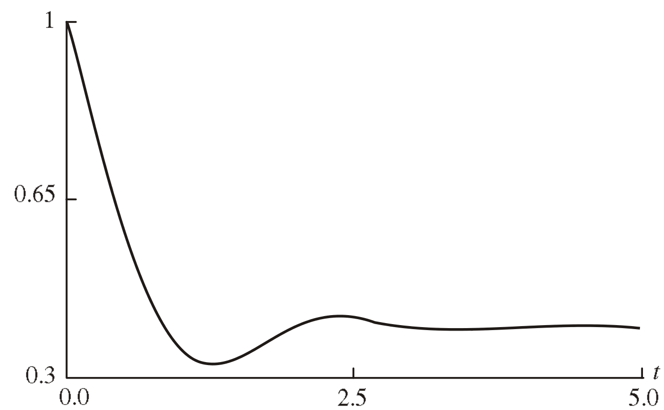


**Figure 1: Graph of $\wp(t)$. Case $\lambda_s = 0.5$**

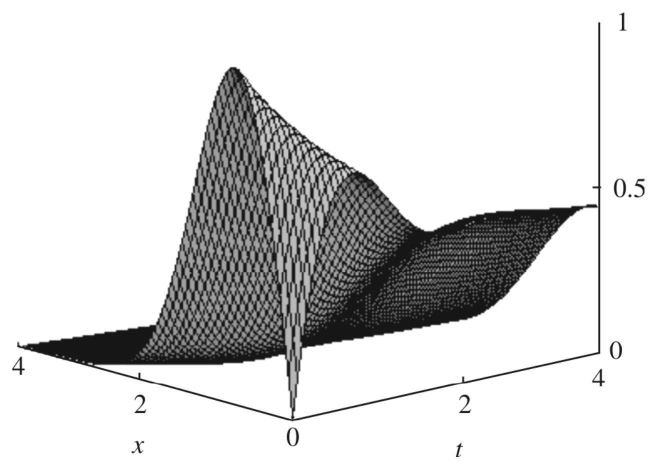


**Figure 2: Graph of $\wp(t, x)$. Case $\lambda_s = 0.5$**

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