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STABILITY AND BIFURCATION OF A DISCRETE-TIME THREE-NEURON SYSTEM WITH DELAYS

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***ABSTRACT*:** In this paper, we consider a simple discrete three-neuron network model with delays. The characteristic equation of the linearized system at the trivial solution is a polynomial equation involving very high order terms.We derive some sufficient and necessary conditions on the asymptotic stability of the trivial solution. We also consider the existence of three types of bifurcations: fold bifurcations, flip bifurcations, and Neimark-Sacker bifurcations (also called Hopf bifurcations for map). The stability and direction of Neimark-Sacker bifurcations are studied by applying the normal form theory and the center manifold theorem.

Keywords: Delay, bifurcation, neural network, stability, normal form, center manifold.

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Running title: Delayed discrete three-neuron network

1. INTRODUCTION

In the recent years, more and more attention has been paid to the study of neural networks because of their potential in applications such as optimization, image processing, pattern recognition, and associative memories. In practice, due to the finite speeds of switching and transmission of signals in a network, delayed systems have been the starting point of investigations.

Most of the models are described by delay-differential systems. Due to the high dimensionality of the problem, mathematical analysis has been restricted to special networks with either small number of neurons or simple architecture like a ring of identical neurons (see, for example, [1, 2, 3, 5, 6, 7, 8, 12, 15, 16, 21]). In most cases, delay can destabilize the network and create oscillatory behavior. Thus it is important to study bifurcations such as Hopf bifurcation when there is a loss of stability.

In order to implement the continuous-time network for computer simulation, experimental or computational purposes, one often discretizes the continuous-time network. We refer to [4, 11, 13, 18, 19] for related discussions on the importance and the need for discrete-time analogues to reflect the dynamics of their continuous-time counterparts. Though the discrete-time model inherits some of the dynamical characteristics of the continuous-time network, there are some differences. For example, discrete-time version can possess spurious steady-state solutions and spurious asymptotic behavior which are not inherent in the original continuous systems. The discussion on the importance of discrete-time analogues in preserving the properties of stability and bifurcation of their continuous-time counterparts have been studied by some authors. Though much has been done on bifurcations for delay-differential systems, only bifurcations for discrete networks of two neurons have been studied (see [9, 17, 18, 19, 20]).

Motivated by the above discussion, in this paper, we consider the following delayed discrete system

$$\begin{cases} x_{n+1} = ax_n + w_1 f_1(y_{n-k_1}), \\ y_{n+1} = ay_n + w_2 f_2(z_{n-k_2}), \\ z_{n+1} = az_n + w_3 f_3(x_{n-k_3}), \end{cases} \tag{1}$$

where x_n, y_n, and z_n denote the respective activations of the three neurons, $a \in (0, 1)$ is the internal decay of the neurons, w_1, w_2, and w_3 are connection weights, the nonnegative integers $k_j \in \mathbb{Z}$ ($j = 1, 2, 3$) denote the synaptic transmission delays, $f_j : \mathbb{R} \to \mathbb{R}$ ($j = 1, 2, 3$) are the activation functions. Throughout this paper, we always assume that $w_1 w_2 w_3 \neq 0$ (otherwise the system is decoupled), $k_1 + k_2 + k_3 = 3k$, and f_j ($j = 1, 2, 3$) are at least C^1-smooth with $f_j(0) = 0$. Without loss of generality, we also assume $f_j'(0) = 1,\ j = 1, 2, 3$.

System (1) can be regarded as a discrete analogy of the differential system

$$\begin{cases} \dot{x} = -kx + w_1 f(y(t-\tau_1)), \\ \dot{y} = -ky + w_2 f(z(t-\tau_2)), \\ \dot{z} = -kz + w_3 f(x(t-\tau_3)), \end{cases} \tag{2}$$

where $\kappa > 0$ characterizes the decay rate of the neurons with which each neuron will reset its activation to the resting state in isolation when disconnected from other neurons, τ_j ($j = 1, 2, 3$) are non-negative and denote the synaptic transmission delays. Existence, multiplicity, stability, and even the spatio-temporal patterns of periodic solutions of (2) have been studied by Guo and Huang [5] and Wei and Li [14].

The purpose of this paper is to study the asymptotic stability of the trivial solution and discuss possible bifurcations. Because of the normalization we will use the product of the connection weights (more precisely, the cubic root of the product) as the bifurcation parameter. It turns out that there exist three types of bifurcations: fold bifurcations, flip bifurcations and Neimark-Sacker bifurcations. We should mention that there is no flip bifurcations in the discrete networks of two neurons in the above mentioned literature. Moreover, we shall study the direction and stability of Neimark-Sacker bifurcation by using the techniques developed by Kuznetsov [10]. These theoretical results are important to complement the experimental and numerical observations made in living neural systems and artificial neural networks for better understanding the mechanisms underlying them.

This paper is organized as follows: In Section 2, we discuss the associated characteristic equation with the linearized system. Followed in Section 3 are the linear stability and possible bifurcations with focus on the direction and stability analysis of the Neimark-Sacker bifurcation.

2. THE CHARACTERISTIC EQUATION

Linearizing (1) at the origin (the trivial solution of (1)) leads to

$$\begin{cases} x_{n+1} = ax_n + w_1 y_{n-k_1}, \\ y_{n+1} = ay_n + w_2 z_{n-k_2}, \\ z_{n+1} = az_n + w_3 x_{n-k_3}, \end{cases} \tag{3}$$

The characteristic matrix of (3) is

$$\Delta(\lambda) = \begin{pmatrix} \lambda - a & -w_1 \lambda^{-k_1} & 0 \\ 0 & \lambda - a & -w_2 \lambda^{-k_2} \\ -w_3 \lambda^{-k_3} & 0 & \lambda - a \end{pmatrix}$$

and hence the characteristic equation is

$$\det\Delta(\lambda) = (\lambda - a)^3 - b^3 \lambda^{-3k} = 0,$$

where $b \in \mathbb{R}$ such that $b^3 = w_1 w_2 w_3$. The characteristic equation can be decomposed as

$$\det \Delta(\lambda) = \prod_{j=0}^{2} [\lambda - a - \omega^j b \lambda^{-k}] = 0, \tag{4}$$

where $\omega = \exp\left\{\frac{2\pi}{3} i\right\}$ and $i = \sqrt{-1}$ (this is the reason why we assume that $k_1 + k_2 + k_3 = 3k$). Regarding b as the bifurcation parameter, we first determine when (4) has a root on the unit circle. It is well known that the trivial solution of the nonlinear system (1) is locally asymptotically stable if all roots of (4) lie inside the unit circle.

If $k = 0$, which implies $k_1 = k_2 = k_3 = 0$, then the three roots of (4) are $a + b$, $a + \omega b$, and $a + \omega^2 b$.

Theorem 1 *Assume that* $k = 0$. *Then the following statements hold.*

(i) $\det \Delta(\lambda)$ *is of Schur type* (*i.e., all zeros are inside the unit circle*) *if and only* if $a^2 + b^2 - 1 < ab < \frac{1}{2}(1 - a^2 - b^2)$.

(ii) *If* $a + b = 1$ (*respectively,* -1), *then* $\det\Delta(\lambda)$ *has a zero* 1 (*respectively,* -1) *and the other two zeros are inside* (*respectively, outside*) *the unit circle.*

(iii) *If* $a^2 + b^2 - ab = 1$, *then on the unit circle* $\det\Delta(\lambda)$ *has a pair of complex conjugate zeros* $a + \omega b$ and $a + \omega^2 b$. *Moreover, the third zero* $a + b$ *is outside* (*respectively, inside*) *the unit circle if* $b > 0$ (*respectively,* < 0).

Proof: Note that $|a + \omega b|^2 = |a + \omega^2 b|^2 = a^2 - ab + b^2$. Also note that $ab < \frac{1}{2}(1 - a^2 - b^2)$ is equivalent to $(a + b)^2 < 1$. Then (i) follows immediately.

Now, suppose $a + b = 1$. The case where $a + b = -1$ can be dealt with similarity. Then we have $\det\Delta(1) = 0$ and $|a + \omega b|^2 = |a + \omega^2 b|^2 = a^2 - ab + b^2 = a^2 - a(1 - a) + (1 - a)^2 = 3a^2 - 3a + 1 < 1$ since $a \in (0, 1)$. This proves (ii).

Finally, we assume that $a^2 + b^2 - ab = 1$. Then $|a + \omega b| = |a + \omega^2 b| = 1$, *i.e.,* $\det\Delta(\lambda)$ has two zeros $a + \omega b$ and $a + \omega^2 b$ on the unit circle. Moreover, we have $b > 1$ or $-1 < b < 0$, otherwise $a + b = a^3 + b^3 < a + b$, which is absurd. If $b > 0$ and hence $b > 1$, then $a + b > 1$; while if $b \in (-1, 0)$, then it is easy to see that $|a + b| < 1$. This completes the proof.

In the sequel, we consider the following polynomials with $k \geq 1$, $a \in (0, 1)$, and $b \in \mathbb{R}$,

$$P_j(\lambda) = \lambda^{k+1} - a\lambda^k - \omega^j b, \quad j = 0, 1, 2. \tag{5}$$

Define a parametric curve Σ with

$$\begin{cases} u(t) = \cos(k+1)t - a\cos kt, \\ v(t) = \sin(k+1)t - a\sin kt. \end{cases} \tag{6}$$

Let $\theta(t) = v(t)/u(t)$. Then

$$\begin{aligned} \theta'(t) &= u^{-2}(t)[v'(t)\, u(t) - u'(t)\, v(t)] \\ &= u^{-2}(t)\, [(a^2 + 1)k + 1 - a(2k + 1) \cos t] \\ &\geq u^{-2}(t)\, [(a^2 + 1)k + 1 - a(2k + 1) \\ &= u^{-2}(t)[k(1 - a) + 1](1 - a) \\ &> 0 \end{aligned}$$

for all $t \in \mathbb{R}$ such that $u(t) \neq 0$. Therefore, as t increases from 0 to π, the corresponding point $(u(t), v(t))$ on the curve Σ moves anticlockwise around the origin. Moreover, it follows from $u^2(t) + v^2(t) = 1 + a^2 - 2a \cos t$ that the part of the curve Σ with parameter $t \in [0, \pi]$ is simple, i.e., it can not intersect itself.

Let $0 \leq \varphi_0 < \varphi_1 < \ldots < \varphi_k < \varphi_{k+1} \leq \pi$ be the $k + 2$ zeros of $v(t)$ in the interval $[0, \pi]$. Obviously, we have $\varphi_0 = 0$, $\varphi_1 \in (0, \pi/(k + 1))$, and $\varphi_{k+1} = \pi$. Then the curve Σ intersects the u-axis at points $(u(\varphi_j), 0)$, $j \in \mathbb{Z}(0, k + 1)$.

Here and in the sequel, for $m \le n \in \mathbb{Z}$, $\mathbb{Z}(m, n) = \{m, m+1, \ldots, n\}$. It follows from the anticlockwise property of the curve Σ that $(-1)^j u(\varphi_j) > 0$ for all $j \in \mathbb{Z}(0, k+1)$. In addition, we have $|u(\varphi_j)| = \sqrt{a^2 + 1 - 2a\cos\varphi_j}$. Hence $|u(\varphi_j)|$ is increasing in j, and

$$u(\varphi_j) = (-1)^j \sqrt{a^2 + 1 - 2a\cos\varphi_j}.$$

In particular, $u(\varphi_0) = 1 - a$ and $u(\varphi_{k+1}) = (-1)^{k+1}(1 + a)$. Moreover, we further claim that

$$(-1)^j v'(\varphi_j) > 0 \text{ for } j \in \mathbb{Z}(0, k+1) \text{ and } (-1)^j u'(\varphi_j) > 0 \text{ for } j \in \mathbb{Z}(1, k). \tag{7}$$

In fact, the first inequality follows from the anticlockwise property of the curve Σ. Since $u^2(t) + v^2(t)$ is increasing in $t \in (0, \pi)$, $u'(t)u(t) + v'(t)v(t) > 0$ for all $t \in (0, \pi)$. It follows that $u'(\varphi_j)u(\varphi_j) > 0$ for all $j \in \mathbb{Z}(1, k)$, which combined with $(-1)^j u(\varphi_j) > 0$, produces the second inequality in (7).

Now, we apply the above results to $P_0(\lambda)$ to obtain the following results.

Lemma 1 *Assume that* $k \ge 1$.

(i) $P_0(\lambda)$ *has a zero of modulus* 1 *if and only if* $b = u(\varphi_j)$ *for some* $j \in \mathbb{Z}(0, k+1)$. *Moreover, if* $b = u(\varphi_j)$ *for some* $j \in \mathbb{Z}(0, k+1)$ *then all the zeros of* $P_0(\lambda)$ *of modulus* 1 *are* $e^{\pm i\varphi_j}$, *which are simple.*

(ii) $P_0(\lambda)$ *has a simple zero* $\lambda = 1$ *on the unit circle and all other zeros of* $P_0(\lambda)$ *are inside the unit circle if and only if* $b = 1 - a$.

(iii) $P_0(\lambda)$ *has a simple zero* $\lambda = -1$ *on the unit circle and all other zeros of* $P_0(\lambda)$ *are inside the unit circle if and only if* $b = (-1)^{k+1}(a + 1)$.

(iv) *For a fixed* $j \in \mathbb{Z}(0, k+1)$, *there exist* $\delta > 0$ and a C^1-*mapping* $\lambda : (u(\varphi_j) - \delta, u(\varphi_j) + \delta) \to \mathbb{C}$ *such that* $\lambda(u(\varphi_j)) = e^{i\varphi_j}$ *and* $\lambda(b)$ *is a zero of* $P_0(\lambda)$ *for all* $b \in (u(\varphi_j) - \delta, u(\varphi_j) + \delta)$. *Moreover,* $(-1)^j \frac{d}{db}|\lambda(b)|\big|_{b=u(\varphi_j)} > 0$.

(v) $P_0(\lambda) = 0$ *has all solutions with modulus less than* 1 *if* $u(\varphi_1) < b < 1 - a = u(\varphi_0)$, *exactly* $2s + 1$ *solutions with modulus greater than* 1 *if* $u(\varphi_{2s}) < b \le u(\varphi_{2(s+1)})$, $s \in \mathbb{Z}(0, [(k-1)/2]\}$; *exactly* $2s$ *solutions with modulus greater than* 1 *if* $u(\varphi_{2s+1}) \le b < u(\varphi_{2s-1})$, $s \in \mathbb{Z}(1, [k/2])$; *exactly* $k + 1$ *solutions with modulus greater than* 1 *if* $b < u(\varphi_{1+2[k/2]})$ *or* $b > u(\varphi_{2[(k+1)/2]})$. *Here* $[\cdot]$ *is the greatest integer function.*

Proof : It is easy to check that if $b = u(\varphi_j)$ then $P_0(\lambda)$ has zeros $e^{\pm i\varphi_j}$. On the other hand, if $e^{i\theta}$ with $\theta \in [0, \pi]$ is a zero of $P_0(\lambda)$, then separating the real and imaginary parts of $P_0(e^{i\theta})$ gives us

$$\cos((k+1)\theta) - a\cos(k\theta) = b \text{ and } \sin((k+1)\theta) - a\sin(k\theta) = 0. \tag{8}$$

It follows that there exists $j_0 \in \mathbb{Z}(0, k+1)$ such that $\theta = \varphi_{j_0}$ and hence $b = u(\varphi_{j_0})$. It also follows from (8) that $|b| = \sqrt{1 + a^2 - 2a\cos\theta}$. Since $\cos\theta$ is monotonic on $[0, \pi]$, it follows that when $b = u(\varphi_j)$ the only zeros of $P_0(\lambda)$ of modulus 1 are $e^{\pm i\varphi_j}$. It is easy to see that $e^{\pm i\varphi_j}$ are simple zeros. This proves (i).

If $P_0(\lambda)$ has a zero $\lambda = 1$, then $P_0(1) = 1 - a - b = 0$ gives $b = 1 - a$. Suppose $b = 1 - a$. Then $P_0(\lambda) = \lambda^{k+1} - 1 - a(\lambda^{k-1})$. This, together with conclusion (i), implies that $\lambda = 1$ is the unique zero of $P_0(\lambda)$ on the unit circle, which is simple. Now, we show that all other zeros of $P_0(\lambda)$ are inside the unit circle. In fact, if there exists a zero λ_0 outside the unit circle, i.e., $|\lambda_0| > 1$, then $|\lambda_0|^{k+1} \le a|\lambda_0|^k + 1 - a$. Thus, $|\lambda_0|^{k+1} - 1 \le a(|\lambda_0|^k - 1)$. It follows that

$$a \geq \frac{|\lambda_0|^{k+1} - 1}{|\lambda_0|^k - 1} > |\lambda_0| > 1,$$

which contradicts with $a \in (0, 1)$. This proves (ii).

The proof of (iii) is similar to that of (ii) by noting

$$P_0(\lambda) = (-1)^{k+1}\{(-\lambda)^{k+1} - 1 - a[(-\lambda)k - 1]\}$$

when $b = (-1)^{k+1}(1 + a)$.

Note that $P_0(e^{i\varphi_j}) = 0$ if $b = u(\varphi_j)$. The existence of δ and the mapping λ follow from the implicit function theorem. Note

$$\frac{d}{db}|\lambda(b)|^2 = \bar{\lambda}(b)\frac{d\lambda(b)}{db} + \lambda(b)\frac{d\bar{\lambda}(b)}{db}$$

$$= \frac{\bar{\lambda}(b)}{(k+1)\lambda^k(b) - ak\lambda^{k-1}(b)} + \frac{\lambda(b)}{(k+1)\bar{\lambda}^k(b) - ak\bar{\lambda}^{k-1}(b)}.$$

Then

$$\frac{d}{db}|\lambda(b)|^2\Big|_{b=u(\varphi_j)} = \mathrm{Re}\{[(k+1)\lambda^{k+1}(u(\varphi_j)) - ak\lambda^k(u(\varphi_j))]^{-1}\}$$

$$= 2\,\mathrm{Re}\frac{e^{-i\varphi_j}}{(k+1)e^{ik\varphi_j} - ake^{i(k-1)\varphi_j}}$$

$$= 2\,\mathrm{Re}\frac{1}{(k+1)e^{i(k+1)\varphi_j} - ake^{ik\varphi_j}}$$

$$= \frac{2[(k+1)\cos((k+1)\varphi_j) - ak\cos(k\varphi_j)]}{a^2k^2 + (k+1)^2 - 2ak(k+1)\cos\varphi_j}$$

$$= \frac{2v'(\varphi_j)}{a^2k^2 + (k+1)^2 - 2ak(k+1)^2 - 2ak(k+1)\cos\varphi_j}.$$

Since $a^2k^2 + (k+1)^2 - 2ak(k+1)\cos\varphi_j > 0$, the sign of $\frac{d}{db}|\lambda(b)|^2\Big|_{b=u(\varphi_j)}$ is determined by $v'(\varphi_j)$, whose sign was given in (7). It follows that $(-1)^j\frac{d}{db}|\lambda(b)|^2\Big|_{b=u(\varphi_j)} > 0$ and hence $(-1)^j\frac{d}{db}|\lambda(b)|\Big\|_{b=u(\varphi_j)} > 0$. This proves (iv).

Finally, we come to prove (*v*). First observe that, for each $b \in \mathbb{R}$, $P_0(\lambda)$ has $k + 1$ zeros, which can be regarded as C^1-functions of b according to the implicit function theorem. It is easy to see that all these zeros are inside the unit circle when $b = 0$. As b decreases and passes through $u(\varphi_1)$, it follows from conclusion (i) and $\frac{d}{db}|\lambda(b)|\Big\|_{b=u(\varphi_1)} < 0$ that only two of these $k + 1$ zeros move from the interior onto the boundary and then to the exterior of the unit circle. Similarly, as b further decreases and passes through $u(\varphi_3)$, another two of these $k + 1$ zeros move from the interior onto the boundary and then to the exterior of the unit circle. Thus, there exist exactly two zeros of $P_0(\lambda)$ outside the unit circle if $u(\varphi_3) \le b < u(\varphi_1)$. One can continue in this manner to finish the remaining proof also with the help of conclusions (ii) and (iii).

Let ψ_j, $j \in \mathbb{Z}_{\pm(k+1)} := \{\pm1, \pm2, \pm(k+1)\}$, be the $2k+2$ zeros of

$$v(t) + \sqrt{3}u(t) = 2\sin\left[(k+1)t + \frac{\pi}{3}\right] - 2a\sin\left[kt + \frac{\pi}{3}\right]$$

in the interval $(-\pi, \pi)$ with $-\pi < \psi_{-k-1} < \ldots < \psi_{-2} < \psi_{-1} < 0 < \psi_1 < \ldots < \psi_k < \psi_{k+1} < \pi$. Obviously, we have $\psi_{-1} \in (-\pi/(k+1), 0)$ and $\psi_1 \in (0, \pi/(k+1))$. This means that the curve Σ intersects the line $v + \sqrt{3}u = 0$ at points $(u(\psi_j), v(\psi_j))$, $j \in \mathbb{Z}_{\pm(k+1)}$. Moreover, we have $v(-\psi_j) - \sqrt{3}u(-\psi_j) = -[v(\psi_j) + \sqrt{3}u(\psi_j)] = 0$, which implies that the curve Σ intersects the line $v - \sqrt{3}u = 0$ at points $(u(-\psi_j), v(-\psi_j))$, $j \in \mathbb{Z}_{\pm(k+1)}$. Therefore, it follows from the anticlockwise property of the curve Σ that

$$\varphi_{s-1} < -\psi_{-s} < \psi_s < \varphi_s < -\psi_{-s-1} < \psi_{s+1} < \varphi_{s+1},\ s \in \mathbb{Z}(1, k) \tag{9}$$

and

$$\begin{gathered}(-1)^j u(\psi_j)\,\mathrm{sign}\,(j) > 0,\\ (-1)^j v(\psi_j)\,\mathrm{sign}\,(j) < 0,\\ (-1)^j \,\mathrm{si}\,gn(j)[v'(\psi_j) + \sqrt{3}u'(\psi_j)] > 0\end{gathered} \tag{10}$$

for all $j \in \mathbb{Z}_{\pm(k+1)}$. In addition, we have $\sqrt{u^2(\psi_j) + v^2(\psi_j)} = \sqrt{a^2 + 1 - 2a\cos\psi_j}$.

Let

$$b_j = (-1)^{j+1}\sin gn(j)\sqrt{a^2 + 1 - 2a\cos\psi_j}\,.$$

It follows from (9) that

$$u(\varphi_{2(s-1)}) < -b_{1-2s} < b_{2s-1} < -u(\varphi_{2s-1}) < b_{-2s} < -b_{2s} < u(\varphi_{2s}) \tag{11}$$

for $s \in \mathbb{Z}(1, [k/2])$.

Applying the above discussion to $P_1(\lambda)$ gives the following results.

Lemma 2 Assume that $k \ge 1$.

(i) *$P_1(\lambda)$ has a zero of modulus 1 if and only if $b = b_j$ for some $j \in \mathbb{Z}_{\pm(k+1)}$. Moreover, if $b = b_j$ for some $j \in \mathbb{Z}_{\pm(k+1)}$ then $P_1(\lambda)$ only has one simple zero $e^{i\psi_j}$ with modulus 1.*

(ii) *For a fixed $j \in \mathbb{Z}_{\pm(k+1)}$, there exist $\delta > 0$ and a C^1-mapping $\lambda : (b_j - \delta, b_j + \delta) \to \mathbb{C}$* such that *$\lambda(b_j) = e^{i\psi_j}$ and $\lambda(b)$ is a zero of $P_1(\lambda)$ for all $b \in (b_j - \delta, b_j + \delta)$. Moreover, $(-1)^j \,\mathrm{si}\,gn(j)\frac{d}{db}|\lambda(b_j)| < 0$.*

(iii) *$P_1(\lambda)$ has all zeros with modulus less than 1 if $b_{-1} < b < b_1$; exactly s zeros with modulus greater than 1 if $b_{2s-1} < b \le b_{-2s}$ or $b_{2s} \le b < b_{1-2s}$, $s \in \mathbb{Z}(1, [(k+1)/2])$. Moreover, all the $k+1$ zeros are outside the unit circle if $b < b_{-\varepsilon(k+1)}$ or $b > b_{\varepsilon(k+1)}$, where $\varepsilon = (-1)^k$.*

Proof : To prove (i), first note that any zero of $P_1(\lambda)$ with modulus 1 is simple. If $b = b_j$, one can check that $P_1(\lambda)$ has a zero $e^{i\psi_j}$. Now suppose $P_1(\lambda)$ has a zero $\lambda = e^{i\theta}$ with $\theta \in (-\pi, \pi)$. It follows from $e^{ik\theta}(e^{i\theta} - a) = \omega b$ that

$$u(\theta) = -\frac{1}{2}b \quad \text{and} \quad v(\theta) = \frac{\sqrt{3}}{2}b.$$

Then $v(\theta) + \sqrt{3}u(\theta) = 0$ and hence $\theta = \psi_j$ for some $j \in \mathbb{Z}_{\pm(k+1)}$. Now, suppose $b = b_j$ and $e^{i\theta}$ is a zero to $P_1(\lambda)$. Then there exists $s \in \mathbb{Z}_{\pm(k+1)}$ such that $\theta = \psi_s$. We claim that $s = j$. Otherwise, $s \neq j$, which together with (9), implies that $|b_j| = \sqrt{a^2 + 1 - 2a\cos\psi_j} \neq \sqrt{a^2 + 1 - 2a\cos\theta} = |b_j|$. This proves (i).

To prove (ii), note that $P_1(e^{i\psi_j}) = 0$ if $b = b_j$. The existence of δ and the mapping λ follow from the implicit function theorem. From

$$\frac{d}{db}|\lambda(b)|^2 = \bar{\lambda}(b)\frac{d\lambda(b)}{db} + \lambda(b)\frac{d\bar{\lambda}(b)}{db}$$

$$= \frac{\omega\bar{\lambda}(b)}{(k+1)\lambda^k(b) - ak\lambda^{k-1}(b)} + \frac{\bar{\omega}\lambda(b)}{(k+1)\bar{\lambda}^k(b) - ak\bar{\lambda}^{k-1}(b)},$$

we have

$$\frac{d}{db}|\lambda(b_j)|^2 = -2\frac{(k+1)\cos\left[(k+1)\psi_j + \frac{\pi}{3}\right] - ak\cos\left[k\psi_j + \frac{\pi}{3}\right]}{(k+1)^2 + a^2k^2 - 2ak(k+1)\cos\psi_j}$$

$$= -\frac{v'(\psi_j) + \sqrt{3}u'(\psi_j)}{(k+1)^2 + a^2k^2 - 2ak(k+1)\cos\psi_j}.$$

The denominator $(k + 1)^2 + a^2k^2 - 2ak(k + 1)\cos\psi_j$ is strictly positive. Therefore, the sign of $\frac{d}{db}|\lambda(b_j)|^2$ and hence the sign of $\frac{d}{db}|\lambda(b_j)|$ is determined by the numerator, whose sign was given in (10). Then $(-1)^j \,\mathrm{sign}(j)\frac{d}{db}|\lambda(b_j)| < 0$ and (ii) is proved.

Finally, we come to prove (iii). We first observe that, for each $b \in \mathbb{R}$, $P_1(\lambda)$ has $k + 1$ zeros, which can be regarded as C^1-functions of b according to the implicit function theorem. It is easy to see that all these zeros are inside the unit circle when $b = 0$. In the following, we only consider the case where b increases from 0 to ∞ as the case where b decreases from 0 to $-\infty$ can be dealt with similarity. As b increases and passes through b_1, it follows from $\frac{d}{db}|\lambda(b_1)| > 0$ (see conclusion (ii) just proved) that only one of these $k + 1$ zeros moves from the interior onto the boundary and then to the exterior of the unit circle. As b further increases and passes through b_{-2}, it follows from $\frac{d}{db}|\lambda(b_{-2})| > 0$ that another zero moves from the interior onto the boundary and then to the exterior of the unit circle. Thus, there exists exactly one zero of $P_1(\lambda)$ outside the unit circle if $b_1 < b \leq b_{-2}$. By induction, there exist s zeros of $P_1(\lambda)$ outside the unit circle if $b_{(-1)^{s+1}s} < b \leq b_{(-1)^s(s+1)}$, $s \in \mathbb{Z}(1, k)$. As b eventually passes through $b_{\varepsilon(k+1)}$, it follows from $\frac{d}{db}|\lambda(b_{\varepsilon(k+1)})| > 0$ that all the $k + 1$ zeros move to the exterior of the unit circle. This completes the proof.

We observe that if λ is a zero of $P_1(\lambda)$ then λ is a zero of $P_2(\lambda)$ and vice versa. In view of Lemma 2, we have

Lemma 3 *Assume that* $k \geq 1$.

(i) $P_2(\lambda)$ *has a zero of modulus* 1 *if and only if* $b = b_j$ *for some* $j \in \mathbb{Z}_{\pm(k+1)}$. Moreover, if $b = b_j$ for some $j \in \mathbb{Z}_{\pm(k+1)}$ then $P_2(\lambda)$ only has one simple zero $e^{-i\psi_j}$ *with modulus* 1.

(ii) *For a fixed* $j \in \mathbb{Z}_{\pm(k+1)}$, *there exist* $\delta > 0$ *and a* C^1*-mapping* $\lambda : (b_j - \delta, b_j + \delta) \to \mathbb{C}$ *such that* $\lambda(b_j) = e^{-i\psi_j}$ *and* $\lambda(b)$ *is a zero of* $P_2(\lambda)$ *for all* $b \in (b_j - \delta, b_j + \delta)$. *Moreover,* $(-1)^j \, \mathrm{si}\,gn(j) \frac{d}{db} |\lambda(b_j)| < 0$.

(iii) $P_2(\lambda)$ *has all zeros with modulus less than* 1 *if* $b_{-1} < b < b_1$; *exactly s zeros with modulus greater than* 1 *if* $b_{2s-1} < b \le b_{-2s}$ *or* $b_{2s} \le b < b_{1-2s}$, $s \in \mathbb{Z}(1, [(k+1)/2])$. *Moreover, all the* $k + 1$ *zeros are outside the unit circle if* $b < b_{-\varepsilon(k+1)}$ or $b > b_{\varepsilon(k+1)}$, where $\varepsilon = (-1)^k$.

With the help of Lemmas 1, 2, and 3, we obtain the following results.

Theorem 2 *Suppose* $k \ge 1$.

(i) *All zeros of* $\det\Delta(\lambda)$ *are inside the unit circle if and only if* $b_{-1} < b < 1 - a$.

(ii) *If* $b = b_j$, $j \in \mathbb{Z}_{\pm(k+1)}$, *on the unit circle* $\det\Delta(\lambda)$ *has a pair of simple zeros* $e^{\pm i\psi_j}$. *In particular, when* $b = b_{-1}$, *except for a pair of simple zeroes* $\exp\{\pm i\psi_{-1}\}$, *all zeros of* $\det\Delta(\lambda)$ *are inside the unit circle; when* $b = 1 - a$, except for a simple zero $\lambda = 1$, *all zeros of* $\det\Delta(\lambda)$ *are inside the unit circle.*

(iii) *There exist* $\delta > 0$ *and a* C^1*-mapping* $\lambda : (b_j - \delta, b_j + \delta) \to \mathbb{C}$ *such that* $\lambda(b_j) = e^{i\psi_j}$ and $\lambda(b)$ is a zero of $\det\Delta(\lambda)$ *for all* $b \in (b_j - \delta, b_j + \delta)$. *Moreover,* $(-1)^j \, sign(j) \frac{d}{db} |\lambda(b_j)| < 0$.

(iv) $\det\Delta(\lambda)$ *has a simple zero* $\lambda = 1$ *on the unit circle if and only if* $b = 1 - \mathrm{a}$.

(v) $\det\Delta(\lambda)$ *has a simple zero* $\lambda = .1$ *on the unit circle if and only if* $b = (-1)^{k+1}(a + 1)$.

(vi) *For a fixed* $j \in \mathbb{Z}(0, k + 1)$, *there exist* $\delta > 0$ *and a* C^1*-mapping* $\lambda : (u(\varphi_j) - \delta, u(\varphi_j) + \delta) \to \mathbb{C}$ *such that* $\lambda(u(\varphi_j)) = e^{i\varphi_j}$ *and* $\lambda(b)$ *is a zero of* $P_0(\lambda)$ *for all* $b \in (u(\varphi_j) - \delta, u(\varphi_j) + \delta)$. *Moreover,* $(-1)^j \frac{d}{db} |\lambda(b_j)| \Big|_{b=u(\varphi_j)} > 0$.

Proof : Here, we only verify conclusion (i) because others follow easily from Lemmas 1, 2, and 3. It is easy to see that all zeros of $\det\Delta(\lambda)$ are inside the unit circle if and only if $u(\varphi_1) < b < 1 - a$ and $b_{-1} < b < b_1$. It then follows from (11) that $-u(\varphi_1) > b_1 > -b_{-1} > 1 - a$. Thus, all zeros of $\det\Delta(\lambda)$ are inside the unit circle if and only if $b_{-1} < b < 1 - a$. This completes the proof.

3. BIFURCATION ANALYSIS

In view of Theorems 1(i) and 2(i), we can obtain the linear stability of the trivial solution of (1).

Theorem 3 *The trivial solution of* (1) *is linearly asymptotically stable if and only if one of the following two conditions holds.*

(i) $k = 0$ and $a^2 + b^2 - 1 < ab < \frac{1}{2}(1 - a^2 - b^2)$.

(ii) $k \ge 1$ and $b_{-1} < b < 1 - a$.

If $b = 1 - a$, then it follows from Theorems 1(ii) and 2(iv), $\det\Delta(\lambda)$ has a simple zero $\lambda = 1$ on the unit circle, and all other zeros are inside the unit circle. If $b = (-1)^{k+1}(a + 1)$, then it follows from Theorems 1(iii) and 2(v), on the unit circle $\det\Delta(\lambda)$ has only a simple zero $\lambda = -1$. Therefore, we have established

Theorem 4 Near $b = 1 - a$, *a fold bifurcation* (*or saddle-node bifurcation*) *occurs in system* (1). *Moreover, near* $b = (-1)^{k+1}(a + 1)$, *a flip bifurcation* (*also referred to as period-doubling or subharmonic bifurcation*) *occurs in system* (1).

If $b = b_{-1}$ and $k \geq 1$, then it follows from Theorem 2 that on the unit circle $\det\Delta(\lambda)$ has only a pair of simple complex conjugate zeros, which take the form of $\exp\{\pm i\psi_{-1}\}$, all other zeros of $\det\Delta(\lambda)$ are inside the unit circle. Since $\psi_{-1} \in (-\pi/(k+1), 0) \subseteq (-\pi/2, 0)$, $\exp\{is\psi_{-1}\} \neq 1$ for all $s = 1, 2, 3, 4$. This, together with Theorem 2(iii), implies the occurrence of Neimark-Sacker bifurcation in system (1) near the origin and $b = b_{-1}$. Thus, we have reached the following conclusion.

Theorem 5 *Assume that* $k \geq 1$. *Then, near* $b = b_{-1}$, *a Neimark-Sacker bifurcation occurs in system* (1), i.e., *a unique closed invariant curve bifurcates from the origin.*

Remark 1 *For system* (1) *without delays, i.e.,* $k_1 = k_2 = k_3 = 0$, *there exists no Neimark-Sacker bifurcations for all b.*

In order to analyze the Neimark-Sacker bifurcation stated in Theorem 5, we compute the reduced system on the center manifold associated with the pair of complex conjugate solutions $\exp\{\pm i\psi_{-1}\}$ of the characteristic equation (4). The reduced system enables us to determine the bifurcation direction, i.e., supercritical bifurcation ($b > b_{-1}$) or subcritical bifurcation ($b < b_{-1}$). Recall that $\psi_{-1} \in (-\pi/(k+1), 0)$. For convenience, let $b_0 = b_{-1}$ and $\lambda_0 = \exp\{-i\psi_{-1}\}$. We further assume that f_j satisfies

$$f_j \in C^3(\mathbb{R},\mathbb{R}), (f_j(0) = f_j''(0) = 0,\ f_j''(0) \neq 0 \text{ for all } j = 1,2,3.$$

We can rewrite (1) as $X_{n+1} = F(X_n)$, where X_n is a $(3k + 3)$-dimensional vector with the j-th component defined as

$$X_n^j = \begin{cases} x_{n-j+1}, & 1 \leq j \leq k_3 1, \\ y_{n-j+k_3+2}, & k_3 + 2 \leq j \leq k_1 + k_3 + 2, \\ z_{n-j+k_1+k_3+3}, & k_1 + k_3 + 3 \leq j \leq 3(k+1), \end{cases}$$

and $F = (F_1, F_2, \ldots, F_{3(k+1)})^T : \mathbb{R}^{3(k+1)} \to \mathbb{R}^{3(k+1)}$ is defined by

$$F_j(X_n) = \begin{cases} aX_n^1 + w_1 f_1(X_n^{k_1+k_3+2}), & j = 1 \\ aX_n^{k_3+2} + w_2 f_2(X_n^{3(k+1)}, & j = k_3 + 2 \\ aX_n^{k_1+k_3+2} + w_3 f_3(X_n^{k_3+1}), & j = k_1 + k_3 + 2, \\ X_n^{j-1}, & j \neq 1, j \neq k_3 + 2 \text{ and } j \neq k_1 + k_3 + 2. \end{cases}$$

Let $\mathcal{A} = DF(0)$, $\mathcal{B} = D^2F(0)$, and $C = D^3F(0)$. It is easy to see that

$$\det(\lambda Id_{3(k+1)} - \mathcal{A}) = \lambda^{3k} \det\Delta(\lambda). \tag{12}$$

Namely, eigenvalues of $\mathcal{A}$ are the zeros of $\det\Delta(\lambda)$. From Theorem 2(ii), $\det\Delta(\lambda)$ has exactly a pair of simple zeros $\lambda_0^{\pm 1}$ at b_0, while all the other zeros are inside the unit circle. This means that on the unit circle $\mathcal{A}$ has exactly one pair of simple zeros $\lambda_0^{\pm 1}$ at b_0. Moreover, it easy to see that $\mathcal{A}$ continuously depends on b. Then we regard $\mathcal{A}$ as a continuous map with respect to b and rewrite $\mathcal{A}$ as $\mathcal{A}(b)$. Let

$$c_1 = w_1\lambda_0^{-k_1},\ c_2\lambda_0 - a,\ c_3 = (\lambda_0 - a)^2 \lambda_0^{k_2} / w_2.$$

Then $c = (c_1, c_2, c_3)^T \in \mathbb{R}^3$ satisfies $\Delta(\lambda_0)c = 0$. Consider $q^0 \in \mathbb{R}^{3(k+1)}$ with the s-th component given by

$$q_s^0 = \begin{cases} c_1\lambda_0^{1-s}, & 1 \le s \le k_3+1 \\ c_2\lambda_0^{k_3+2-s}, & k_3+2 \le s \le k_1+k_3+2, \\ c_3\lambda_0^{k_1+k_3+3-s}, & k_1+k_3+3 \le s \le 3(k+1). \end{cases}$$

Then q^0 is an eigenvector for $\mathcal{A}(b_0)$ corresponding to the eigenvalue λ_0; namely

$$\mathcal{A}(b_0)q^0 = \lambda_0 q^0.$$

The adjoint equation of (1) is

$$\begin{cases} x_{n-1} = ax_n + w_3 f_3(z_{n+k_3}), \\ y_{n-1} = ay_n + w_1 f_1(x_{n+k_1}), \\ z_{n-1} = az_n + w_2 f_2(y_{n+k_2}), \end{cases} \tag{13}$$

The associated characteristic matrix of the linearized system of (13) at the origin is

$$\Delta^*(\lambda) = \begin{pmatrix} \lambda^{-1}-a & 0 & -w_3\lambda^{k_3} \\ -w_1\lambda^{k_1} & \lambda^{-1}-a & 0 \\ 0 & -w_2\lambda^{k_2} & \lambda^{-1}-a \end{pmatrix}.$$

Therefore,

$$\det \Delta^*(\lambda) = (\lambda^{-1} - a)^3 - b^3\lambda^{3k} = 0. \tag{14}$$

It is easy to see that $\Delta^*(\lambda^{-1}) = \Delta^T(\lambda)$. Then $\lambda \in \mathbb{C}$ satisfies $\det \Delta^*(\lambda) = 0$ if and only if $\det\Delta(\lambda^{-1}) = 0$. Therefore, at b_0, $\det \Delta^*(\lambda)$ has exactly a pair of simple zeros $\lambda_0^{\pm 1}$ on the unit circle while all the other zeros are outside the unit circle. Let $d = (d_1, d_2, d_3)^T \in \mathbb{C}$ satisfy $d^{-T}c = 1$ and $d^{-T}\Delta(\lambda_0) = 0$. Indeed, we have

$$d_1 = \bar{D}w_3\lambda_0^{k_3},\ d_2 = \bar{D}(\lambda_0^{-1}-a)^2\lambda_0^{-k_2}/w_2,\ d_3 = \bar{D}(\lambda_0^{-1}-a),$$

where $D = \frac{1}{3}w_1^{-1}w_3^{-1}\lambda_0^{k_1+k_3+1}[(k+1)\lambda_0 - ak]^{-1}$. Then $\Delta^*(\lambda_0)\,d = 0$. Consider $p^0 \in \mathbb{R}^{3(k+1)}$ with the s-th component given by

$$p_s^0 = \begin{cases} d_1, & s = 1, \\ d_1(\lambda_0^{-1}-a)\lambda_0^{2-s}, & 2 \le s \le k_3+1, \\ d_2, & s = k_3+2, \\ d_2(\lambda_0^{-1}-a)\lambda_0^{k_3+3-s}, & k_3+3 \le s \le k_1+k_3+2, \\ d_{3,} & s = k_1+k_3+3, \\ d_3(\lambda_0^{-1}-a)\lambda_0^{k_1+k_3+3-s}, & k_1+k_3+3 \le s \le 3k+3. \end{cases}$$

Then p^0 is an eigenvector for AT (b_0) corresponding to the eigenvalue λ_0^{-1}; namely

$$\mathcal{A}^T(b_0)p^0 = \bar{\lambda}_0 p^0$$

Then $< p^0, q^0 > = 1$ and $< p^0, \overline{q}^0 > = 0$, where $< . , .>$ means the standard scalar product in $\mathbb{C}^{3(k+1)}$: $< p, q > = \overline{p}^T q$. Any vector $X \in \mathbb{R}^{3(k+1)}$ can be uniquely represented for b near b_0 as

$$X = zq(b) + \overline{zq(b)}$$

for some complex z, where $p, q : \mathbb{R} \to \mathbb{C}^{3(k+1)}$ are smooth in b with $q(b_0) = q^0$ and $p(b_0) = p^0$. Obviously, $z = < p(b), X>$. Thus the mapping $F : \mathbb{R}^{3(k+1)} \to \mathbb{R}^{3(k+1)}$ can be transformed for b near b_0 into the following form:

$$z \mapsto \lambda(b)z + g(z, \overline{z}, b), \tag{15}$$

where $\lambda(b)$ can be written as $\lambda(b) = [1 + \phi(b)]e^{i\theta(b)}$ and $\phi(b)$ and $\theta(b)$ are smooth functions with $\phi(b_0) = 0$ and $\theta(b_0) = -\psi_{-1}$, and

$$g(z, \overline{z}, b) = \sum_{s+l \geq 2} \frac{1}{s!l!} g_{sl}(b) z^s \overline{z}^l .$$

Let $z = re^{i\theta}$. Then we can rewrite the normal form (15) as

$$\begin{aligned} r &\mapsto r + Cr(b - b_0) + Ar^3 + \text{h.o.t.}, \\ \theta &\mapsto \theta + B + Er^2 + \text{h.o.t.}, \end{aligned} \tag{16}$$

where $B = |\arg(\lambda_0)|, C = \frac{d}{db}|\lambda(b_0)|$, and

$$A = \operatorname{Re}(\overline{\lambda}_0 g_{21}(b_0)) - \operatorname{Re}\left[\frac{(1-2)\lambda_0^{-2}}{1-\lambda_0} g_{11}(b_0) g_{20}(b_0)\right] - \frac{1}{2}|g_{11}(b_0)|^2 - |g_{02}(b_0)|.$$

Therefore, the direction and stability of the Neimark-Sacker bifurcation of (1) can be determined by the signs of the coefficients A and C; B and E give the asymptotic information on rotation numbers. More precisely, if $AC < 0$ (respectively, > 0), then the Neimark-Sacker bifurcation of (1) at b_0 is supercritical (respectively, subcritical) and the unique closed invariant curve bifurcating from (0, 0, 0) for b near b_0 has the same stability as the trivial solution had before the occurrence of bifurcation (respectively, unstable). In fact, in view of Theorem 2(iii), we have $C < 0$.

Under the condition (H), by a direct computation, we have

$$g_{20}(b_0) = 0,\ g_{11}(b_0) = 0,\ g_{02}(b_0) = 0,$$

and

$$\begin{aligned} g_{21}(b_0) &= b_1 f_1'''(0)\overline{d_1} c_2 c_2 \overline{c_2} \lambda_0^{-k_1} + b_2 f_2'''(0)\overline{d_2} c_3 c_3 \overline{c_3} \lambda_0^{-k_2} + b_3 f_3'''(0)\overline{d_3} c_1 c_1 \overline{c_1} \lambda_0^{-k_3} \\ &= D\overline{\omega}\{b_0^3 w_1 w_3 f_1'''(0) + b_0^8 w_2^{-3} f_2'''(0) + b_0 w_1^3 w_3 f_3'''(0)\}\lambda_0^{-k_1-k_3-k} \\ &= D\overline{\omega}\{w_1^2 w_2 w_3^2 f_1'''(0) + b_0^{\frac{8}{3}} w_2^{-\frac{1}{3}} w_3^{\frac{8}{3}} f_2'''(0) + w_1^{\frac{10}{3}} w_2^{\frac{1}{3}} w_3^{\frac{4}{3}} f_3'''(0)\}\lambda_0^{-k_1-k_3-k} \\ &= \frac{1}{3}\overline{\omega}\{w_1 w_2 w_3 f_1'''(0) + w_1^{\frac{5}{3}} w_2^{-\frac{1}{3}} w_3^{\frac{5}{3}} f_2'''(0) + w_1^{\frac{7}{3}} w_2^{\frac{1}{3}} w_3^{\frac{1}{3}} f_3'''(0)\}. \end{aligned}$$

$$\cdot[(k+1)\lambda_0 - ak]^{-1}\lambda_0^{1-k}$$

$$= \frac{1}{3}\bar{\omega} b_0^3 K[k+1)\lambda_0 - ak]^{-1}\lambda_0^{1-k},$$

where

$$K = f_1'''(0) + b_0^2 w_2^{-2} f_2'''(0) + b_0^{-2} w_1^2 f_2'''(0). \tag{17}$$

Therefore,

$$\begin{aligned}
\text{sign}(AC) &= -\text{sign}(A)] \\
&= -\text{si}\,gn\{\text{Re}\,\bar{\lambda}_0 g_{21}(b_0))\} \\
&= -sign\left\{ b_0 K\,\text{Re}\frac{\bar{\omega}\lambda_0^{-k}}{(k+1)\lambda_0 - ak} \right\} \\
&= sign\left\{ K\left[(k+1)\cos\left[-(k+1)\psi_{-1} + \tfrac{2\pi}{3}\right] - ak\cos(-k\psi_{-1} + \tfrac{2\pi}{3})\right]\right\} \\
&= -\text{si}\,gn\left\{ K\left[v'(\psi_{-1}) + \sqrt{3}u'(\psi_{-1})\right]\right\} \\
&= -sign\{K\}.
\end{aligned}$$

The last equality follows from (10). Therefore, we have the following results.

Theorem 6 *Assume that* $k \geq 1$ *and* (*H*) *holds. Then, at* $b = b_{-1}$, *system* (1) *undergoes a Neimark-Sacker bifurcation. The direction of bifurcation and the stability of the bifurcating closed invariant curve are determined by K. More precisely, if* $K > 0$ (*respectively,* < 0), *then the bifurcation is supercritical* (*respectively, subcritical*), *i.e., the bifurcating closed invariant curve exists for* $b > b_{-1}$ (*respectively,* $< b_{-1}$), *and the bifurcating closed invariant curve is orbitally asymptotically stable* (*respectively, unstable*).

Remark 2 *In particular, if system* (1) *has the same activation functions, i.e.,* $f_1(x) = f_2(x) = f_3(x) = f(x)$, *then* $\text{sign}\{K\} = sign\{f'''(0)\}$. *Therefore, The direction of bifurcation and the stability of the bifurcating closed invariant curve are determined by* $f'''(0)$. *More precisely, if* $f'''(0) > 0$ (*respectively,* < 0), *then the bifurcation is supercritical* (*respectively, subcritical*), *i.e., the bifurcating closed invariant curve exists for* $b > b_{-1}$ (*respectively,* $< b_{-1}$), *and the bifurcating closed invariant curve is orbitally asymptotically stable* (*respectively, unstable*).

Remark 3 *In this section, we only consider the Neimark-Sacker bifurcation at* $b = b_{-1}$. *In fact, we can obtain the existence of Nemark-Sacker bifurcating closed invariant curves with b at other Neimark-Sacker bifurcation points, such as* $u(\varphi_j)$ $(j \in \mathbb{Z}(1, k))$ and b_s $(s \in \mathbb{Z}_{\pm(k+1)} \setminus \{-1\})$. *Moreover, these bifurcating closed invariant curves are all unstable because of the unstable manifold containing the origin. Similarly, we can use the above discussion to obtain their bifurcation direction.*

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