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ANALYTICAL INVESTIGATION OF ICE FLOE DRIFT IN THE MARGINAL ICE ZONE

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ABSTRACT: An analytical solution was constructed to investigate the ice floe drift, velocity field, and trajectories. The mathematical model considers the balance of atmosphere and ocean drag forces on ice floe, including skin and body drag forces from wind, waves, and currents. We have obtained analytical solutions of air-ice and water-ice skin stresses, water-ice form stress, and wave radiation stress. Graphical solutions are presented for ice floe drift due to wind stress. Mathematical formulations are presented for the ice floe drift due to Eulerian current, water-ice form stress, and wave radiation pressure. We systematically presented in this paper the classical solutions of the ice floe drift, velocity, and trajectories considering the effects of wind, Eulerian current, water-ice form stress, and the wave radiation stress. The mathematical models developed here, will be tested with available experimental data.

Keywords: Ice Floe, Marginal Ice Zone, MIZ, Ice Drift, Ocean Surface Waves, Energy Balance Equation, Wave Spectrum, Ice Floe Velocity Components, Ice Floe Trajectories

1. INTRODUCTION

An ice floe is a floating chunk of sea ice that is less than 10 kilometers in its greatest dimension. Marginal Ice Zone (MIZ) is an interfacial region of ice floes which forms at the boundary of open water and the continuous ice pack. Wadhams (1986) describes MIZ as “that part of the ice cover which is close enough to the open ocean boundary to be affected by its presence”. This definition is generally applied to that region of ice pack which is significantly affected by the ocean swell. The MIZ is essentially an area of enhanced ice drift and deformation. Figure 1 depicts a typical MIZ situation including ice floes and wave induced ice fracture at the ice edge. In the Antarctic, MIZ region may extend hundreds of kilometres from the ice edge.

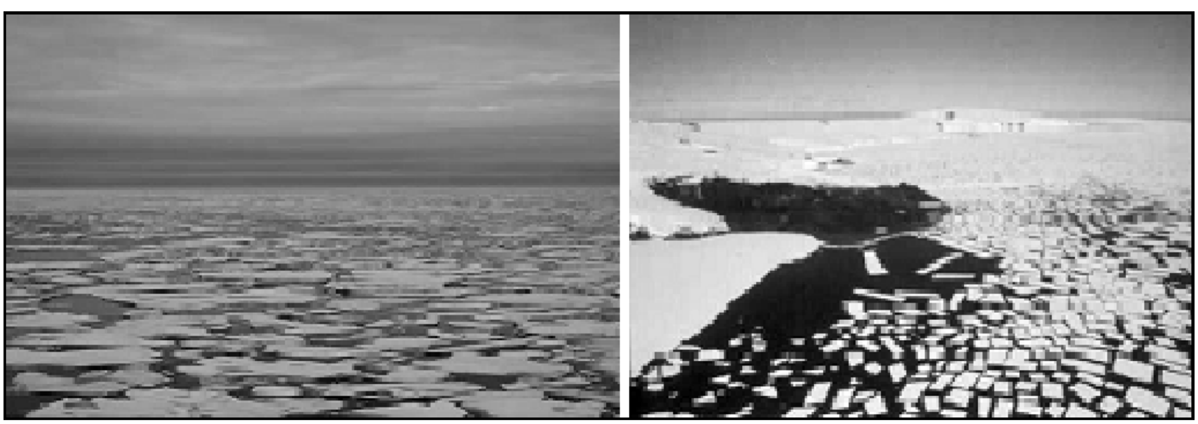

Figure 1: Marginal Ice Zone in the Antarctic, and Wave Induced Ice Fracture at the Ice Edge. (Courtesy: National Science Foundation, USA, and Squire *et al.* [8])

Interactions between ice, wind, waves, and current in the marginal ice zone can dramatically move the ice distribution of the ice floes. Moreover, compared to the ice free situation, currents, waves, and the associated planetary boundary layer are altered by the MIZ. This paper considers the mathematical model for the ice edge and ice floe trajectories which is based on a balance equation for forces due to wind, waves, and currents impinging on the ice, as described by Tang and Fissel (1991)[10], Steele *et al.* (1989)[9] and Jenkins (1989)[2].

Ocean surface waves are central to the atmosphere-ocean coupling dynamics at the air-sea interface. The dominating physical processes that determine ocean surface waves are, input of energy due to wind S_{in}, nonlinear transfer between spectral components due to wave-wave interactions S_{nl}, and energy dissipation due to white capping and wave breaking S_{ds}. Operational wave models combine these processes in the energy balance equation, which may be written as

$$\frac{\partial E(f,\theta)}{\partial t} + C_g \cdot \nabla E(f,\theta) = S_{in} + S_{ds} + S_{nl} \tag{1}$$

where the two dimensional wave spectrum $E(f, \theta)$ is a function of frequency f, direction θ, time t, and position $\mathbf{x}$ and where $\mathbf{C}_g$ is the group velocity. Examples of how S_{in}, S_{ds}, and S_{nl} may be parameterized are presented in Hasselmann et al. [1] and Perrie and Hu [4]. Following Hasselmann's approach, we computed various parameters of a JONSWAP Wave Energy Spectrum. Figure 2 shows a JONSWAP Wave Energy Spectrum plot with peak frequency of 0.3 Hz, at 10 m/s wind speed developed over 3.25 hours.

This paper systematically illustrates the mathematical formulations of the ice floe drift velocity due to wind forcing effects, the Eulerian currents, water ice form stress and the wave radiation. The ice floe trajectories and the ice floe velocity fields are illustrated in a clear cut way. Linearizing the governing equations with zero initial conditions and using the Laplace transform method, we have obtained simple solutions to simulate the real field

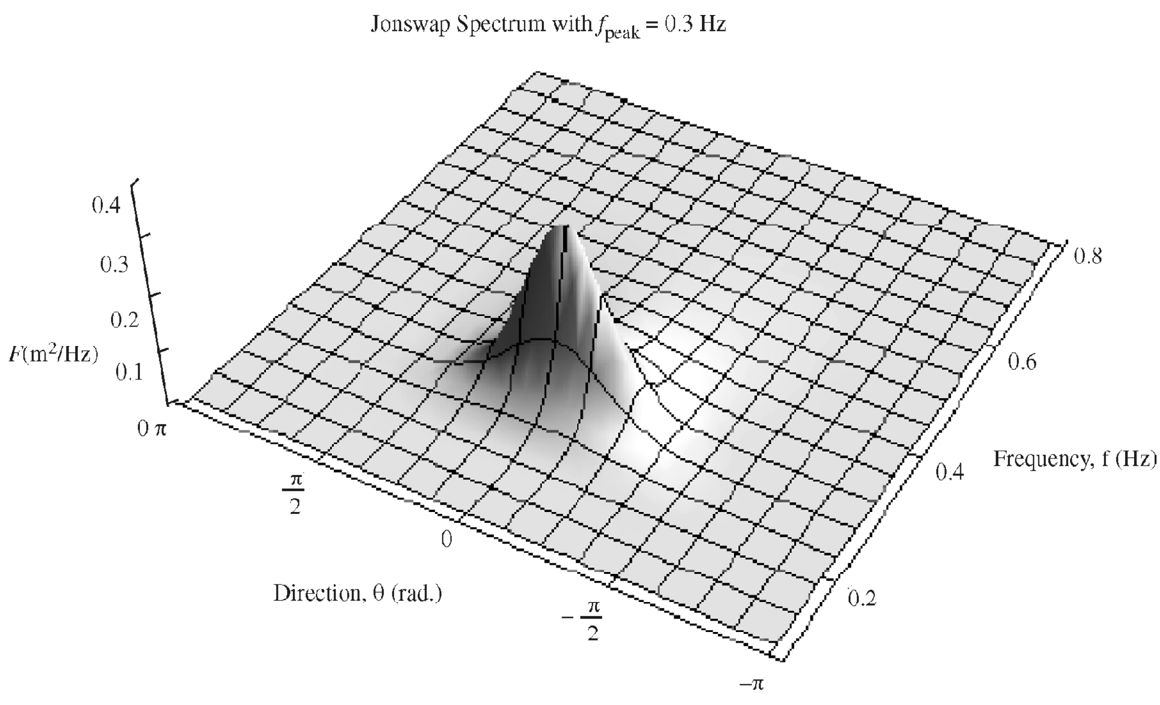

Figure 2: JONSWAP Wave Energy Spectrum

conditions. Graphical solutions are displayed in case of the external wind stress which causes the ice floe drift from one place to another. These highly simplified results seem to agree quite well with real field data. This investigation should be treated as a benchmark study to acquire scientific estimations of the real world situations.

2. MATHEMATICAL FORMULATION

The mathematical equation of motion for an ice floe in the marginal ice zone due to wind, waves, and current can be written as

$$m\left(\frac{\partial u'}{\partial t}+f\times\mathbf{u}'\right)=A\left(\tau_{\text{air}}^{\text{skin}}+\tau_{\text{water}}^{\text{skin}}+\tau_{\text{air}}^{\text{form}}+\tau_{\text{water}}^{\text{form}}+\tau_{\text{rad}}^{\text{wave}}\right)$$

$$-\text{mgr}\,\nabla\xi+\mathbf{F} \tag{2}$$

where g is the acceleration due to gravity, m is the ice mass, A is the ice floe surface area, ξ is the sea surface elevation, $\mathbf{F}$ is the ice internal stress gradient, $\mathbf{u}'$ is the absolute ice velocity, $\tau_{\text{air}}^{\text{skin}}$ is the wind stress on the top surface of the ice floe, $\tau_{\text{water}}^{\text{skin}}$ is the water stress on the bottom surface of the ice floe, $\tau_{\text{air}}^{\text{form}}$ is the air-ice form stress, $\tau_{\text{water}}^{\text{form}}$ is the water-ice form stress, and $\tau_{\text{rad}}^{\text{wave}}$ is the wave radiation pressure.

If the ice concentration is low, the internal stress gradient $\mathbf{F}$ is essentially zero. Replacing $-mg\nabla\xi$ by the geostropic current $mf\times\mathbf{Ug}$ and neglecting $\mathbf{F}$, equation (2) can be expressed as

$$m\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{f}\times\mathbf{u}\right)=A\left(\tau_{\text{air}}^{\text{skin}}+\tau_{\text{water}}^{\text{skin}}+\tau_{\text{air}}^{\text{form}}+\tau_{\text{water}}^{\text{form}}+\tau_{\text{rad}}^{\text{wave}}\right), \tag{3}$$

which gives the ice floe velocity, $\mathbf{u}=\mathbf{u}'-\mathbf{Ug}$, relative to the geostropic current $\mathbf{Ug}$.

The stresses $\tau_{\text{air}}^{\text{skin}}$ and $\tau_{\text{water}}^{\text{skin}}$ are caused by skin friction. It is assumed that $|\tau_{\text{air}}^{\text{form}}|<<|\tau_{\text{water}}^{\text{form}}|$. Therefore, the final time-dependent equation of motion for an ice floe becomes

$$\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{f}\times\mathbf{u}\right)=\frac{A}{m}\left(\tau_{\text{air}}^{\text{skin}}+\tau_{\text{water}}^{\text{skin}}+\tau_{\text{water}}^{\text{form}}+\tau_{\text{rad}}^{\text{wave}}\right), \tag{4}$$

To solve this equation, we must know the mathematical expressions of $\tau_{\text{air}}^{\text{skin}}$, $\tau_{\text{water}}^{\text{skin}}$ $\tau_{\text{water}}^{\text{form}}$ and $\tau_{\text{rad}}^{\text{wave}}$. Perrie and Hu [4] (1997) have described these expressions with various types of parameters effecting these stresses. We will produce the mathematical relationship as given below.

In a calm sea condition, the drift of the ice floe can be assumed to be only due to the eect of the wind. The air-ice-skin friction stress $\tau_{\text{air}}^{\text{skin}}$, thus, is usually represented by a quadratic formula in terms of the wind speed $\mathbf{U}_{10}$,

$$\tau_{\text{air}}^{\text{skin}}=\rho_a C_{ai}^s\left|\mathbf{U}_{10}-\mathbf{u}\right|(\mathbf{U}_{10}-\mathbf{u}) \tag{5}$$

where ρ_a is the air density and C_{ai}^s is the air-ice-skin friction drag coffecient. Following Steele et al [9], we used $C_{ai}^s\approx 3\times 10^{-3}$.

Analogous to (5), the water-ice-skin friction stress $\tau_{\text{water}}^{\text{skin}}$ may be represented as

$$\tau_{water}^{skin}\equiv\rho_w C_{wi}^s|\mathbf{u}-\mathbf{U_e}|(\mathbf{u}-\mathbf{U_e}) \tag{6}$$

where ρ_w is the water density, C_{wi}^s is the water-ice-skin friction drag coefficient, and $\mathbf{U_e}$ is the Eulerian current at the z grid point just below $z=-D$, where D is the ice draft.

The water-ice form stress $\tau_{\text{water}}^{\text{form}}$ describes the normal force acting on the leading face of an ice floe as it moves through the water at relative velocity $\mathbf{u}-\mathbf{U_e}$. Following Steele *et al.* [9], the general form for $\tau_{\text{water}}^{\text{form}}$ is given by

$$\tau_{\text{water}}^{\text{form}}=-\frac{2}{\pi}\rho_w C_{wi}^f\frac{D}{L}\left|\mathbf{u}-\bar{\mathbf{U}}_\mathbf{e}\right|(\mathbf{u}-\bar{\mathbf{U}}_\mathbf{e})\Gamma \tag{7}$$

where C_{wi}^f is the water-ice form drag coefficient, L is the ice floe diameter, DL is the ice floe cross-section, and $\mathbf{U_e}$ is the Eulerian current, vertically averaged over the leading face of the floe

$$\bar{\mathbf{U}}_{\mathbf{e}} = \frac{1}{D}\int_{-D}^{0} \mathbf{U}_{\mathbf{e}}(z)dz. \tag{8}$$

The parameter Γ describes the reduction in drag due to the wake effect

$$\Gamma = \left(1 - \sqrt{\frac{D}{L_f}}\right)^2, \tag{9}$$

where L_f is the effective average fetch between ice floes.

The wave radiation pressure $\tau_{\text{rad}}^{\text{wave}}$ represents the force exerted on ice floes by reflected and diffracted waves. Following Wadhams [11] and Steele et al. [9], the force on the floe of diameter L, due to perfect reflection of surface waves, is

$$F_{\text{rad}} = \frac{1}{2}\rho_w g a^2 L, \tag{10}$$

where a is the wave amplitude. Equation (10) can be reduced to

$$\tau_{\text{rad}}^{\text{wave}} = 3.2 \times 10^{-4}\, \frac{\rho_w}{\rho_a}\sqrt{\frac{1}{\pi}\left(\frac{1}{fi} - 1\right)}\,\tau, \tag{11}$$

where fi is the ice cover concentration. In arriving at the above result, we have used the parametric values of a^2 and L_f as follows:

$$a^2 = 3.2 \times 10^{-4}\left(\frac{\tau}{\rho_a}\right)\left(\frac{L_f}{g}\right)$$

$$L_f = \frac{L}{2}\sqrt{\pi\left(\frac{1}{fi} - 1\right)}$$

Thus, $\tau_{\text{rad}}^{\text{wave}}$ depends on wind stress $\tau = \rho_a C_D U_{10}^2$ and ice cover concentration.

3. ICE FLOE DRIFT DUE TO WIND STRESS: MODEL I

We simplify the governing partial differential equation with the initial condition and obtain,

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= \alpha U_{10} - \alpha u \\ \frac{\partial v}{\partial t} + fu &= -\alpha v \end{aligned} \tag{12}$$

where $\mathbf{u} = (u, v, 0)$ are the velocity components of the ice floe in a horizontal plane, $f = (0, 0, f)$ are the Coriolis force components, and $\alpha = \frac{A}{m}\rho_a C_{ai}^s |\mathbf{U}_{10} - \mathbf{u}|$. Also, we have $(\mathbf{U}_{10} - \mathbf{u}) = (\mathbf{U}_{10} - u, -v, 0)$. The wind velocity vector $\mathbf{U}_{10}$ is assumed to be parallel to the positive x-direction. Here we assume that $U_{10} >> |\mathbf{u}|$, i.e., the wind speed is much greater than the that of the ice drift and so we can safely assume that is a constant parameter. The initial conditions at $t = 0$ are assumed as (when there is no wind):

$$u(0) = 0,\ v(0) = 0. \tag{13}$$

Using Laplace transform $\mathcal{L}\{u\} = \int_0^\infty u(t)e^{-st}dt$ and $\mathcal{L}\{v\} = \int_0^\infty v(t)e^{-st}dt$ with the initial conditions (13), the simultaneous differential equations (12) can be transformed as

$$(s+\alpha)\mathcal{L}\{u\} - f\mathcal{L}\{v\} = \frac{\alpha U_{10}}{s}$$

$$f\mathcal{L}\{u\} + (s+\alpha)\mathcal{L}\{v\} = 0$$

Solving these two algebraic equations by Cramer's rule, and using the residue calculus of complex variables, we obtain the solutions as

$$u(t) = (\alpha U_{10})\left(\frac{\alpha}{\alpha^2+f^2} + e^{-\alpha t}\left\{\frac{f\sin ft - \alpha\cos ft}{\alpha^2+f^2}\right\}\right) \tag{14}$$

$$v(t) = (\alpha U_{10})\left(\frac{f}{\alpha^2+f^2} + e^{-\alpha t}\left\{\frac{f\cos ft - \alpha\sin ft}{\alpha^2+f^2}\right\}\right) \tag{15}$$

It is easily verified that at $t = 0$, the initial conditions are satisfied. For large time, when t goes to infinity, the drift velocity becomes

$$u(t) = \frac{\alpha^2 U_{10}}{\alpha^2+f^2} \quad \text{and} \quad v(t) = -\frac{\alpha f U_{10}}{\alpha^2+f^2}.$$

This implies that the ice floe moves with a constant speed for large time as long as the wind speed persists. We represent the equations (14) and (15) in non-dimensional forms as follows:

$$U - a = e^{-\alpha t}\{b\sin ft - a\cos ft\} \tag{16}$$

$$V + b = e^{-\alpha t}\{b\cos ft + a\sin ft\} \tag{17}$$

where U, V, a, and b are given by

$$\mathrm{U} = \frac{u}{U_{10}}, \quad V = \frac{v}{U_{10}}, \quad a = \frac{\alpha^2}{\alpha^2+f^2} \quad \text{and} \quad \frac{\alpha f}{\alpha^2+f^2}.$$

Thus, the velocity field of the ice floe, i.e., the $U - V$ plot, can be described by the following circular spiral type solution as a function of time

$$(U-a)^2 + (V+b)^2 = (a^2+b^2)e^{-2\alpha t} \tag{18}$$

The equation (18) reveals that the radius of the circle at $t = 0$ becomes simply $\sqrt{a^2+b^2}$ but when the time progresses the radius starts to decrease exponentially and at very large time, i.e., when $t \to \infty$, the radius of the circle becomes zero implying that the circle shrinks to zero at the center $(a, -b)$. This simulated behavior of the drift of the ice floe is not unusual in a real field situation. The graphical representation of our mathematical model in Figure 3 confirms this analytical conjecture of the velocity field of the ice floe. The computations were carried out by assuming a cylindrical shape for ice floe with diameter L and a thickness of T. The air-ice skin friction drag coefficient was set to $C_{ai}^s = 3\times10^{-3}$, following Steele et al. [9], the Coriolis parameter was set to $f = 1.07 \times 10^{-4}s^{-1}$, and the wind speeds at $10m$ above the surface (U_{10}) were varied between $10m/s$ and $25m/s$ with $5m/s$ increments.

To further verify this conjecture, we determine the ice floe trajectory. We replace u and v in terms of derivatives of x and y with respect to time, such that $u = \frac{\partial x}{\partial t}$ and $v = \frac{\partial y}{\partial t}$. Thus (12) will take the following form:

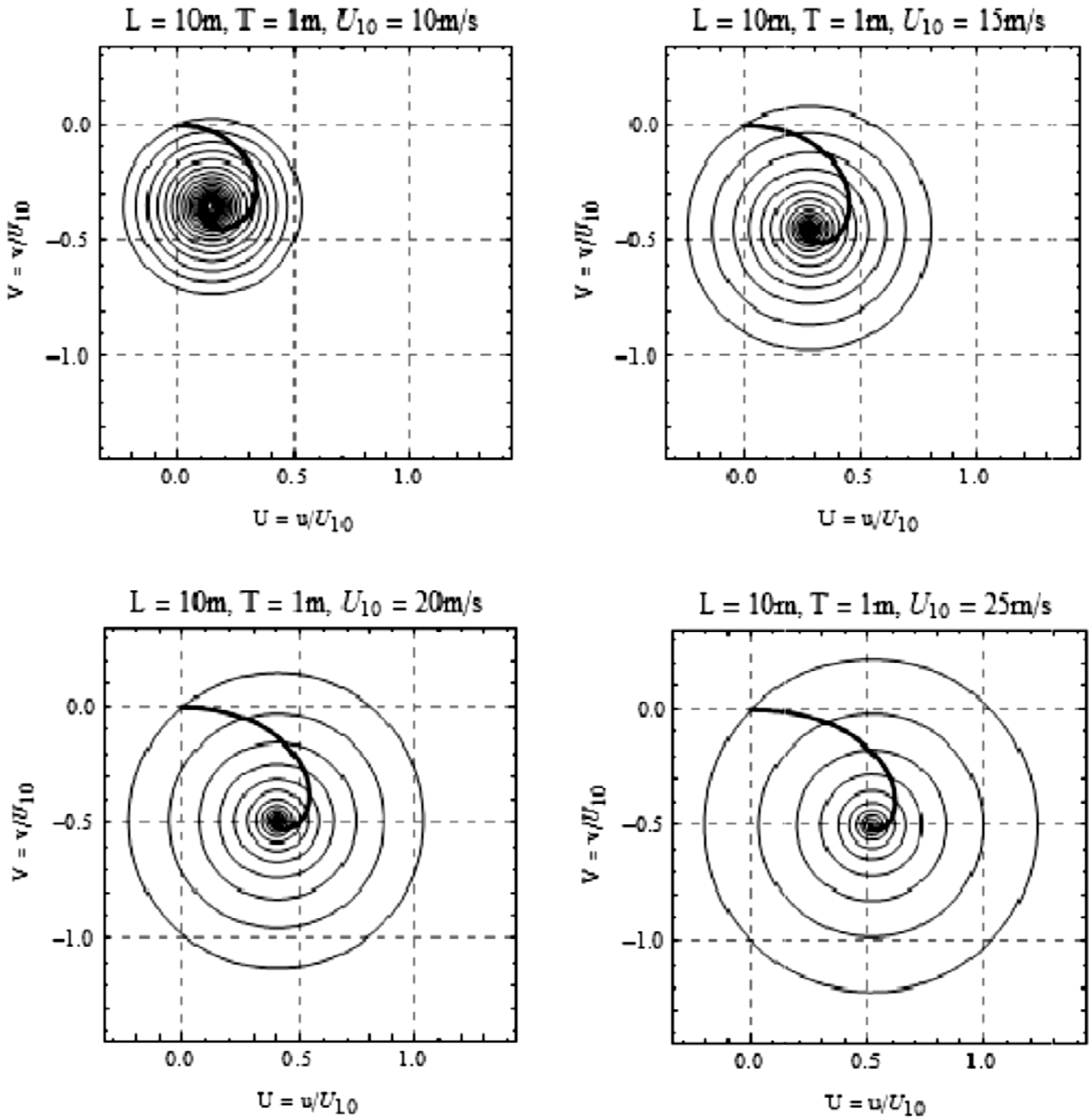

Figure 3: Non-dimensional Ice Floe Velocity Field and Phase Diagrams

$$\frac{\partial^2 x}{\partial t^2} - f\frac{\partial y}{\partial t} = \alpha U_{10} - \alpha\frac{\partial x}{\partial t} \tag{19}$$

$$\frac{\partial^2 y}{\partial t^2} + f\frac{\partial x}{\partial t} = -\alpha\frac{\partial y}{\partial t} \tag{20}$$

The initial conditions are $x(0) = 0$, $\frac{\partial x}{\partial t}(0) = 0$. Also, $y(0) = 0$, $\frac{\partial y}{\partial t}.(0) = 0$. The solutions can be obtained by using the Laplace transform method. Hence,

$$(s^2 + s\alpha)\ \mathcal{L}\{x\} - fs\mathcal{L}\{y\} = \frac{\alpha U_{10}}{s}$$

$$fs\ \mathcal{L}\{x\} + (s_2 + s)\ \mathcal{L}\ \{y\} = 0.$$

Solving these two equations by Cramer's rule and using the residue calculus of complex variables, we obtain the solutions as

$$\frac{x(t)}{(\alpha U_{10})} = -\frac{\alpha^2 - f^2}{(\alpha^2 + f^2)^2} + \frac{\alpha t}{\alpha^2 + f^2} + e^{\alpha t}\left\{\frac{(\alpha^2 + f^2)\cos ft - 2\alpha f \sin ft}{(\alpha^2 + f^2)^2}\right\} \tag{21}$$

$$\frac{y(t)}{(\alpha U_{10})} = \frac{2\alpha f}{(\alpha^2 + f^2)^2} - \frac{ft}{\alpha^2 + f^2} - e^{\alpha t}\left\{\frac{(2\alpha f)\cos ft + (\alpha^2 - f^2)\sin ft}{(\alpha^2 + f^2)^2}\right\} \tag{22}$$

The non-dimensional forms of (21) and (22) can be written as:

$$X(t) = -\frac{\alpha^2 - f^2}{(\alpha^2 + f^2)} + \alpha t + e^{-\alpha t}\left\{\frac{(\alpha^2 - f^2)\cos ft - 2\alpha f \sin ft}{(\alpha^2 + f^2)}\right\} \tag{23}$$

$$Y(t) = -\frac{2\alpha f}{(\alpha^2 + f^2)} - ft - e^{-\alpha t}\left\{\frac{(2\alpha f)\cos ft + (\alpha^2 - f^2)\sin ft}{(\alpha^2 + f^2)}\right\} \tag{24}$$

where

$$X(t) = \frac{x(t)}{\alpha U_{10}/(\alpha^2 + f^2)}$$

and

$$Y(t) = \frac{y(t)}{\alpha U_{10}/(\alpha^2 + f^2)}.$$

With these definitions, the ice floe trajectories can be obtained as

$$(X - c)^2 + (Y - d)^2 = e^{-2\alpha t} \tag{25}$$

where $c = \frac{\alpha^2 + f^2}{\alpha^2 + f^2} + \alpha t$ and $d = \frac{2\alpha f}{\alpha^2 + f^2} - ft$, respectively. It can be easily seen that the ice floe path is a circle with the center (c, d) and radius $e^{-\alpha t}$. The parameters c, d and the radius are all dependent on time t. Thus the ice floe will move in a circular path with exponentially decreasing radius with respect to time. Further more, for large time, the floe trajectory will follow a linear path with the linearly dependent coordinates of the center of the circle with respect to time. At the initial stage, i.e., at $t = 0$, the trajectory will be an unit circle with center at $\left(\frac{\alpha^2 - f^2}{\alpha^2 + f^2}, \frac{2\alpha f}{\alpha^2 + f^2}\right)$. The X–coordinate may be positive or negative according to $\alpha^2 < f^2$ or $\alpha^2 > f^2$, respectively. However, the Y–coordinate is always a negative number. The graphical simulations of the non-dimensional trajectory of the ice floe with varying wind speeds and their corresponding phase diagrams following our derived mathematical formulations are shown in Figure 4. The trajectories are circular spirals starting with a unit circle at $t = 0$ and ending with a point circle at $t \to \infty$; but the center is moving according to law of order $O(\alpha t)$ such that $X \equiv \alpha t$ and $Y \equiv -ft$. Our computation shows that as time passes, the orbital motion of the ice floe, due to the earth's angular motion, gravitational pull, and the constant wind effect, eventually becomes linear. Also, the result of this simplified approach tends to display more displacement along the y-axis than that along the x-axis. However, this can be corrected with further mathematical computation. One such computation is given below.

4 ICE FLOE DRIFT DUE TO WIND STRESS: MODEL II

The equation of motion of an ice floe (5) can be rewritten in the form,

$$\begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \\ 0 \end{bmatrix} + \begin{bmatrix} i & j & k \\ 0 & 0 & f \\ u & v & 0 \end{bmatrix} = A\rho_a C_{ai}^s \begin{bmatrix} \sqrt{(U_{10}-u)^2+v^2}\,(U_{10}-u) \\ -v\sqrt{(U_{10}-u)^2+v^2} \\ 0 \end{bmatrix} \tag{26}$$

We obtain,

$$\begin{bmatrix} \frac{\partial u}{\partial t} - fv = \frac{A\rho_a C_{ai}^s}{m}(U_{10}-u)^2\sqrt{1+\frac{v^2}{(U_{10}-u)^2}} \\ \frac{\partial v}{\partial t} + fu = \frac{A\rho_a C_{ai}^s}{m}(U_{10}-u)\sqrt{1+\frac{v^2}{(U_{10}-u)^2}} \end{bmatrix} \tag{27}$$

Using the binomial expansion, the above equation can be written as,

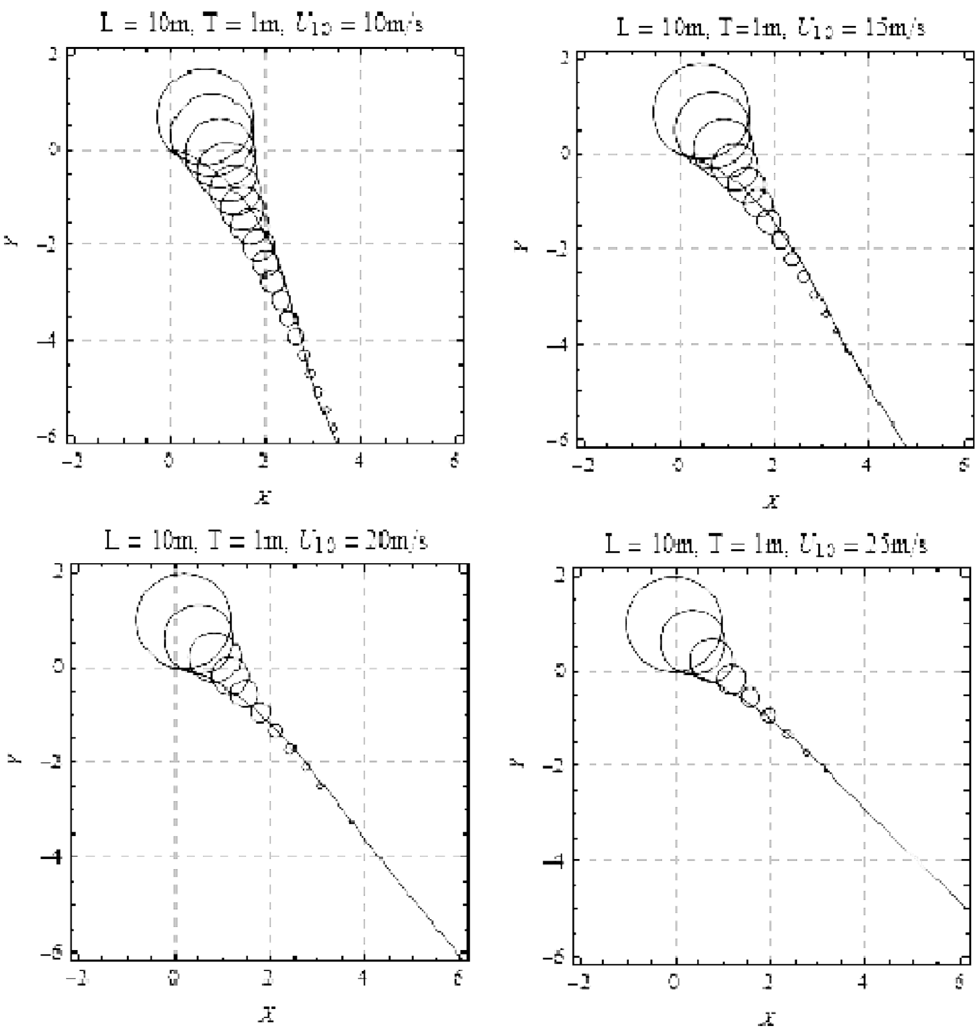

Figure 4: Non-dimensional Ice Floe Trajectories with Phase Diagrams

$$\begin{cases} \frac{\partial u}{\partial t} - fv = \frac{A\rho_a C_{ai}^s}{m}(U_{10}-u)^2\left[1+\frac{1}{2}\left(\frac{v}{U_{10}-u}\right)^2 - \frac{1}{8}\left(\frac{v}{U_{10}-u}\right)^4 + \frac{1}{16}\left(\frac{v}{U_{10}-u}\right)^6 + \dots\right] \\ \frac{\partial v}{\partial t} + fu = -\frac{A\rho_a C_{ai}^s}{m}v(U_{10}-u)\left[1+\frac{1}{2}\left(\frac{v}{U_{10}-u}\right)^2 - \frac{1}{8}\left(\frac{v}{U_{10}-u}\right)^4 + \frac{1}{16}\left(\frac{v}{U_{10}-u}\right)^6 + \dots\right] \end{cases} \tag{28}$$

Considering wind speed is much greater than that of the ice floe drift, the above expansion can be linearized in the form,

$$\begin{cases} \frac{\partial u}{\partial t} - fv = \frac{A\rho_a C_{ai}^s U_{10}}{m}(U_{10}-2u) \\ \frac{\partial v}{\partial t} + fu = -\frac{A\rho_a C_{ai}^s U_{10}}{m}v \end{cases} \tag{29}$$

Following same approach as before, we represent equation (29) in non-dimensional form as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - fv = \alpha(1-2U) \\ \frac{\partial v}{\partial t} + fu = -\alpha V \end{cases} \tag{30}$$

Using Laplace transform with initial conditions, equation (30) can be transformed as

$$\begin{cases} (s+2\alpha)\mathcal{L}(U) - f\mathcal{L}(V) = \frac{\alpha}{s} \\ f\mathcal{L}(U) + (s+\alpha)\mathcal{L}(V) = 0 \end{cases} \tag{31}$$

Solving these two algebraic equations, we obtain,

$$\begin{cases} \mathcal{L}(U) = \frac{\alpha(s+\alpha)}{s\left[(s+\alpha)(s+2\alpha)+f^2\right]} \\ \mathcal{L}(V) = \frac{\alpha f}{s[(s+\alpha)(s+2\alpha)+f^2]} \end{cases} \tag{32}$$

Using the residue calculus of complex variables, we obtain the Laplace inverse as

$$\begin{cases} U = \frac{\alpha^2}{2\alpha^2+f^2} + \frac{e^{-3\alpha t/2}}{\beta}\left[-\frac{\beta\alpha^2}{2\alpha^2+f^2}\cos\left(\frac{\beta}{2}t\right) + \frac{\alpha(\alpha^2+2f^2)}{2\alpha^2+f^2}\sin\left(\frac{\beta}{2}t\right)\right] \\ V = \frac{\alpha f}{2\alpha^2+f^2} + \frac{e^{-3\alpha t/2}}{\beta}\left[\frac{\alpha\beta f}{2\alpha^2+f^2}\cos\left(\frac{\beta}{2}t\right) + \frac{3\alpha^2 f}{2\alpha^2+f^2}\sin\left(\frac{\beta}{2}t\right)\right] \end{cases} \tag{33}$$

where $\alpha^2 - 4f^2 = -\beta^2 < 0$.

The non-dimensional velocity components of the ice floe with the simplifications introduced in Model-I and Model-II have been computed and shown in fig. 5. Interestingly, Model-II shows further improvements and its computational results tend to agree more closely with those of Perrie and Hu [4].

5. ICE FLOE DRIFT DUE TO EULERIAN CURRENT

An ice floe can be drifted by the effects of Eulerian currents. In this situation, the mathematical equations will be considered as follows:

$$\frac{\partial u}{\partial t} + \mathbf{f} \times \mathbf{u} = \beta(\mathbf{u} - \mathbf{U_e}) \tag{34}$$

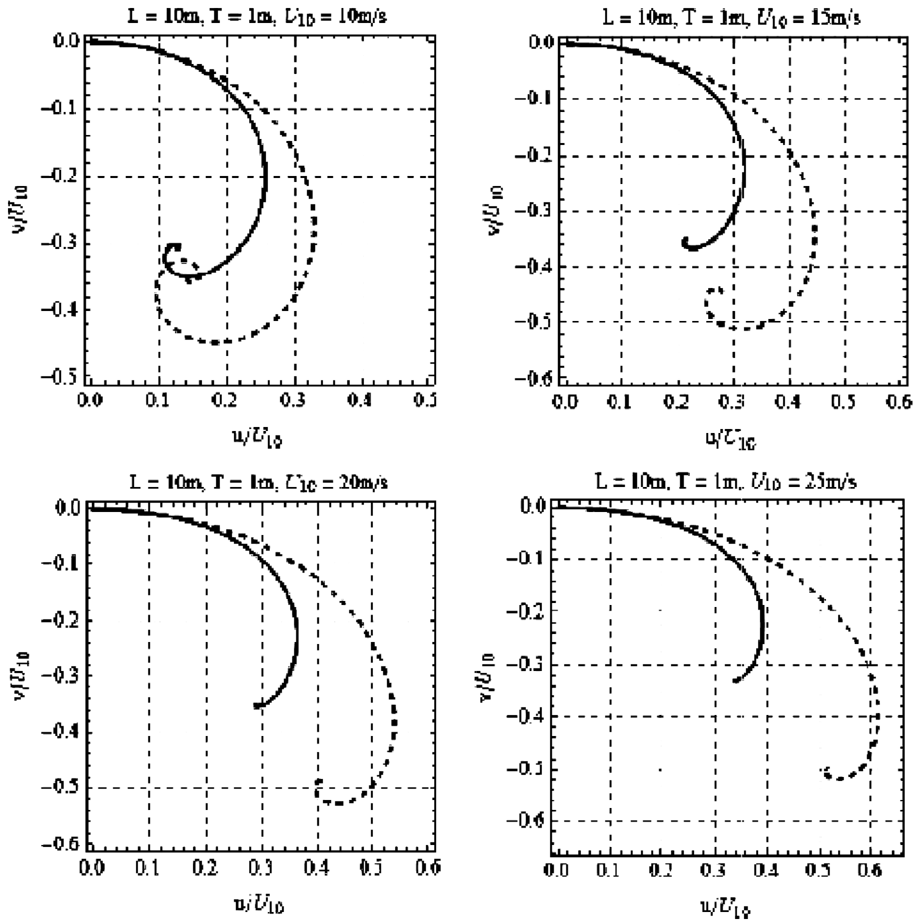

Figure 5: Non-dimensional Velocity Components of the Ice Floes (Dashed Line: Model I, Solid Line: Model-II.

where $\beta = -(\frac{A}{m})\rho_w C_w^s\,|u - \mathbf{U}_e|$ and the Eulerian current is $\mathbf{U}_e = (U_e, V_e, o)$. Eulerian current is usually function of z and t. Thus equation (34) can be cast in to complex variable $w(t) = u(t) + iv(t)$ and $W(t) = U_e(t) + iV_e(t)$ such that it becomes

$$\frac{\partial w}{\partial t} + (if)w = \beta(w - W_e) \tag{35}$$

Taking the Laplace transform as before, and after reduction yields

$$L\{w\} = -\frac{\beta}{s - (\beta - if)}\mathcal{L}(W_e) \tag{36}$$

Using convolution integral the inverse Laplace transform of (36) can be obtained as

$$w(t) = -\beta\int_0^t e^{(\beta - if)(t-\lambda)} W_e(\lambda)d\lambda \tag{37}$$

Now equating the real and the imaginary parts yields

$$u(t) = -\beta\int_0^t e^{\beta(t-\lambda)}\left[U_e(\lambda)\cos f(t+\lambda) + V_e(\lambda)\sin f(t-\lambda)\right]d\lambda \tag{38}$$

$$v(t) = -\beta\int_0^t e^{\beta(t-\lambda)}\left[V_e(\lambda)\cos f(t-\lambda) + U_e(\lambda)\sin f(t-\lambda)\right]d\lambda\,. \tag{39}$$

It is evident that the Eulerian currents are functions of time t. The velocity field of the ice floe can be completely determined provided the Eulerian currents are known, a priori. Theses functions have been determined by Rahman [6]. Thus substituting the mathematical expressions in the above two equations, we can determine the velocity field of an ice floe.

The ice floe trajectories can then be obtained by integration of $u(t)$ and $v(t)$ with respect to time. Because $u = \frac{\partial x}{\partial t}$, and $v = \frac{\partial y}{\partial t}$, respectively, which yield

$$x(t) = -\beta\int_0^t\int_0^{t'} e^{\beta(t'-\lambda)}[U_e(\lambda)\cos f(t'-\lambda) + V_e(\lambda)\sin f(t'-\lambda)]d\lambda dt' \tag{40}$$

$$y(t) = -\beta\int_0^t\int_0^{t'} e^{\beta(t'-\lambda)}[V_e(\lambda)\cos f(t'-\lambda) + U_e(\lambda)\sin f(t'-\lambda)]d\lambda dt' . \tag{41}$$

6. ICE FLOE DRIFT DUE TO WATER-ICE FORM STRESS

The eect of the water-ice form stress may be responsible in drifting an ice floe from one place to another. In this situation, the mathematical equations will be considered as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f}\times\mathbf{u} = \gamma(\mathbf{u} - \bar{\mathbf{U}}_\mathbf{e}) \tag{42}$$

where $\gamma = -\left(\frac{A}{m}\right)\frac{2}{\pi}\rho_w C_w^f\left|\mathbf{u} - \bar{\mathbf{U}}_\mathbf{e}\right|\Gamma$ and the averaged Eulerian current is $\bar{\mathbf{U}}_\mathbf{e} = (\bar{U}_e, \bar{V}_e, 0)$. This averaged Eulerian current is assumed to be constant. Thus equation (42) can be cast in to complex variable $w(t) = u(t) + iv(t)$ and $\bar{W} = \bar{U}_e, i\bar{V}_e$ such that it becomes

$$\frac{\partial w}{\partial t} + (if)w = \gamma(w - \bar{W}_e) \tag{43}$$

Taking the Laplace transform as before, and after reduction yields

$$\mathcal{L}\{w\} = -\frac{\gamma\bar{W}_e}{s(s-(\gamma - if))} \tag{44}$$

The Laplace inverse of (44) can be obtained as

$$w(t) = -\frac{\gamma\bar{W}_e}{\gamma - if}\left(e^{(\gamma - if)t} - 1\right). \tag{45}$$

Now equating the real and the imaginary parts yields

$$u(t) = p - e^{\gamma t}\{p\cos ft + q\sin ft\} \tag{46}$$

$$v(t) = q - e^{\gamma t}\{q\cos ft - p\sin ft\} \tag{47}$$

where $p = \frac{\gamma^2\bar{U}_e - \gamma f\bar{V}_e}{\gamma^2 + f^2}$, and $q = \frac{\gamma f\bar{U}_e + \gamma^2\bar{V}_e}{\gamma^2 + f^2}$. It can be easily determined the velocity field as

$$(u - pt)^2 + (v - qt)^2 = (p^2 + q^2)e^{2\gamma t} \tag{48}$$

The ice floe trajectories are obtained by integration of $u(t)$ and $v(t)$ with respect to time, as before. Because $u = \frac{\partial x}{\partial t}$, and $v = \frac{\partial y}{\partial t}$, respectively, which yield

$$x(t) = pt - \int_0^t e^{\gamma t}\,(p\cos ft + q\sin ft)\,dt \tag{49}$$

$$y(t) = qt - \int_0^t e^{\gamma t}\,(\mathrm{q}\cos ft - p\sin ft)\,dt \tag{50}$$

Performing these integrations the equations of the ice floe paths can be obtained as

$$x(t) = pt - \frac{p\gamma - qf}{\gamma^2 + f^2} + \frac{e^{\gamma t}}{\gamma^2 + f^2}\{(p\gamma - qf)\cos ft + (pf + q\gamma)\sin ft\} \tag{51}$$

$$y(t) = qt - \frac{pf + q\gamma}{\gamma^2 + f^2} + \frac{e^{\gamma t}}{\gamma^2 + f^2}\{pf + q\gamma)\cos ft - (p - qf)\sin ft\} \tag{52}$$

It can be easily seen that at $t = 0$ the initial conditions are satisfied. The ice floe trajectories are given by

$$(x - (pt - A))^2 + (y - (qt - B))^2 = (A^2 + B^2)\,e^{2\gamma t} \tag{53}$$

where $A = \frac{p\gamma - qf}{\gamma^2 + f^2}$, and $B = \frac{pf - q\gamma}{\gamma^2 + f^2}$. The paths are circles of radii $\sqrt{A^2 + B^2 e^{\gamma t}}$ with centers at $((pt - A), (qt - B))$. Both the radius and the center are function of time, and therefore, the trajectories will be circular spirals.

7. ICE FLOE DRIFT DUE TO WAVE RADIATION PRESSURE

In this situation the external force is considered to be the wave radiation stress. This force depends primarily on the ice floe concentration and the wind velocity squared. The mathematical model is similar to the above case except that stress is constant but not dependent on the ice floe velocity. Hence, the governing equation can be written in complex form as

$$\frac{\partial w}{\partial t} + (if)w = \gamma \tag{54}$$

Taking the Laplace transform and using the initial condition $w(0) = 0$, and after inversion, we obtain

$$w(t) = \left(\frac{\lambda}{f}\right)\{\sin(ft) - i(1 - \cos(ft))\} \tag{55}$$

where = $\lambda = \frac{A}{m}\tau_{\mathrm{rad}}^{\mathrm{wave}}$. The real and imaginary parts yield

$$u(t) = \left(\frac{\lambda}{f}\right)\sin(ft) \tag{56}$$

$$v(t) = \left(\frac{\lambda}{f}\right)(\cos(ft) - 1) \tag{57}$$

from which the velocity field can be obtained as

$$u^2 + \left(v + \frac{\lambda}{f}\right)^2 = \left(\frac{\lambda}{f}\right)^2. \tag{58}$$

The ice floe trajectories are given by

$$x(t) = \frac{\lambda}{f^2}[1 - \cos(ft)] \tag{59}$$

$$y(t) = \frac{\lambda}{f^2}[\sin(ft) - (ft)] \tag{60}$$

and the equation is

$$(x - a)^2 + (y + b)^2 = a^2,$$

where $a = \frac{\lambda}{f^2}$ and $b = \frac{\lambda t}{f}$.

It is worth noting that the radius is a constant but the center is a function of time. The trajectories are circular paths.

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