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A DOMINANT TRANSFER APPROXIMATION FOR THE NONLINEAR WAVE-WAVE INTERACTIONS IN WIND WAVE SPECTRA

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ABSTRACT: The exact computation of nonlinear wave-wave interactions is time consuming because it includes the nonlinear wave-wave interactions over all of wavenumber space. A more efficient method to do the computation is developed in this study. Instead of using the entire spectrum, the new method uses only a specific wave number which gives the dominant transfer and a scaling factor, F_d, which we denote the 'dominant factor'. In other words we represent the nonlinear transfer as $S_{nl} = F_d \times S_{nl}(k_d, \theta)$. However, the new method is intended for deep water waves only. This approach reduces the computation time and also gives a more accurate calculation than methods based on the well-known discrete interaction approximation (DIA).

***AMS Mathematical Subject Classification*:** 76B15, 76M15

***Keywords and phrases*:** wind waves, action density, nonlinear interactions, quadruplets, dominant transfer, dominant factor.

1. INTRODUCTION

Wind-generated waves are described by the energy spectrum, $E(f, \theta)$, which obeys an energy conservation relation, whereby energy is input to the spectrum by wind and removed by wave-breaking dissipation,

$$\frac{dE}{dt} = S_{tot}. \tag{1}$$

In deep water, S_{tot} can be represented as:

$$S_{tot} = S_{inp} + S_{wcap} + S_{nl4} \tag{2}$$

where S_{inp} is the energy input by wind, S_{wcap} is the energy dissipation by white-capping and wave breaking, and S_{nl4} represents the non-linear quadruplet wave-wave interactions.

The nonlinear interactions play an important role in the evolution of wind waves, representing a mechanism for shifting wave energy to lower and higher frequencies within the spectrum. In deep water, the nonlinear interactions describe the resonant exchange of energy, momentum, and action between four spectral components with wave number vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and $\mathbf{k}_4$ and corresponding frequencies ω_1, ω_2, ω_3, and ω_4. Efficient computation of the nonlinear term is hampered by the complexity of the functional form and its computation is several orders of magnitude more expensive than all other terms in equation (2).

The basic equation describing S_{nl4} is the Boltzmann integral or kinetic equation, proposed by Hasselmann (1962). Although Hasselmann and Hasselmann (1981) developed the pioneering systematic computation of S_{nl4}, their method is not practical for operational applications. Therefore Hasselmann et al. (1985) developed the Discrete Interaction Approximation, DIA, which dramatically increases the computational speed. DIA enabled the development of third generation wave prediction models such as WAM and SWAN. However, DIA has a number of shortcomings (Van Vledder, 2000):

- DIA compares poorly with full integrations of S_{nl4} for many types of spectra.
- DIA's estimated spectral width is too large compared with measurements and full integrations of S_{nl4}.

- DIA produces too much transfer towards higher frequencies (Fig. 3.6, Komen *et al.*, 1984), which has impact on the tuning that has to be imposed on the source terms for wind input and dissipation.

Since DIA's development, improvements to the nonlinear term computation method have been attempted, consisting of two major approaches:

- More efficient calculation of the full S_{nl4} expression, for example
  - Special grid and scaling symmetries utilized by Tracy and Resio (1982), Resio and Perrie (1991)
  - Reduced interaction approximation by Lin and Perrie (1999)
  - Neural network methodologies by Krasnopolsky, *et al.* (2002)
- Enhancement on DIA, for example
  - Multiple DIA or MDIA by Hashimoto *et al.* (2002)
  - Optimal multiple DIA by Polnikov (2003)
  - DIA with four representative quadruplets by Tolman (2004).

The present study attempts to speed up the calculation of the full S_{nl4} expression using the *Webb-Tracy-Resio* (WTR) formulation and selecting only the dominant transfer in the computation. Related papers such as Van Vledder (2006) present detailed comparisons with other approaches mentioned above.

The basic methodology of the WTR formulation is described in Section 2. In Section 3, the principal features of the new method are presented. Results and comparisons are presented in Section 4, and conclusions in Section 5.

2. WTR METHOD

The Boltzmann integral describes the rate of change of action density at a particular wave number due to resonant interactions between quadruplets of wave numbers. To interact, these wave numbers must satisfy the following resonance conditions:

$$\omega_1 + \omega_2 = \omega_3 + \omega_4 \tag{3}$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \tag{4}$$

where ω_i is the radian frequency and $\mathbf{k}_i$ is the wave number. The frequency and the wave number are related by the dispersion relation:

$$\omega^2 = gk \tanh(kd) \tag{5}$$

where g is the gravitational acceleration and d is the water depth. The rate of change of action density n_1 at wave number $\mathbf{k}_1$ due to all quadruplet interactions involving $\mathbf{k}_1$ is given by the following six-fold Boltzmann integral:

$$\frac{dn_1}{dt} = \iiint C(\mathbf{k}_1, \mathbf{k}_2 \mathbf{k}_3, \mathbf{k}_4)[n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)] \tag{6}$$

$$\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\, \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$$

where n_i is the action density $n(\mathbf{k}_i)$ at wave number $\mathbf{k}_i$, ω_i is the angular frequency at $\mathbf{k}_i$, the $\delta(\ldots)$ is the Dirac delta function and the term C is the coupling coefficient, for which expressions have been given by Webb (1978) and Tracy and Resio (1982). The delta functions in equation (6) reflect the resonance conditions in equations (3)-(4).

Webb (1978) introduced a transfer function $T(\mathbf{k}_1, \mathbf{k}_3)$

$$\frac{dn_1}{dt} = \int T(\mathbf{k}_1, \mathbf{k}_3) d\mathbf{k}_3 \tag{7}$$

where

$$T(\mathbf{k}_1, \mathbf{k}_3) = 2\int\int C(\mathbf{k}_1, \mathbf{k}_2\mathbf{k}_3, \mathbf{k}_4)[n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)]$$
$$\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \tag{8}$$
$$\Theta(|\mathbf{k}_1 - \mathbf{k}_4| - |\mathbf{k}_1 - \mathbf{k}_3|)\, d\mathbf{k}_2 d\mathbf{k}_4$$

and

$$\Theta(x) = 1 \quad \text{if } x > 0$$
$$\Theta(x) = 0 \quad \text{if } x \le 0 \tag{9}$$
$$x = |\mathbf{k}_1 - \mathbf{k}_4| - |\mathbf{k}_1 - \mathbf{k}_3| .$$

The $\Theta(x)$ function reduces the domain of computation by half, because it is assumed that the spectrum is symmetric around 0°. The delta function limits the wave number configuration to $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$, so that the four wave-number vectors form a parallelogram.

In order to evaluate the integral numerically, we must fix values for $\mathbf{k}_1(x, y)$ and $\mathbf{k}_3(x, y)$ and find $\mathbf{k}_2(x, y)$ and $\mathbf{k}_4(x, y)$ that satisfy the resonance conditions (3) and (4). The set of solutions can be represented as an egg-shaped two-dimensional locus in a Cartesian coordinate system in $\mathbf{k}_2$ space where $\mathbf{k}_{2x}$ is the x-axis and $\mathbf{k}_{2y}$ is the y-axis. On this locus, **n** is the normal or radial direction and **s** is in the increasing θ or tangential direction.

Defining a new vector, $\mathbf{P} = \mathbf{k}_1 - \mathbf{k}_3$ and using a (**s**, **n**) coordinate system for $W(\mathbf{k}_2)$, Tracy and Resio (1982) stated the transfer integral, equation (8), as:

$$T(\mathbf{k}_1, \mathbf{k}_3) = 2\oint [n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)]$$
$$\times C(\mathbf{k})\Theta(x)\frac{1}{|\nabla W(k_{2x}, k_{2y})|} ds \tag{10}$$

where

$$|\nabla W(k_{2x}, k_{2y})| = \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right]^{1/2} \tag{11}$$

and

$$\frac{\partial W}{\partial x} = \sqrt{g}\left\{\frac{x}{2}(x^2 + y^2)^{-3/4} - \frac{(P_x + x)}{2}[(P_x + 2)^2 + (P_y + 2)^2]^{-3/4}\right\} \tag{12}$$

$$\frac{\partial W}{\partial y} = \sqrt{g}\left\{\frac{y}{2}(x^2 + y^2)^{-3/4} - \frac{(P_y + y)}{2}[(P_y + 2)^2 + (P_y + 2)^2]^{-3/4}\right\}. \tag{13}$$

P_x and P_y are the rectangular components of **P** and x and y are the rectangular components of $\mathbf{k}_2$.

Finally, in polar coordinates, the nonlinear energy transfer, equation (7), can be written as:

$$\frac{dn_1}{dt} = \int_0^\infty \int_0^{2\pi} T(\mathbf{k}_1, \mathbf{k}_3)\, k_3 d\theta_3 dk_3. \tag{14}$$

3. FORMULATION OF THE MODEL

As seen in equation (10), to find the transfer function, $T(\mathbf{k}_1, \mathbf{k}_3)$, we have to integrate along the loci, over a finite number of points, which we represent as p. The nonlinear energy transfer must then be computed over the entire spectrum, from the smallest k_3 to the highest k_3 and from $\theta = 0$ to $\theta = 2\pi$ as shown on equation (14). A typical wave model uses a spectral discretization of 30 wave number bins and 36 angular bins, where $wn(1) = k_0$, $wn(2) = \lambda\ wn(1)$, $wn(3) = \lambda\ wn(2)$, and so on, where λ is the incremental factor. Suppose there are 36 points on the resonance locus. Therefore to compute one point of nonlinear interaction needs $30 \times 36 \times 36$ or 38880 cycles. The entire spectrum will require $38880 \times 30 \times 36$ or more than 4×10^7 cycles.

In this study we show that certain sets of (k_1, k_3) give the dominant transfer. On those sets, the maximum nonlinear interaction occurs, much greater than the nonlinear transfer from other contributions. Experimental computations are conducted for wave conditions represented by snapshot JONSWAP spectra, including the range of peakedness from $\gamma = 1$ to $\gamma = 7$. Figure 1 suggests that there is a set of (k_1, k_3) which gives the maximum or dominant transfer in the nonlinear process in these cases. In terms of the polar coordinates, we use (k, θ) with wavenumbers spaced logarithmically in 'rings', $k_{i+1} = \lambda\ k_i$, where λ is usually in the range about 1.05 to 1.21. We find through numerical experiments that given wavenumbers $k_{1(i)}$ and $k_{3(j)}$, with polar 'ring' indices i and j respectively, the 'ring' of dominant transfer can be stated as:

$$j = i + m_d \tag{15}$$

and m_d is defined as:

$$m_d = \begin{cases} 1 \text{ if } m \le 1 \\ \text{nearest integer of } m \text{ if } m > 1, \end{cases}$$

$$m = \frac{\ln \lambda_0}{\ln \lambda} \tag{16}$$

where λ is an incremental factor of $\mathbf{k}$ and λ_0 is the λ that is used in most applications of operational third-generation wave models, which equals 1.21. See for example Tolman (1999). In this paper $\lambda = 1.21$ also.

With this result, we propose a new method to approximate the Boltzmann integral, which we denote as the *Dominant Transfer Approximation* (DTA). Rather than integrating over the entire spectral domain from $k_3 = 0$ to $k_3 = \infty$, we use only the value for k_3 which gives maximum transfer as:

$$\frac{dn_1}{dt} = F_d \int_0^{2\pi} T(\mathbf{k}_1, \mathbf{k}_{3_d})\ \Delta k_{3_d} k_{3_d} d\theta_3 \tag{17}$$

where

$$F_d = \frac{Max(Snl_{all-k})}{Max(Snl_{d-k})} \tag{18}$$

where F_d or the dominant factor is a scaling factor. In this approximation, $Snl_{all\text{-}k}$ is pre-computed from the entire spectrum (see Eqn.(14)) and Snl_{d-k} is computed from the dominant set.

$$Snl_{d-k} = \int_0^{2\pi} T(\mathbf{k}_1, \mathbf{k}_{3_d})\ \Delta k_{3_d} k_{3_d} d\theta_3. \tag{19}$$

The method for finding F_d is presented in the next section. With equation (17), computing one point of nonlinear transfer for one time step of the 30×36 spectral grid with 36 locus of resonance bins needs 1080 cycles only.

4. FINDING F_D

In this section, we are going to discuss F_d, the key to the DTA method. We want to see how F_d behaves if major JONSWAP parameters change. Before we implement this method to a real wave model, we must do some experiments to collect the data to find F_d. First we set up the grid that is common for the wave model. Then we compute S_{nl4} with the WTR method, equation (14). Next, we compute the same case with equation (19). Finally, we determine F_d based on the positive lobe of the nonlinear transfer, following equation (18), because this drives the spectral downshifting during wave development (Komen *et.al.*, 1994). Once we find F_d, we can use it in a wave model to compute S_{nl4} with equation (17).

The results from the DTA method are compared to the results from the full integration method of Tracy and Resio (1982) and Resio and Perrie (1991). Before doing comparisons between the DTA method and the WTR method, we must examine the behaviour of F_d as a function of the key JONSWAP parameters. We consider a variety of cases defined by prescribed JONSWAP input spectra, with Phillip's α coefficient varying from 0.001 to 0.04, f_p varying from 0.2 to 0.4 Hz, peakedness varying as γ = 1, 3, 5, and 7 and spectral spreading factor = $f(\cos^{2n}\theta)$ where n = 1, 2 and 6, representing windsea as well as swell cases.

Figures 2 and 3a - 3b show that while F_d is a function of γ and spectral spreading $\cos^{2n}\theta$, the Phillips' α coefficient and peak frequency, f_p, have little influence on F_d. This result suggests that the F_d is function of γ and spreading factor only.

These tests assume a JONSWAP spectrum with parameters (unless otherwise varied) $\alpha = 0.01$, peak spectral width parameters σ_a = 0.07 and σ_b = 0.09, f_p = 0.3 Hz and spectral spreading factor = $f(\cos^2\theta)$ for differing peakedness values γ = 1, 3, 5, and 7 respectively.

The average time required to compute the 30*x*18*x*36 (*wavenumber bins-half circle of directional bins-locus of resonance bins*) spectral grid is 12.163 seconds for the full integration method (Exact method). With a special grid, the WTR method time is 4.864 seconds, compared to 0.624 seconds for the DTA program. However, DIA only takes 0.001 seconds, which is comparatively much less, as shown in Fig. 4.

The two-dimensional and one-dimensional comparisons of DTA to the full integration following the Webb-Tracy-Resio (WTR) formulation are given in Figs. 5-6. These figures show that the DTA results compare well with the full integration WTR formulation.

5. CONCLUSION

We have shown that the nonlinear transfer due to 4-wave interactions can be approximated by the dominant transfer related to selected wavenumbers and a scaling factor F_d. The latter is shown to follow a well-defined variation, depending on wave maturity, as specified in terms of wave peakedness γ and the spectral spreading function and varies in the range 3.7 to its limit of 1.5 for old waves. For a variety of JONSWAP input spectra, we show that this dominant transfer approximation (DTA) formulation compares well with the full integration of the nonlinear transfer due to wave-wave interactions, and is competitive with DIA.

Once F_d is determined, we show that DTA represents an efficient approximate computation of the full integration. In actual computations, the WTR method needs five loops to compute the entire spectral grid, whereas the new DTA program requires four loops, and gives similar results. However, although DTA is 20 times faster than the original full integration method, it is still slower compared to DIA (about 600 times slower). Further work is needed to find the angular dependence of the scaling factor F_d and to investigate possible optimizations along the loci of resonance. Thus, we can optimize the computation.

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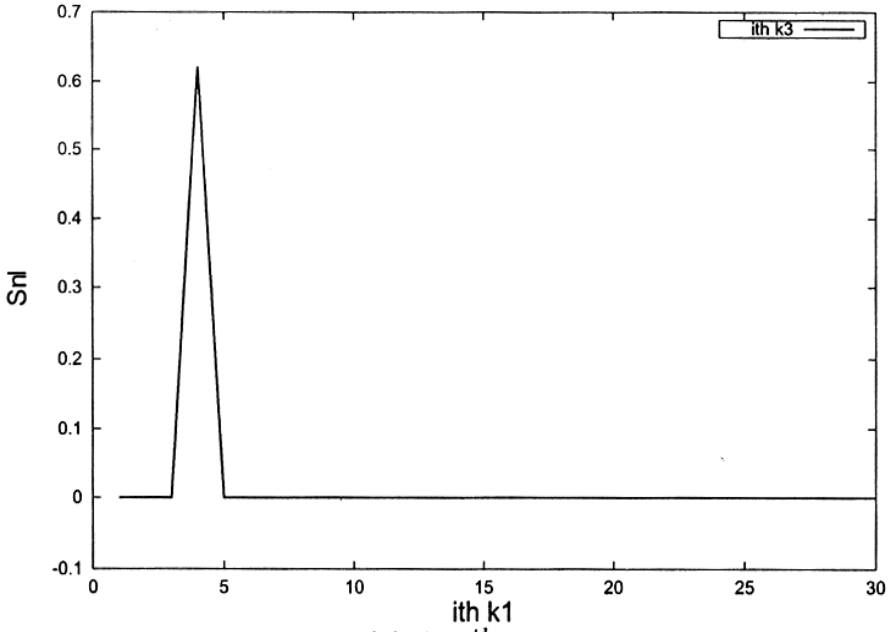
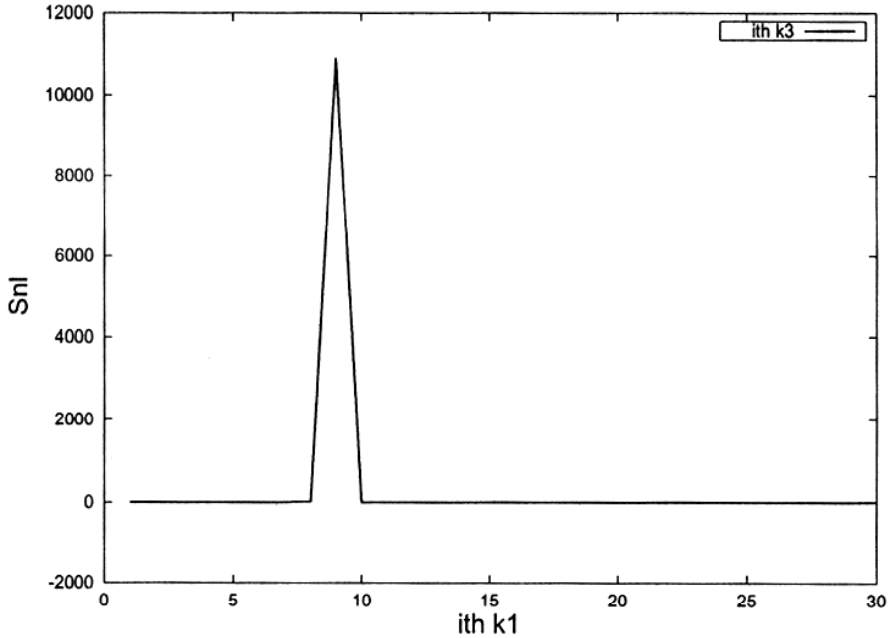
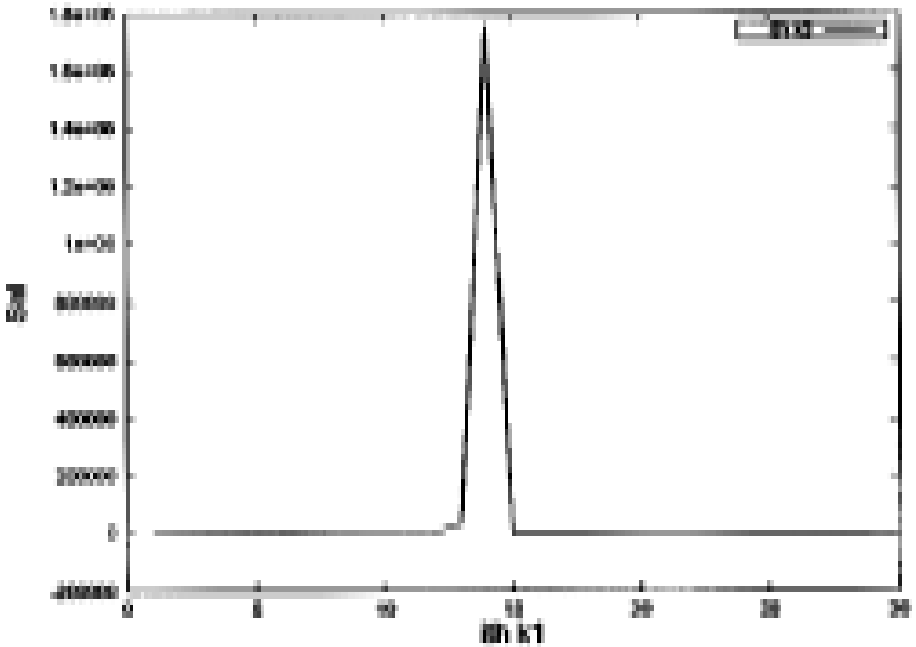
FIGURES



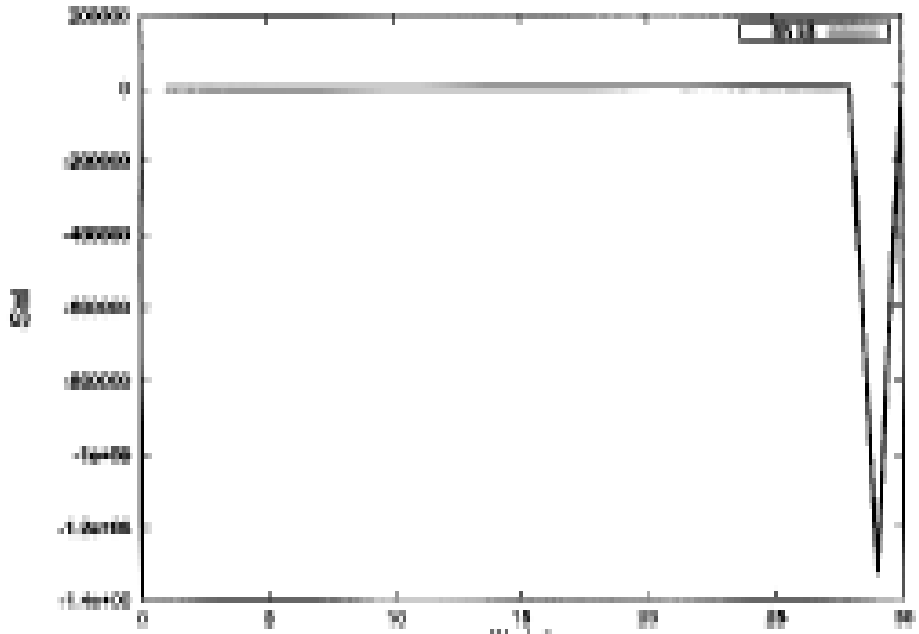
(a) At $i_{k1}^{th} = 5$



(b) At $i_{k1}^{th} = 10$



(c) At $i_{k1}^{th} = 18$



(d) At $i_{k1}^{th} = 30$

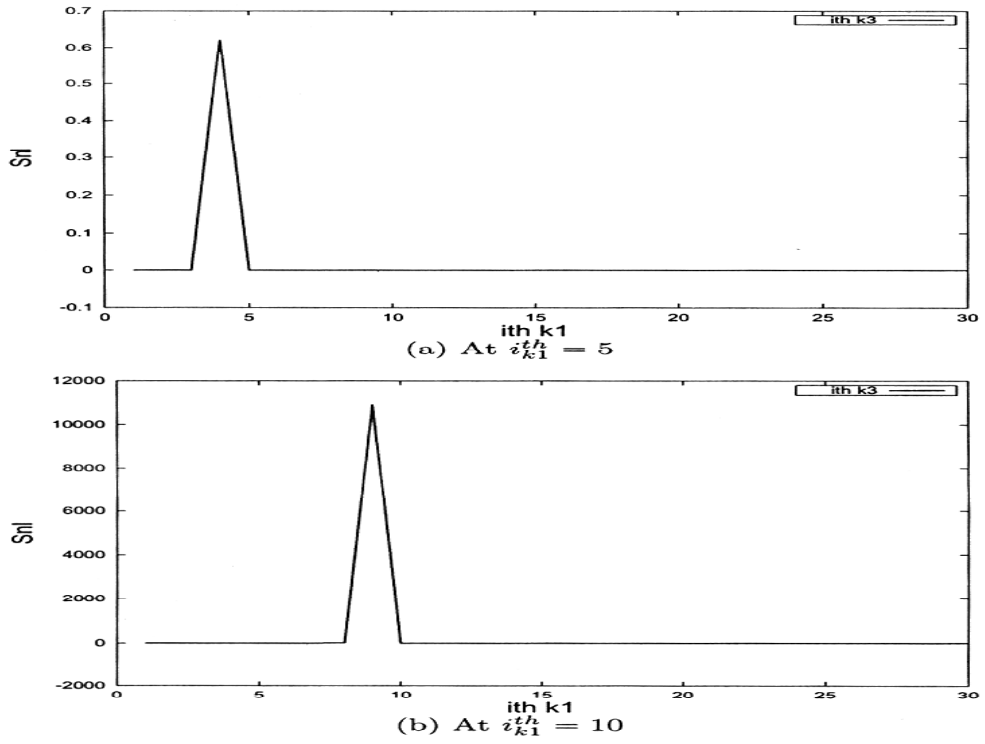


Figure 1: Dominant transfer for certain set of k_1 and k_3, where i_{k1}^{th} states the i^{th} ring of wavenumber k_1. Panel (e) presents the corresponding S_{nl4}.

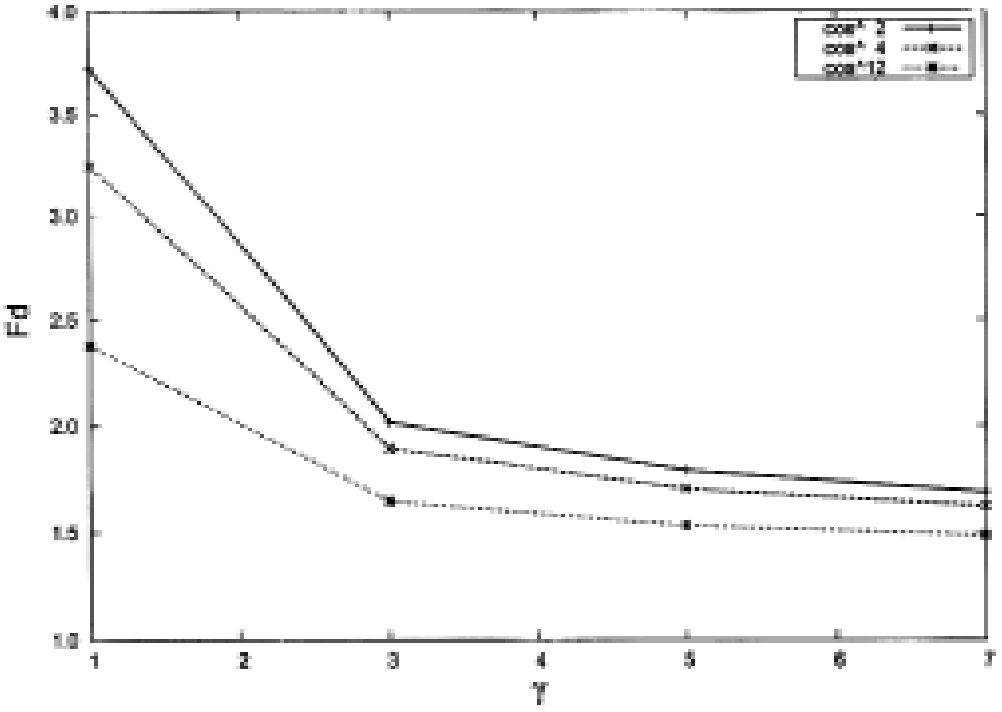
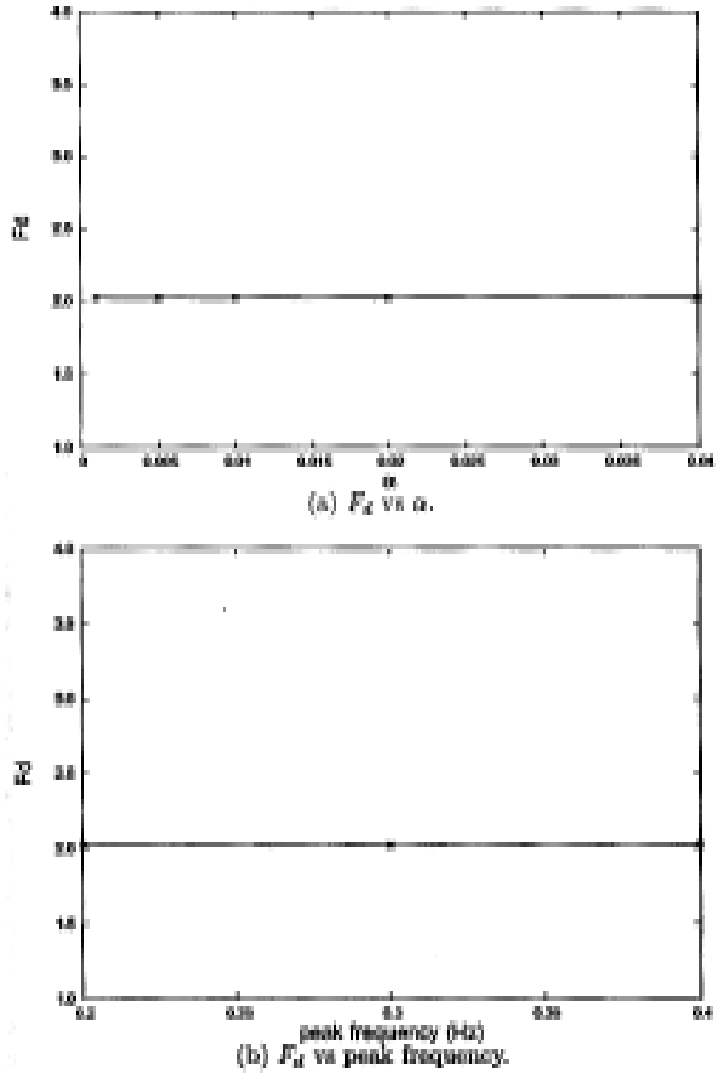


Figure 2: F_d as function of γ and spreading factor.



(a) F_d vs α.

(b) F_d vs peak frequency.

Figure 3: F_d for different α and peak frequeny.

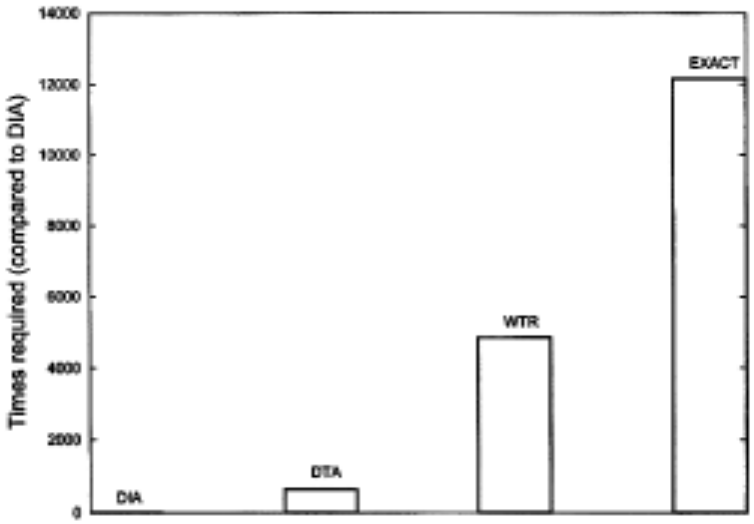
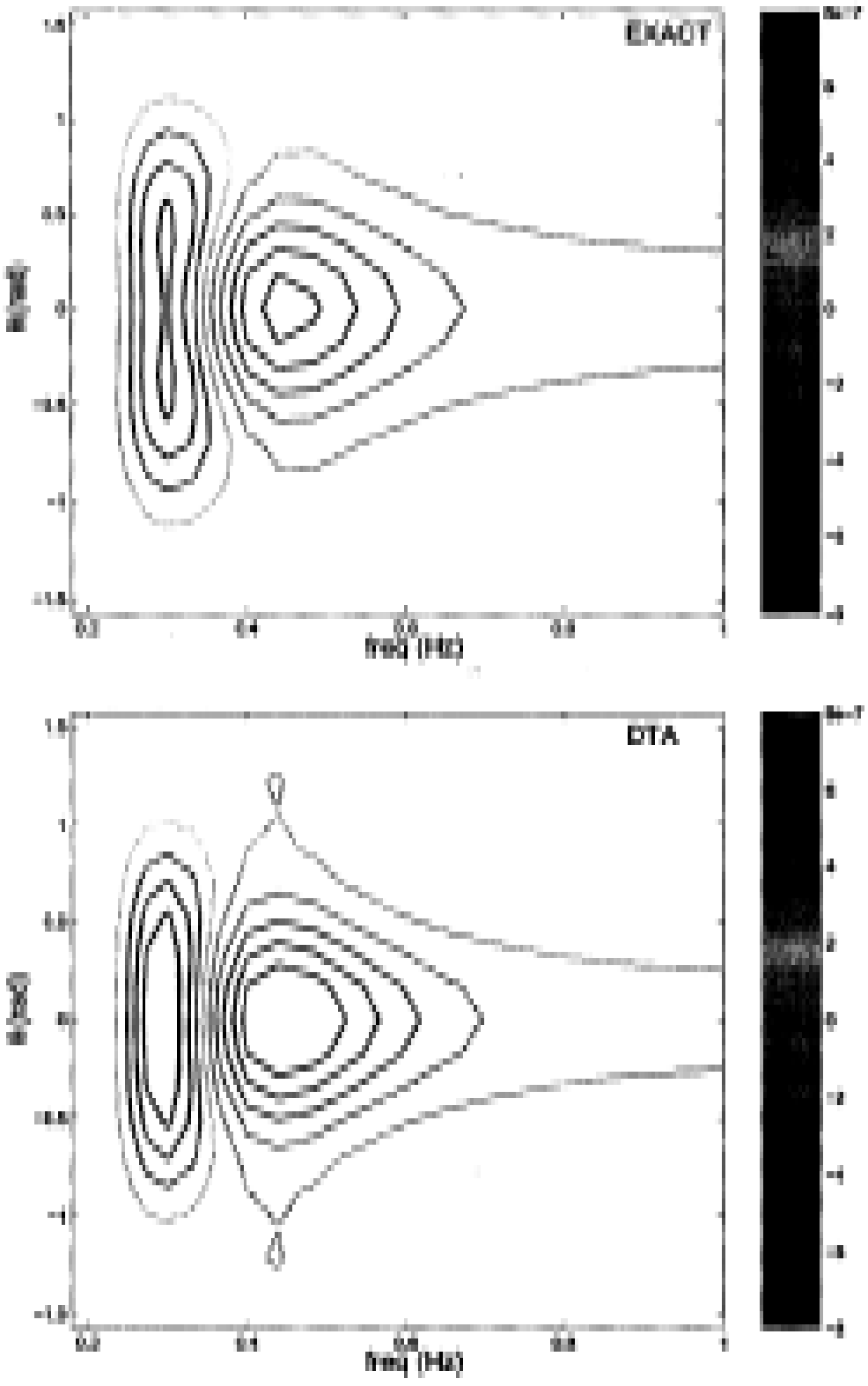
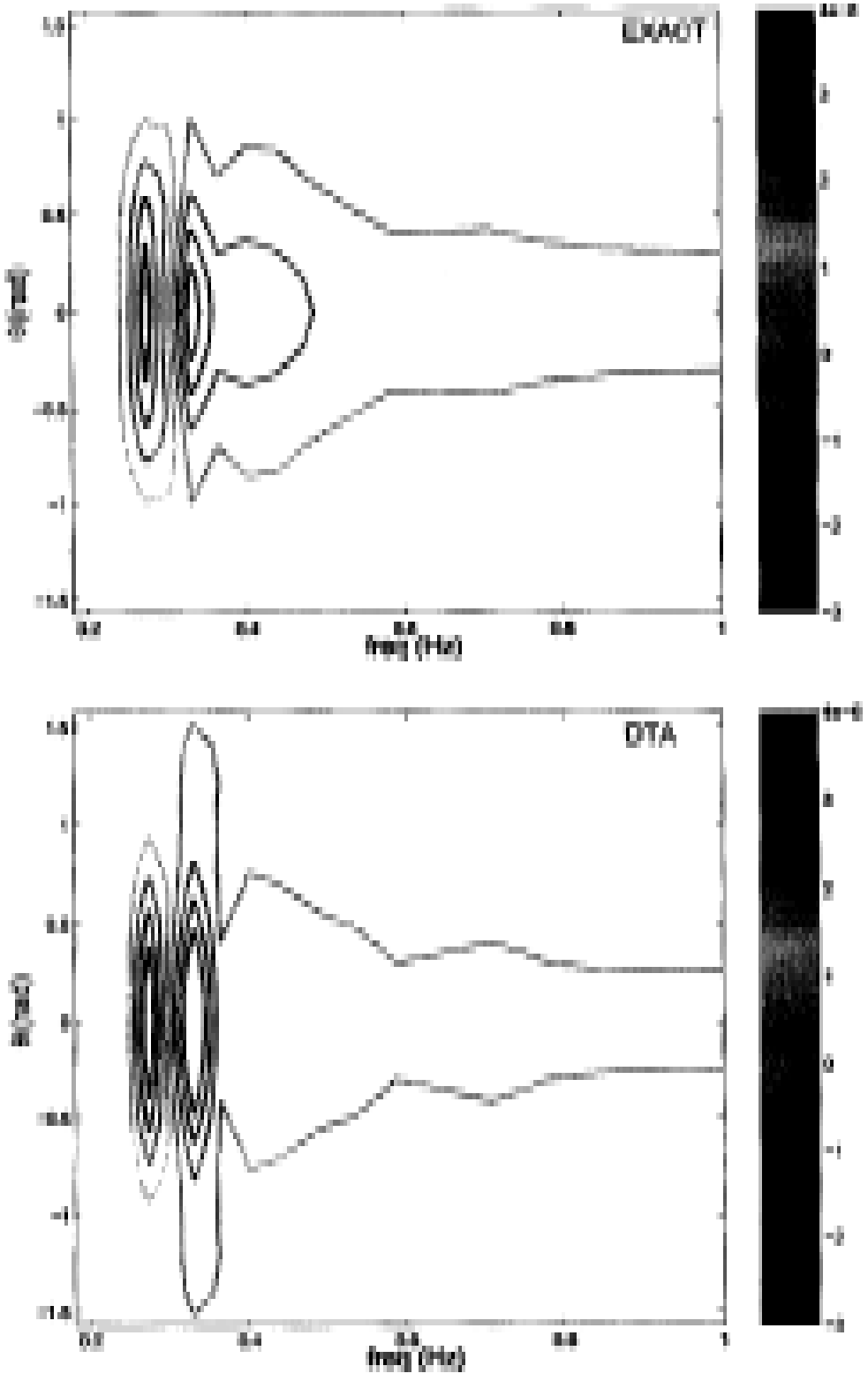


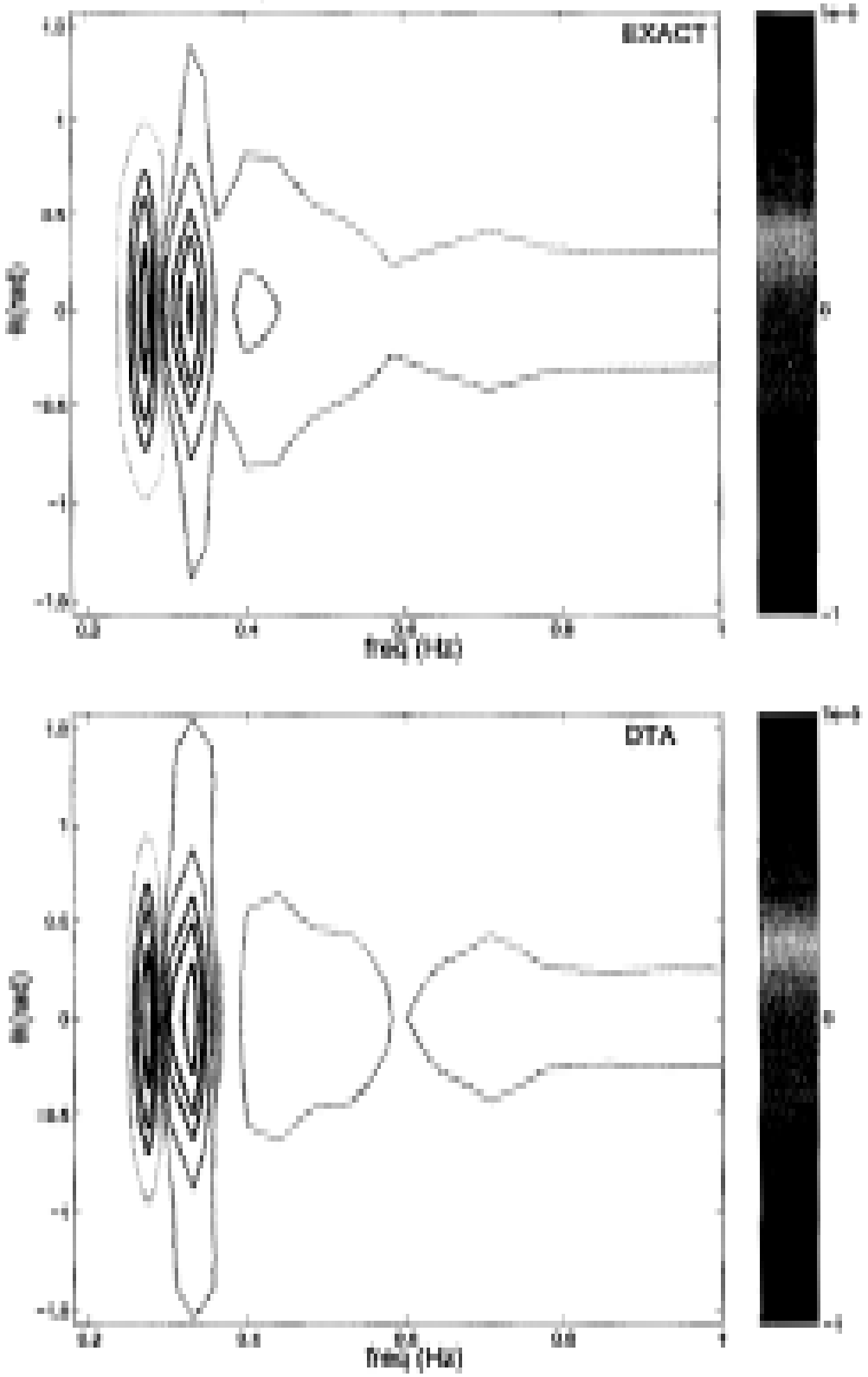
Figure 4: Comparisons of required computing time for each method.



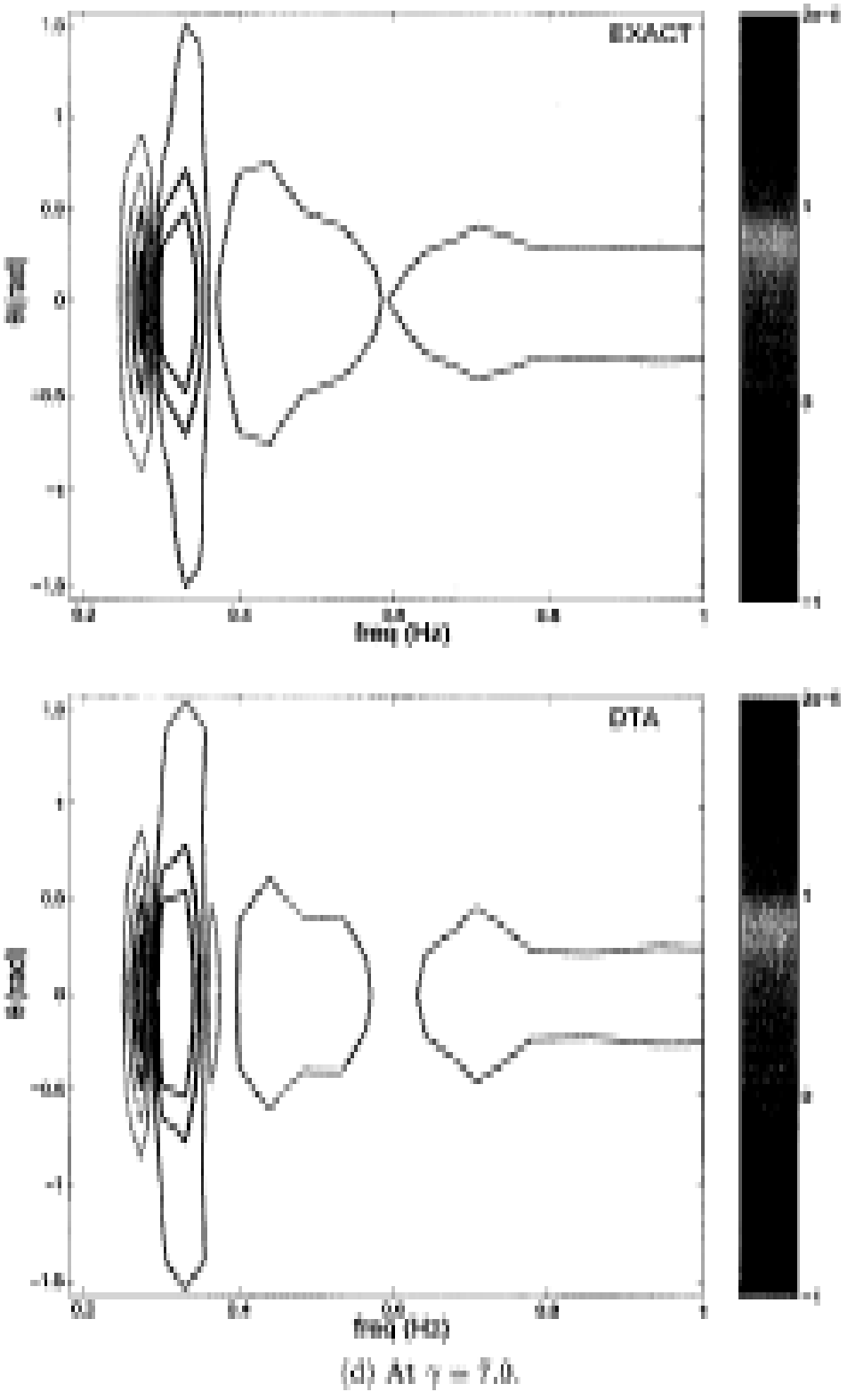
(a) At $\gamma = 1.0$.



(b) At $\gamma = 3.0$.

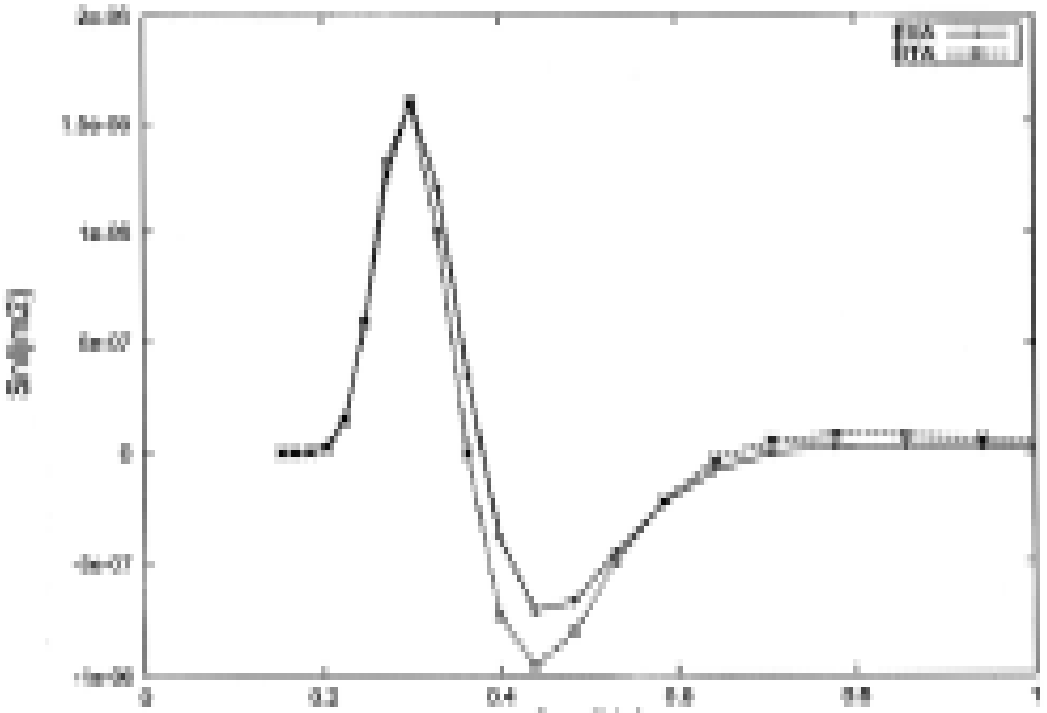


(c) At $\gamma = 5.0$.

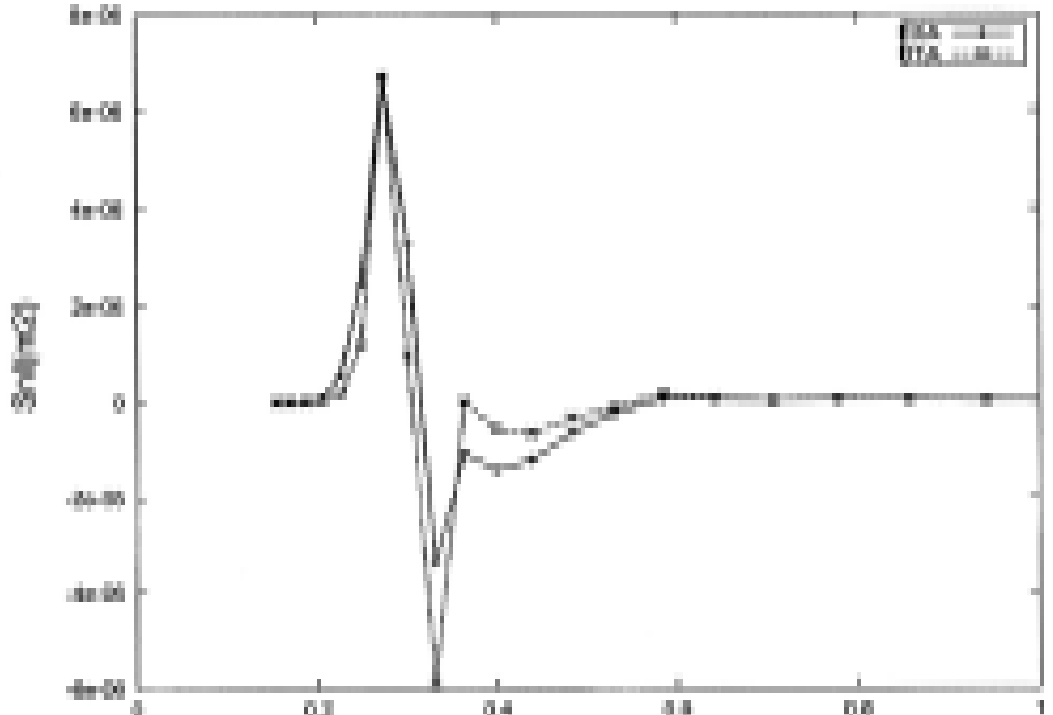


(d) At $\gamma = 7.0$.

Figure 5: 2D Snl for different γ.



(a) At $\gamma = 1.0$



(b) At $\gamma = 3.0$

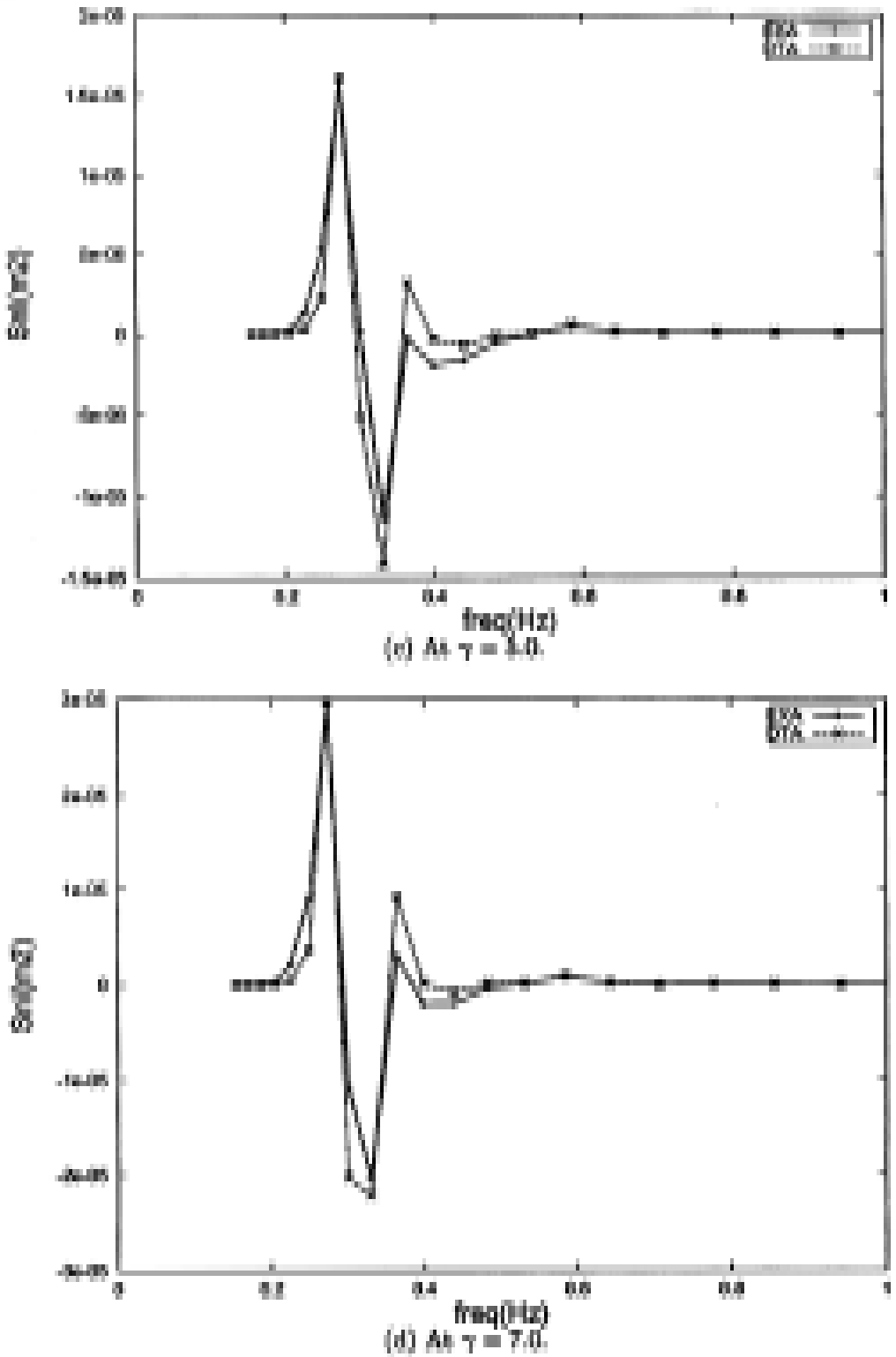


Figure 6: ID Snl for different γ.

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