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VARIATIONAL ITERATION TECHNIQUE FOR SOLVING BOUSSINESQ EQUATIONS

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ABSTRACT: In this paper, we implement a relatively new analytical technique which is called as the variational iteration method for the solution of Boussinesq equations which commonly describe the propagation of small amplitude long wave in several physical contents. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. Several examples are given to verify the efficiency of the suggested technique. The fact that variational iteration method solves nonlinear problems without using the so-called Adomian's polynomials is a clear advantage of this technique over the decomposition method.

***Keywords*:** Variational iteration method, Boussinesq equations, nonlinear equations, error estimates.

1. INTRODUCTION

The importance of soliton producing nonlinear wave equations is well understood among theoretical physicists and applied mathematicians. An equation admitting soliton solutions which has received comparatively little attention in the literature is

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxx} \tag{1}$$

It is referred as the "bad" Boussinseq or the nonlinear beam equation and describes the motion of long waves in shallow water under gravity in one-dimentional nonlinear lattices; see [1-5, 11, 13, 14, 24, 25]. Equation (1) admits the solitary wave solution

$$u\,(x,\ t) = A\ Sec\ h^2(\sqrt{A/6}(x-ct)), \tag{2}$$

where A and $c = \pm\sqrt{1+2A/3}$ are the amplitude and the speed of the solitary wave, respectively. These features of equation. (1) are quite reminiscent of the properties of the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{3}$$

in that they both poses solitary wave solutions, except that the KdV equation allows only one-directional wave propagation and the Boussinseq equation describes bi-directional wave propagation. Recently, a great deal of research has been conducted in the study of equation (1) from various points of view, see [1-5, 11, 13, 14, 24, 25] and the references therein. An exact formula for the interaction of solitary waves is given in Manoranjan et al [14]. Hirota [11] has deduced conservation laws and has examined N-soliton interaction. The representation of periodic waves as sums of solitons has been given by Whitham [25] and the modified decomposition method was used by Wazwaz [23] to construct soliton solution of equation (1) subject to the initial conditions

$$u(x,\ 0) = f(x),\ u_t\,(x,\ 0) = g\,(x). \tag{4}$$

El-Sayed and Kaya [3] studied the solitary wave solutions by using the decomposition method of the (2+1)-dimensional Boussinesq equation

$$u_{tt} = u_{xx} + u_{yy} + (u^2)_{xx} {}^= u_{xxxx}$$

More recently, Hajji used modified decomposition method for solving such equations, see [5]. Inspired and motivated by the ongoing research in this area, we use the variational iteration method for finding the series solution of a regularized version of equation (1) via the singularity perturbed (sixth-order) Boussinesq equation

$$u_{tt} + (p\,(u))_{xx} + \alpha u_{xxxx} + \beta u_{xxxxxx} \tag{5}$$

where α and β are real numbers (β is small). This equation was originally introduced by Daripa and Hua [1]. The sixth order derivative term provides dispersive regularization. The physical relevance of equation (5) in the context of water waves was recently addressed by Dash and Daripa [2]. It was shown that equation (5) actually describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water. So, it is closely related to the singularly perturbed (fifth-order) KdV equation.

$$u_t + uu_x + u_{xxx} + \varepsilon^2 u_{xxxxx}$$

which can be derived from equation (5) by using suitable transformations, see [5] and the references therein. The fifth-order KdV equation has been studied by Kaya [13] where soliton solutions were found using the Adomian's decomposition method. Since equation (1) has solitary wave solutions, the natural question arises whether (5) also admits solitary Boussinesq equation (1) to describe bi-directional wave propagation on the surface of shallow water. In this paper, we consider the generalized Boussinesq equation

$$u_{tt} = \sum_{i=0}^{m} b_i u_{(2i+2)x} + [Q(u)]_{xx}, \tag{6}$$

where $q(u) = u + b_0 u^r$, r and b_i $(i = 1, 2, \ldots, m)$ are all real constants and $u_{(2i+2)x}$ denotes the $(2i+2)$ and derivative of u with respect to x. Note that the choices $m = 1$, $b_0 = 1$, $b_1 = 1$ and $r = 2$ yield (1) and for choices $m = 2$, $b_0 = 1$, $b_1 = \alpha$, $b_2 = \beta$, $m = 2$, and $p(u) = u^r$, equation (6) becomes the singularly perturbed sixth-order Boussinesq equation (5). The basic motivation of this paper is to approach the singularly perturbed Boussinesq equation (5) by implementing the variational iteration method. It is shown that the variational iteration method provides the solution in a rapid convergent series with easily computable components. We write the correct functional for the singularly perturbed Boussinesq equation and calculate the Lagrange multiplier optimally via variational theory. The use of Lagrange multiplier reduces the successive application of the integral operator and minimizes the computational work. Moreover, the selection of the initial value is done by introducing an essential modification which increases the efficiency of the proposed algorithm. The VIM solves effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. Several examples are given to illustrate the reliability and performance of the proposed method.

2. VARIATIONAL ITERATION METHOD

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{7}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method [6-10, 12, 15-22, 24], we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds, \tag{8}$$

where λ is a Lagrange multiplier [6-10], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; (8) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [6-10, 12]. In this method, it is required first to determine the Lagrange multiplier λ optimally via variational theory. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$. For the convergence criteria and error estimates of variational iteration method, see Ramos [22].

3. NUMERICAL APPLICATIONS

In this section, we apply the variational iteration method for solving the singularly perturbed Boussinesq model equation (5). In particular, we consider the equation for sufficiently small values of β and apply the technique presented in the previous section. We introduce a slight modification in the selection of the initial value which increases the efficiency of the proposed iterative scheme. For the sake of comparison, we take the same examples as in [5].

Example 3.1: Consider the following singularly perturbed sixth order Boussinesq equation

$$u_{tt} = u_{xx} + (p\,(u))_{xx} + \alpha u_{xxxx} + \beta u_{xxxxx}, \tag{5}$$

taking $\beta = 1$, $\beta = 0$ and $p(u) = 3u^2$, the model equation is given as

$$u^{tt} = u_{xx} + 3(u^2)_{xx} = u_{xxxx},$$

with initial conditions

$$u(x,0) = \frac{2ak^2e^{kx}}{(1+ae^{kx})^2}, \qquad u_t(x,0) = \frac{2ak^3\sqrt{1+k^2}\,(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3},$$

where a and k are arbitrary constants. The exact solution $u(x, t)$ of the problem is given as [5]

$$u(x,t) = 2\frac{ak^2\exp(kx+k\sqrt{1+k^2t})}{\left(1+a\exp(kx+k\sqrt{1+k^2t})\right)^2}.$$

The correct functional is given by

$$u_{n+1}(x,t) = \frac{2ak^2e^{kx}}{(1+ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}\,(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t + \int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial t^2} - \left((\tilde{u}_n)_{xx} + 3(\tilde{u}_n{}^2)_{xx} + (\tilde{u}_n)_{xxxx}\right)\right)ds.$$

Making the correct functional stationary, the Lagrange multipliers can easily be identified as $\lambda = s - x$, consequently, the correct functional is given as

$$u_{n+1}(x,t) = \frac{2ak^2e^{kx}}{(1+ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}\,(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t + \int_0^t (s-x)\left(\frac{\partial^2 u_n}{\partial t^2} - \left((\tilde{u}_n)_{xx} + 3(\tilde{u}_n{}^2)_{xx} + (\tilde{u}_n)_{xxxx}\right)\right)ds.$$

The following approximants are obtained

$$u_0(x,t) = \frac{2e^x}{(1+e^x)^2},$$

$$u_1(x,t) = \frac{2e^x}{(1+e^x)^2} + \frac{2ak^3\sqrt{1+k^2}\,(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t + \frac{2e^x\left(1-4e^x+e^{2x}\right)}{\left(1+e^x\right)^4}t^2,$$

$$u_2(x,t) = \frac{2e^x}{(1+e^x)^2} + \frac{2ak^3\sqrt{1+k^2}\,(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t + \frac{2e^x(1-4e^x+e^{2x})}{(1+e^x)^4}t^2$$
$$-\frac{2\sqrt{2}e^x(-1+e^x)(1-10e^x+e^{2x})}{3(1+e^x)^5}t^3 + \frac{e^x(1-4e^x+e^{2x})(1-44e^x+78e^{2x}-44e^{3x}+e^{4x})}{3(1+e^x)^8}t^4,$$

$$u_3(x,t)=\frac{2e^x}{(1+e^x)^2}+\frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t+\frac{2e^x\left(1-4e^x+e^{2x}\right)}{\left(1+e^x\right)^4}t^2$$

$$-\frac{2\sqrt{2}e^x\left(-1+e^x\right)\left(1-10e^x+e^{2x}\right)}{3\left(1+e^x\right)^5}t^3+\frac{e^x\left(1-4e^x+e^{2x}\right)\left(1-44e^x+78e^{2x}-44e^{3x}+e^{4x}\right)}{3\left(1+e^x\right)^8}t^4$$

$$+\frac{8e^{2x}\left(1-10e^x+20e^{2x}-10e^{3x}+e^{4x}\right)}{\left(1+e^x\right)^8}t^4-\frac{\sqrt{2}e^x\left(-1+e^x\right)\left(1-56e^x+246e^{2x}-56e^{3x}+e^{4x}\right)}{15\left(1+e^x\right)^7}t^5$$

$$+\frac{e^x\left(1-452e^x+19149e^{2x}-207936e^{3x}+807378e^{4x}-1256568e^{5x}\right)}{45\left(1+e^x\right)^{12}}t^6$$

$$+\frac{e^x\left(807378e^{6x}-207936e^{7x}+19149e^{8x}-452e^{9x}+e^{10x}\right)}{45\left(1+e^x\right)^{12}}t^6,$$

$\vdots$

The series solution is given as

$$u(x,t)=\frac{2e^x}{(1+e^x)^2}+\frac{2ak^3\sqrt{1+k^2}\left(1-ae^{kx}\right)e^{kx}}{(1+ae^{kx})^3}t+\frac{2e^x\left(1-4e^x+e^{2x}\right)}{(1+e^x)^4}t^2$$

$$-\frac{2\sqrt{2}e^x\left(-1+e^x\right)\left(1-10e^x+e^{2x}\right)}{3(1+e^x)^5}t^3+\frac{e^x\left(1-4e^x+e^{2x}\right)\left(1-44e^x+78e^{2x}-44e^{3x}+e^{4x}\right)}{3(1+e^x)^8}t^4$$

$$+\frac{8e^{2x}\left(1-10e^x+20e^{2x}-10e^{3x}+e^{4x}\right)}{(1+e^x)^8}t^4-\frac{\sqrt{2}e^x\left(-1+e^x\right)\left(1-56e^x+246e^{2x}-56e^{3x}+e^{4x}\right)}{15(1+e^x)^7}t^5$$

$$+\frac{e^x\left(1-452e^x+19149e^{2x}-207936e^{3x}+807378e^{4x}-1256568e^{5x}\right)}{45(1+e^x)^{12}}t^6$$

$$+\frac{e^x\left(807378e^{6x}-207936e^{7x}+19149e^{8x}-452e^{9x}+e^{10x}\right)}{45(1+e^x)^{12}}t^6+\cdots,$$

which is in full agreement with [5].

Table 1 exhibits the absolute errors between the series and the exact solutions by using the variational iteration method. Higher accuracy can be obtained by adding some more components of the series solution.

Table 1
Error Estimates

x_i	t_k					
	0.01	0.02	0.04	0.1	0.2	0.5
–1	2.80886 E-14	1.79667 E-12	1.15235 E-10	2.83355 E-8	1.83899 E-6	4.74681 E-4
–0.8	6.27276 E-14	4.01362 E-12	2.57471 E-10	6.33178 E-8	4.10454 E-6	1.04489 E-3
–0.6	6.08402 E-14	3.90188 E-12	2.25663 E-10	6.18024 E-8	4.02299 E-6	1.03093 E-3
–0.4	1.16573 E-14	7.41129 E-13	4.82756 E-11	1.23843 E-8	8.53800 E-6	2.46302 E-4
–0.2	5.53446 E-14	3.53395 E-12	2.25663 E-10	5.47485 E-8	3.47264 E-6	8.35783 E-4
0	8.63198 E-14	5.53357 E-12	2.54174 E-10	8.65197 E-8	5.54893 E-6	1.37353 E-3
0.2	5.56222 E-14	3.55044 E-12	2.27779 E-10	5.60362 E-8	3.63600 E-6	9.29612 E-4
0.4	1.14353 E-14	7.14928 E-13	4.49107 E-11	1.03370 E-8	5.93842 E-7	9.61260 E-5
0.6	6.06182 E-14	3.87551 E-12	2.47218 E-10	5.97562 E-8	3.76275 E-6	8.79002 E-4
0.8	6.23945 E-14	3.99519 E-12	2.55127 E-10	6.18881 E-8	3.92220 E-6	9.36404 E-4
1	2.79776 E-14	1.78946 E-12	1.14307 E-10	2.77684 E-8	1.76607 E-6	4.28986 E-4

Example 3.2: Consider the following singularly perturbed sixth order Boussinesq equation

$$u_{tt} = u_{xx} + (u^2)_{xx} - u_{xxxx} + \frac{1}{2}u_{xxxxxx},$$

with initial conditions

$$u(x,0) = -\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right), \qquad u_t(x,0) = \frac{-210\sqrt{\frac{194}{13}}\sec h^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}.$$

The exact solution of the problem is given as

$$u(x,t) = -\frac{105}{169}\sec h^4\left[\sqrt{\frac{1}{26}}(x - \sqrt{\frac{97}{169}}t)\right].$$

The correct functional is given by

$$u_{n+1}(x,t) = -\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right) + \frac{-210\sqrt{\frac{194}{13}}\sec h^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}t$$

$$+\int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial t^2} - \left((\tilde{u}_n)_{xx} + (\tilde{u}_n{}^2)_{xx} - (\tilde{u}_n)_{xxxx} + \frac{1}{2}(\tilde{u}_n)_{xxxxxx}\right)\right)ds.$$

Making the correct functional stationary, the Lagrange multipliers can easily be identified as consequently, we obtain the following correct functional

$$u_{n+1}(x,t)=-\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right)+\frac{-210\sqrt{\frac{194}{13}}\sec h^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}t$$

$$+\int_0^t (s-x)\left(\frac{\partial^2 u_n}{\partial t^2}-\left(\left(\tilde{u}_n\right)_{xx}+(\tilde{u}_n{}^2)_{xx}-\left(\tilde{u}_n\right)_{xxxx}+\frac{1}{2}\left(\tilde{u}_n\right)_{xxxxxx}\right)\right)ds.$$

The following approximants are obtained

$$u_0(x,t)=-\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right),$$

$$u_1(x,t)=-\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right)-\frac{105\sqrt{\frac{194}{13}}\sec h^6\left(\frac{x}{\sqrt{26}}\right)\sinh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)}{2197}t$$

$$-\frac{105}{371293}\left(-291+194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right)\sec h^6\frac{x}{\sqrt{26}}t^2,$$

$$u_2(x,t)=-\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right)-\frac{105\sqrt{\frac{194}{13}}\sec h^6\left(\frac{x}{\sqrt{26}}\right)\sinh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)}{2197}t$$

$$-\frac{105}{371293}\left(-291+194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right)\sec h^6\frac{x}{\sqrt{26}}t^2$$

$$+\frac{395\sec h^7\frac{x}{\sqrt{26}}}{52206766144}\left(10816\sqrt{2522}\sinh\frac{x}{\sqrt{26}}-1664\sqrt{2522}\sinh\frac{3x}{\sqrt{26}}\right)t^3$$

$$+\left(-334200\sec h^5\left(\frac{x}{\sqrt{26}}\right)+354247\cosh\left(\frac{2}{\sqrt{13}}x\right)\sec h^5\left(\frac{x}{\sqrt{26}}\right)-47164\cosh\left(\frac{2\sqrt{2}}{\sqrt{13}}x\right)\sec h^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4$$

$$+\left(3201\cosh^3\left(\frac{3\sqrt{2}}{\sqrt{13}}x\right)\sec h^5\left(\frac{x}{\sqrt{26}}\right)-388\cosh\left(\frac{4\sqrt{2}}{\sqrt{13}}x\right)\sec h^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4,$$

$$\vdots$$

The series solution is given as

$$u(x,t)=-\frac{105}{169}\sec h^4\left(\frac{x}{\sqrt{26}}\right)-\frac{105\sqrt{\frac{194}{13}}\sec h^6\left(\frac{x}{\sqrt{26}}\right)\sinh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)}{2197}t$$

$$-\frac{105}{371293}\left(-291+194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right)\sec h^{6}\frac{x}{\sqrt{26}}t^{2}$$

$$+\frac{395\sec h^{7}\frac{x}{\sqrt{26}}}{52206766144}\left(10816\sqrt{2522}\sinh\frac{x}{\sqrt{26}}-1664\sqrt{2522}\sinh\frac{3x}{\sqrt{26}}\right)t^{3}$$

$$+\left(-334200\sec h^{5}\left(\frac{x}{\sqrt{26}}\right)+354247\cosh\left(\frac{2}{\sqrt{13}}x\right)\sec h^{5}\left(\frac{x}{\sqrt{26}}\right)-47164\cosh\left(\frac{2\sqrt{2}}{\sqrt{13}}x\right)\sec h^{5}\left(\frac{x}{\sqrt{26}}\right)\right)t^{4}$$

$$+\left(3201\cosh^{3}\left(\frac{3\sqrt{2}}{\sqrt{13}}x\right)\sec h^{5}\left(\frac{x}{\sqrt{26}}\right)-388\cosh\left(\frac{4\sqrt{2}}{\sqrt{13}}x\right)\sec h^{5}\left(\frac{x}{\sqrt{26}}\right)\right)t^{4}+\cdots.$$

Table 2
Error Estimates

x_i	t_k					
	0.01	0.02	0.04	0.1	0.2	0.5
–1	7.77156 E-16	1.36557 E-14	8.57869 E-13	2.09264 E-10	1.33823 E-8	3.25944 E-6
–0.8	1.11022 E-16	1.99840 E-15	1.12688 E-13	2.73880 E-11	1.74288 E-9	4.14094 E-7
–0.6	2.22045 E-16	1.09912 E-14	7.28861 E-13	1.78030 E-10	1.14025 E-8	2.79028 E-6
–0.4	1.11022 E-16	2.32037 E-14	1.50302 E-12	3.67002 E-10	2.34944 E-8	5.74091 E-6
–0.2	6.66134 E-16	3.23075 E-14	2.04747 E-12	4.99918 E-10	3.19983 E-9	7.81509 E-6
0	4.44089 E-16	3.49720 E-14	2.24365 E-12	5.47741 E-10	3.50559 E-8	8.55935 E-6
0.2	5.55112 E-16	3.19744 E-14	2.04714 E-12	4.99820 E-10	3.19858 E-8	7.80749 E-6
0.4	3.33067 E-16	2.32037 E-14	1.50324 E-12	3.66815 E-10	2.34706 E-8	5.72641 E-6
0.6	3.33067 E-16	1.12133 E-14	7.28528 E-12	1.77772 E-10	1.13695 E-8	2.77022 E-6
0.8	3.33067 E-16	1.99840 E-15	1.13132 E-13	2.76944 E-11	1.78208 E-9	4.41936 E-7
1	7.77156 E-16	1.38778 E-14	8.58313 E-13	2.09593 E-10	1.34244 E-8	3.28504 E-6

Table 2 exhibits the absolute errors between the series and the exact solutions by using the variational iteration method. Higher accuracy can be obtained by adding some more components of the series solution.

4. CONCLUSIONS

In this paper, we have used the variational iteration method for solving Boussinesq equations. The method is implemented in a direct way without using linearization, perturbation or restrictive assumptions. The method gives more realistic series solutions that converge very rapidly in physical problems. rapidly in physical problems. This shows that the variational iteration technique can be considered as an efficient method for solving linear and nonlinear problems. The fact that the variational iteration method solves nonlinear problems without using Adomian's polynomials can be considered as an advantage of this method over the decomposition method.

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