International Journal of Applied Mathematics and Engineering Sciences
Vol. 2 No. 1 (June, 2017)

QUASI VARIATIONAL INEQUALITIES AND NONEXPANSIVE MAPPINGS

Muhammad Aslam Noor & Zhenyu Huang

ABSTRACT: In this paper, we suggest and analyze three-step iterations for finding the common element of the set of fixed points of a nonexpansive mappings and the set of solutions of the quasi variational inequalities. We also study the convergence criteria of three-step iterative method under some mild conditions. Our results include the previous results as special cases and may be considered as an improvement and refinement of the previously known results.

***KEY WORDS*:** Common elements, nonexpansive mappings, relaxed (γ, r)-Cocoercive mappings, nonlinear variational inequalities, Hilbert spaces.

1. INTRODUCTION

Quasi variational inequalities are being used as a mathematical programming tool in modelling various equilibria problems in economics, finance, operations, optimization, enviroment sciences, regional, pure and applied sciences. It combines novel theoretical and algorithmic advances with new domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. As a result of interaction between several branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various algorithms for solving quasi variational inequalities and related optimization problems, see [1-19]. It is well known that the solution of the quasi variational inequalities can be computed using the iterative projection method, the convergence of which requires the strongly monotonicity and Lipschitz continuity of the involved operator. These strict conditions rule out its applications in important problem. To overcome these drawback, we use the concept of the relaxed co-coercive concept, which is weaker than the strongly monotonicity. In this respect our results represent a refinement of the previously known results.

Noor [11] suggested and analyzed several three-step iterative methods for solving different classes of variational inequalities. It has been shown that three-step schemes are numerically better than two-step and one-step methods. Related to the quasi variational inequalities, is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. Motivated by the research going on these fields, we suggest and analyze a new three-step iterative method for finding the common solution of these problems. We also prove the convergence criteria of these new iterative schemes under some mild conditions. Since the quasi variational inequalities include the variational inequalities and implicit complementarity problems as special cases, results obtained in this paper continue to hold for these problems. Results proved in this paper may be viewed as a significant and improvement of the previously known results.

2. BASIC RESULTS

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $||\cdot||$ respectively. S be a nonexpansive operator.

Given a point-to-set mapping $K : u \to K(u)$, which associates a closed convexvalued $K(u)$ of H with any element u of H, consider the problem of finding $u \in K(u)$ such that

[1]This research is supported by the Higher Education Commission, Pakistan, through grant No. 1-28/HEC/HRD/2005/90.
[2]The research is partially sponsored by the Scientific Research Foundation for Returned Overseas Chinese Scholars, State Education Ministry.

$$\langle Tu,\ v-u\rangle \geq 0,\ \ \forall v \in K(u), \tag{2.1}$$

which is known as a quasi variational inequality. To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u'' \geq f(x) & on\ \Omega = [a,b] \\ u \geq M(u) & on\ \Omega = [a,b] \\ [-u'' - f(x)][u - M(u)] = 0 & on\ \Omega = [a,b] \\ u(a) = 0,\ \ u(b) = 0. & \end{array}\right\} \tag{2.2}$$

where $f(x)$ is a continuous function and $M(u)$ is the cost (obstacle) function. The prototype encountered is

$$M(u) = k + \inf_i \{u^i\}. \tag{2.3}$$

In (2.3), k represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator M provides the coupling between the unknowns $u = (u^1, u^2, \ldots, u^i)$. We study the problem (2.2) in the framework of variational inequality approach. To do so, we first define the set K as

$$K(u) = \{u : u \in H_0^1(\Omega) : u \geq M(u),\ \text{on } \Omega\},$$

which is a closed convex-valued set in $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Sobolev (Hilbert) space, see [1-5]. One can easily show that the energy functional associated with the problem (2.2) is

$$\begin{aligned} I[v] &= -\int_a^b \left(\frac{d^2v}{dx^2}\right) v dx - 2\int_a^b f(x)(v)dx, \forall v \in K(u) \\ &= \int_a^b \left(\frac{dv}{dx}\right)^2 dx - 2\int_a^b f(x)(v)dx \\ &= \langle Tv, v\rangle - 2\langle f, u\rangle \end{aligned} \tag{2.4}$$

where

$$\langle Tu, v\rangle = \int_a^b \left(\frac{d^2u}{dx^2}\right)(v)dx = \int_a^b \frac{du}{dx}\frac{dv}{dx}dx \tag{2.5}$$

$$\langle f, u\rangle = \int_a^b f(x)(v)dx.$$

It is clear that the operator T de.ned by (2.5) is linear, symmetric and positive. Using the technique of Noor [13], one can show that the minimum of the functional $I[v]$ defined by (2.4) associated with the problem (2.2) on the closed convex-valued set $K(u)$ can be characterized by the inequality of type

$$\langle Tu,\ v-u\rangle \geq \langle f,\ v-u\rangle,\ \forall v \in K(u), \tag{2.6}$$

which is exactly the quasi variational inequality (2.1). Note that, if $K^*(u)$ is the polar cone of a closed convex-valued cone $K(u)$, then problem (2.1) is equivalent to finding

$$u \in (u),\ Tu \in K^{*}(u), \text{ and } \langle Tu, u\rangle = 0.$$

which is known as the implicit complementarity problem, see the references.

Note that, if $K(u)$ is independent of the solution u, that is, $K(u) = K$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu,\ v - u\rangle \geq 0,\ \forall v \in K, \tag{2.7}$$

which is known *s* variational inequality introduced and studied by Stampacchia [18] in 1964. For the numerical methods, applications and formulations, see [1-19] and the references therein.

We now recall some well known concepts and results.

Lemma 2.1. For a given $z \in H$, $u \in K$ satisfies the inequality

$$\langle u - z,\ v - u\rangle \geq 0,\ \forall v \in K,$$

if and only if

$$u = P_K[z],$$

where P_K is the projection of H onto the closed convex set K. It is also known that the projection operator P_K is nonexpansive.

Using Lemma 2.1, one can show that the quasi variational inequality (2.1) is equivalent to a fixed-point problem. This result is due to Noor [8].

Lemma 2.2. The function $u \in K(u)$ is a solution of the quasi variational inequality (2.1) if and only if $u \in K(u)$ satisfies the relation

$$u = P_{K(u)}[u - \rho T u],$$

where $\rho > 0$ is a constant.

Lemma 2.2 implies that quasi variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the quasi variational inequalities and related optimization problems.

Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $F(S)$ and the set of the solutions of the quasi variational inequalities (1) by $QV\ I(K(u), T)$. We can characterize the problem. If $x^* \in F(S) \cap QV\ I(K(u), T)$, then $x^* \in F(S)$ and $x^* \in QV\ I(K(u), T)$. Thus from Lemma 2.2, it follows that

$$x^* = Sx^* = P_{K(u)}[x^* - \rho Tx^*] = SP_{K(u)}[x^* - \rho Tx^*],$$

where $\rho > 0$ is a constant.

This fixed point formulation is used to suggest the following multi-step iterative methods for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings S and the quasi variational inequalities (2.1).

Algorithm 2.1. For a given $x_0 \in K(x_0)$, compute the approximate solution x_n by the iterative schemes

$$z_n = (1 - c_n)x_n + c_n SP_{K(x_n)}[x_n - \rho Tx_n], \tag{2.8}$$

$$y_n = (1 - b_n)x_n + b_n SP_{K(z_n)}[z_n - \rho Tz_n], \tag{2.9}$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_{K(y_n)}[y_n - \rho Ty_n], \tag{2.10}$$

where a_n, b_n, $c_n \in [0, 1]$ for all $n \geq 0$ and S is the nonexpansive operator. Algorithm 2.1 is a three-step predictor-corrector method. For $S = I$, the identity operator, Algorithm 2.1 appears to be a new one.

Note that for $c_n \equiv 0$, Algorithm 2.1 reduces to:

Algorithm 2.2. For an arbitrarily chosen initial point $x_0 \in K(x_0)$, compute the approximate solution $\{x_n\}$ by the iterative schemes

$$y_n = (1-b_n)x_n + b_n SP_{K(x_n)}[x_n - \rho Tx_n],$$

$$x_{n+1} = (1-a_n)x_n + a_n SP_{K(y_n)}[y_n - \rho Ty_n],$$

where a_n, $b_n \in [0, 1]$ for all $n \geq 0$ and S is the nonexpansive operator. Algorithm 2.2 is called the two-step (Ishikawa iterations) iterative method. For $b_n \equiv 1$, $a_n \equiv 1$, Algorithm 2.2 reduces to:

Algorithm 2.3. For an arbitrarily chosen initial point $x_0 \in K(x_0)$, compute the sequence $\{x_n\}$ by the iterative schemes

$$y_n = SP_{K(x_n)}[x_n - \rho Tx_n],$$

$$x_{n+1} = SP_{K(y_n)}[y_n - \rho Ty_n].$$

For $S = I$, Algorithm 2.3 can be written as

$$x_{n+1} = PK(y_n)[P_{K(x_n)}[x_n - \rho Tx_n] - \rho TP_{K(x_n)}[x_n - \rho Tx_n]],$$

which is called implicit double projection method and this result is mainly due to Noor [12]. For $b_n \equiv 0$, $c_n \equiv 0$, Algorithm 2.1 collapses to the following iterative method.

Algorithm 2.4. For a given $x_0 \in K(x_0)$, compute the approximate solution x_{n+1} by the iterative schemes:

$$x_{n+1} = (1-a_n)x_n + a_n SP_{K(x_n)}[x_n - \rho Tx_n], \tag{2.11}$$

which is known as the Mann iteration (one-step method) and appears to be a new one.

For $K(u) \equiv K$, Algorithm 2.1 reduces to the following three-step iterative methods for solving the problem $F(S) \cap VI(K,T)$, which is due to Noor and Huang [16].

Algorithm 2.5. For a given $x_0 \in K$, compute the approximate solution x_n by the iterative schemes

$$z_n = (1-c_n)x_n + c_n SP_K[x_n - \rho Tx_n], \tag{2.12}$$

$$y_n = (1-b_n)x_n + b_n SP_K[z_n - \rho Tz_n], \tag{2.13}$$

$$x_{n+1} = (1-a_n)x_n + a_n SP_K[y_n - \rho Ty_n], \tag{2.14}$$

where a_n, b_n, $c_n \in [0, 1]$ for all $n \geq 0$ and S is the nonexpansive operator. Algorithm 2.5 is a three-step predictor-corrector method. For the convergence analysis of Algorithm 2.5, see Noor and Huang [16]. It is worth mentioning that three-step methods are also known as Noor iterations. Clearly Noor iterations include Mann-Ishikawa iterations as special cases. In particular, three-step methods suggested in this paper are quite general and include several new and previously known algorithms for solving variational inequalities and nonexpansive mappings.

Definition 2.1. A mapping $T : K \to H$ is called μ-Lipschitzian if for all x, $y \in K$, there exists a constant $\mu > 0$, such that

$$\|Tx - Ty\| \leq \mu\|x - y\|.$$

Definition 2.2. A mapping $T : K \to H$ is called α-inverse strongly monotonic (or co-coercive) if for all x, $y \in K$, there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|Tx - Ty\|^2.$$

Definition 2.3. A mapping $T : K \to H$ is called r-strongly monotonic if for all $x, y \in K$, there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq r||x - y||^2.$$

Definition 2.4. A mapping $T : K \to H$ is called relaxed (γ, r)-cocoercive if for all $x, y \in K$, there exists constants $\gamma > 0, r > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma||Tx - Ty||^2 + r||x - y||^2.$$

Remark 2.1. Clearly a r-strongly monotonic mapping or a γ-inverse strongly monotonic mapping must be a relaxed (γ, r)-cocoercive mapping, but the converse is not true. Therefore the class of the relaxed (γ, r)-cocoercive mappings is the most general class, and hence definition 2.4 includes both the definition 2.2 and the definition 2.3 as special cases.

Lemma 2.3 [11]. Suppose $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:

$$\delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k,\ k \geq 0$$

with $\lambda_k \in [0, 1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$, and $\sigma_k = o(\lambda_k)$. Then $\lim_{k\to\infty}\delta_k = 0$.

In order to consider the convergence analysis of the iterative methods, we need the following assumption, which is mainly due to Noor [9,12].

Assumption 2.1. The projection operator $P_{K(x)}$ satisfies the following condition

$$\|P_{K(x)}(w) - P_{K(u)}(w)\| \leq v\|x - u\|,\ \forall x,u,w \in H,$$

where $v > 0$ is a constant.

Remark 2.2. We remark that Assumption 2.1 is true for the special case, $K(x) = m(x) + K$, which appears in many important applications [7], where m is a point-topoint mapping and K is a closed convex set in H. It is well known that

$$P_{K(x)}(w) = m(x) + P_K[w - m(x)].$$

If m is a Lipschitz continuous with a constant $\tilde{v} > 0$, then

$$\begin{aligned}
&\|P_{K(x)}(w) - P_{K(u)}(w)\| \\
&= \|m(x) - m(u) + P_K[w - m(x)] - P_K[w - m(u)]\| \\
&\leq \|m(x) - m(u)\| + \|P_K[w - m(x)] - P_K[w - m(u)]\| \\
&\leq 2\|m(x) - m(u)\| \leq 2\tilde{v}\|x - u\|,
\end{aligned}$$

which shows that Assumption 2.1 is true for $v = 2\tilde{v} > 0$.

3. MAIN RESULTS

In this section, we investigate the strong convergence of Algorithms 2.1, 2.2 and 2.4 in finding the common element of two sets of solutions of the quasi variational inequalities (2.1) and $F(S)$ and this is the main motivation of this paper.

Theorem 3.1. Let $K(u)$ be a closed convex-valued subset of a real Hilbert space H. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of $K(u)$ into H, and S be a nonexpansive mapping of $K(u)$ into $K(u)$ such that $F(S) \cap QVI(K(u), T) \neq \phi$. Let $\{x_n\}$ be a sequence defined by Algorithm 2.1, for any initial point $x_0 \in K(x_0)$, with conditions

$$\left|\rho-\frac{r-\gamma\mu^2}{\mu^2}\right|<\frac{\sqrt{(r-\gamma\mu^2)^2-\mu^2(2v-v^2)}}{\mu^2} \tag{3.1}$$

$$r_1>\gamma_1\mu^2+\mu_1\sqrt{v(2-v)},\ \ v\in(0,1), \tag{3.2}$$

a_n, b_n, c_n $\in$ [0, 1] and $\sum_{n=0}^{\infty} a_n=\infty$. If Assumption 2.1 holds, then x_n obtained from Algorithm 2.1 converges strongly to $x^* \in F(S) \cap QVI(K(u), T)$.

Proof: Let $x^* \in K(u)$ be the solution of $F(S) \cap QV\, I(K(u), T)$. Then

$$x^*=(1-c_n)x^*+c_nSP_{K(x^*)}[x^*-\rho Tx^*] \tag{3.3}$$

$$=(1-b_n)x^*+b_nSP_{K(x^*)}[x^*-\rho Tx^*] \tag{3.4}$$

$$=(1-a_n)x^*+a_nSP_{K(x^*)}[x^*-\rho Tx^*] \tag{3.5}$$

where a_n, b_n, $c_n \in [0, 1]$ are some constants. To prove the result, we need to evaluate $\|x_{n+1}-x^*\|$ for all $n \geq 0$. From (2.14),(3.5), Assumption 2.1, and the nonexpansive mapping S, we have

$$\|x_{n+1}-x^*\|$$

$$\leq(1-a_n)\|x_n-x^*\|+a_n\|SP_{K(y_n)}[y_n-\rho Ty_n]-SP_{K(x^*)}[x^*-\rho Tx^*]\|$$

$$\leq(1-a_n)\|x_n-x^*\|+a_n\|P_{K(y_n)}[y_n-\rho Ty_n]-P_{K(y_n)}[x^*-\rho Tx^*]\|+\|P_{K(y_n)}[x^*-\rho Tx^*]-P_{K(x^*)}[x^*-\rho Tx^*]$$

$$\leq(1-a_n)\|x_n-x^*\|+a_n\|y_n-x^*-\rho(Ty_n-Tx^*)\|+a_nv\|y_n-x^*\|. \tag{3.6}$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T,

$$\begin{aligned}
&\|y_n-x^*-\rho(Ty_n-Tx^*)\|^2\\
=\ &\|y_n-x^*\|^2-2\rho\langle Ty_n-Tx^*, y_n-x^*\rangle+\rho^2\|Ty_n-Tx^*\|^2\\
\leq\ &\|y_n-x^*\|^2-2\rho[-\gamma\|Ty_n-Tx^*\|^2+\mathrm{r}\|y_n-x^*\|^2]+\rho^2\|Ty_n-Tx^*\|^2\\
\leq\ &\|y_n-y^*\|^2+2\rho\gamma\mu^2\|y_n-x^*\|^2-2\rho r\|y_n-x^*\|^2+\rho^2\mu^2\|y_n-x^*\|^2\\
=\ &[1+2\rho\gamma\mu^2-2\rho r+\rho^2\mu^2]\|y_n-x^*\|^2. \qquad (3.7)
\end{aligned}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}
x_{n+1}-x^*\| &\leq (1-a_n)\|x_n-x^*\|\\
&\quad +a_n\left\{\sqrt{1+2\rho\gamma\mu^2-2\rho r+\rho^2\mu^2}+v\right\}\|y_n-x^*\|\\
&=(1-a_n)\|x_n-x^*\|+a_n\|y_n-x^*\|,
\end{aligned} \tag{3.8}$$

where

$$\theta=\sqrt{1+2\rho\gamma\mu^2-2\rho r+\rho^2\mu^2}+v. \tag{3.9}$$

It follows from (3.1) and (3.2) that $\theta < 1$.

From (2.13), (3.4), Assumption 2.1, and nonexpansivity of S, we have

$$\begin{aligned}\| y_n - x^* \| \leq\ & (1-b_n)\| x_n - x^*\| \\ & + b_n \| SP_{K(z_n)}[z_n - \rho Tz_n] - SP_{K(x^*)}[x^* - \rho Tx^*]\| \\ & \leq (1-b_n)\| x_n - x^*\| + b_n \| P_{K(z_n)}[x^* - \rho Tx^*] - P_{K(x^*)}[x^* - \rho Tx^*]\| \\ & + b_n \| P_{K(z_n)}[z_n - \rho Tz_n] - P_{K(z_n)}[x^* - \rho Tx^*]\| \\ & \leq (1-b_n)\| x_n - x^*\| + b_n v \| z_n - x^*\| \\ & + b_n \| z_n - x^* - \rho(Tz_n - Tx^*)\|. \end{aligned} \tag{3.10}$$

Now from the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T, we have

$$\begin{aligned} & \| z_n - x^* - \rho[Tz_n - Tx^*]\|^2 \\ = \ & \| z_n - x^*\|^2 - 2\rho\langle Tz_n - Tx^*, z_n - x^*\rangle + \rho^2\| Tz_n - Tx^*\|^2 \\ \leq\ & \| z_n - x^*\|^2 - 2\rho[-\gamma \| Tz_n - Tx^*\|^2 + r\| z_n - x^*\|^2] \\ & + \rho^2\| Tz_n - Tx^*\|^2 \\ \leq\ & \| z_n - x^*\|^2 + 2\rho\gamma\mu^2\| z_n - x^*\|^2 - 2\rho r\| z_n - x^*\|^2 \\ & + \rho^2\mu^2\| z_n - x^*\|^2 \\ = \ & [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\| z_n - x^*\|^2. \end{aligned} \tag{3.11}$$

From (3.9), (3.10) and (3.11), we have

$$\| y_n - x^*\| \leq (1-b_n)\| x_n - x^*\| + b_n\theta\| z_n - x^*\|. \tag{3.12}$$

In a similar way, from (2.12), (3.3) and (3.9), it follows that

$$\begin{aligned}\| z_n - x^*\| \leq\ & (1-c_n)\| x_n - x^*\| + c_n\theta\| x_n - x^*\|, \\ = \ & \{(1 - c_n(1-\theta))\}\| x_n - x^*\| \\ \leq\ & \| x_n - x^*\|. \end{aligned} \tag{3.13}$$

From (3.8), (3.12) and (3.13), we obtain

$$\begin{aligned}\| x_{n+1} - x^*\| \leq\ & (1-a_n)\| x_n - x^*\| + a_n\theta\| y_n - x^*\| \\ \leq\ & (1-a_n)\| x_n - x^*\| + a_n\theta\| z_n - x^*\| \\ \leq\ & (1-a_n)\| x_n - x^*\| + a_n\theta\| x_n - x^*\| \\ = \ & [1 - a_n(1-\theta)]\| x_n - x^*\|, \end{aligned} \tag{3.14}$$

and hence by Lemma 2.3, we have $\lim n \to\infty \| x_n - x^*\| = 0$, the required result.

If the convex-valued set $K(x^*)$ is independent of the solution x^*, that is, $K(x^*) \equiv K$, then Theorem 3.1 reduces to the following result, which is due to Noor and Huang [16].

Theorem 3.2. Let K be a closed convex subset of a real Hilbert space H. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of K into H, and S be a nonexpansive mapping of K into K such that $F(S) \cap VI(K, T) \neq \phi$. Let $\{x_n\}$ be a sequence defined by Algorithm 2.5, for any initial point $x_0 \in K$, with conditions

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2,\ \gamma\mu^2 < r,$$

a_n, b_n, $c_n \in [0, 1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, then x_n obtained from Algorithm 2.5 converges strongly to $x^* \in F(S) \cap VI(K, T)$.

If $c_n \equiv 0$, then the following result is a special case of Theorem 3.1.

Theorem 3.3. Let $K(u)$ be a closed convex-valued subset of a real Hilbert space H. Let Assumption 2.1 hold. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of $K(u)$ into H, and S be a nonexpansive mapping of $K(u)$ into $K(u)$ such that $F(S) \cap QVI(K(u),T) \neq \phi$. Let $\{x_n\}$ be a sequence defined by Algorithm 2.2, for any initial point $x_0 \in K(x_0)$, with conditions (3.1) and (3.2). If Assumption 2.1 holds, then x_n obtained from Algorithm 2.2 converges strongly to $x^* \in F(S) \cap QVI(K(u),T)$.

Next we will provide and prove the strong convergence theorem of Algorithm 2.4 under the α-inverse strongly monotonicity.

Theorem 3.4. Let K be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$. Let T be an α-inverse strongly monotonic mapping of $K(u)$ into H, and S be a nonexpansive mapping of $K(u)$ into $K(u)$ such that $F(S) \cap QVI(K(u),T) \neq \phi$. If

$$|\rho - \alpha| \leq \alpha(1 - v), \tag{3.15}$$

then the approximate solution obtained from Algorithm 2.4 converges strongly to $x^* \in F(S) \cap QVI(K(x^*),T)$.

Proof. It is well known that, if T is α-inverse strongly monotonic with the constant $\alpha > 0$, then T is $\frac{1}{\alpha}$–Lipschitzian continuous [9]. Consider

$$\begin{aligned}
& \|x_n - x^* - \rho[Tx_n - Tx^*]\|^2 \\
= \ & \|x_n - x^*\|^2 + \rho^2\|Tx_n - Tx^*\|^2 - 2\rho\langle Tx_n - Tx^*, x_n - x^*\rangle \\
\leq \ & \|x_n - x^*\|^2 + \rho^2\|Tx_n - Tx^*\|^2 - 2\rho\alpha\|Tx_n - Tx^*\|^2 \\
= \ & \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha)\|Tx_n - Tx^*\|^2 \\
\leq \ & \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha).\frac{1}{\alpha^2}\|x_n - x^*\|^2 \\
= \ & (1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2})\|x_n - x^*\|^2 . \quad (3.16)
\end{aligned}$$

From (2.11), (3.5) and Assumption 2.1, we have

$$\begin{aligned}
& \|x_{n+1} - x^*\| \\
\leq \ & (1 - a_n)\|x_n - x^*\| + a_n\|SP_{K(x_n)}[x_n - \rho Tx_n] - SP_{K(x^*)}[x^* - \rho Tx^*]\| \\
\leq \ & (1 - a_n)\|x_n - x^*\| + a_n\|P_{K(x_n)}[x_n - \rho Tx_n] - P_{K(x^*)}[x^* - \rho Tx^*]\| \\
\leq \ & (1 - a_n)\|x_n - x^*\| + a_n\|P_{K(x_n)}[x_n - \rho Tx_n] - P_{K(x_n)}[x^* - \rho Tx^*]\| \\
& + \|P_{K(x_n)}[x^* - \rho Tx^*] - P_{K(x^*)}[x^* - \rho Tx^*]\| \\
\leq \ & (1 - a_n)\|x_n - x^*\| + a_n\|x_n - x^* - \rho(Tx_n - Tx^*)\| + a_n v\|x_n - x^*\| \\
\leq \ & (1 - a_n)\|x_n - x^*\| + a_n\theta_1\|x_n - x^*\| \\
= \ & [1 - a_n(1 - \theta_1)]\|x_n - x^*\|,
\end{aligned}$$

where

$$\theta_1 = \sqrt{1 + \frac{\rho^2 - 2\rho\alpha}{\alpha} + v} \tag{3.17}$$

From (3.15), it follows that $\theta_1 < 1$ and consequently using Lemma 2.3, we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$ the required result .

REFERENCES

[1] C. Baiocchi and A. Capelo, Variational and Quasi Variational Inequalities, Wiley, New York, 1984.

[2] P. Daniele, F. Giannessi and A. Maugeri, Equilibrium Problems and Variational Models, Kluwer Academic Publishers, United Kingdom, 2003.

[3] F. Giannessi and A. Maugeri, Variational Inequalities and Network Equilibrium Problems, Plenum Press, New York, 1995.

[4] F. Giannessi, A. Maugeri and P. M. Pardalos, Equilibrium Problems, Nonsmooth Optimization and Variational Inequalities Problems, Kluwer Academic Publishers, Dordrecht Holland, 2001.

[5] R. Glowinski, J. L. Lions and R. Tremolieres, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, Holland, 1981.

[6] Zhenyu Huang and M. Aslam Noor, Strong Convergence Criteria for Nonexpansive and Related (r, γ)-cocoercive Mappings in Hilbert spaces, Preprint, 2006.

[7] M. Mosco, Implicit Variational Problems and Quasi-Variational Inequalities, in: Nonlinear Operators and the Calculus of Variations, Lecture Notes in Mathematics, Springer Verlag, New York, New York, **543**, pp. 83-126, 1976.

[8] M. Aslam Noor, An Iterative Scheme for a Class of Quasi Variational Inequalities, J. Math. Anal. Appl., **110** (1985), 463-468.

[9] M. Aslam Noor, Sensitivity Analysis for Quasi-variational Inequalities, J. Optim. Theory Appl. **95** (1997), 399-407.

[10] M. Aslam Noor, Some Algorithms for General Monotone Mixed Variational Inequalities, Math. Comput. Modell. **29** (1999), 1-9.

[11] M. Aslam Noor, New Approximation Schemes for General Variational Inequalities, J. Math. Anal. Appl. **251** (2000), 217-229.

[12] M. Aslam Noor, Three-step Iterative Algorithms for Multivalued Quasi Variational Inclusions, J. Math. Anal. Appl. **255** (2001), 589-604.

[13] M. Aslam Noor, Some Developments in General Variational Inequalities, Appl. Math. Comput. **152** (2004) 199-277.

[14] M. Aslam Noor, Fundamentals of Mixed Quasi Variational Inequalities, Inter. J. Pure Appl. Math. **15** (2004), 137-258.

[15] M. Aslam Noor, K. Inayat Noor and T. M. Rassias, Some Aspects of Variational Inequalities, J. Comput. Appl. Math., **47** (1993), 285-312.

[16] M. Aslam Noor and Zhenya Huang, Three-step Methods for Nonexpansive Mappings and Variational Inequalities, Appl. Math Comput (2006), in press.

[17] M. Patriksson, Nonlinear Programming and Variational Inequalities: A Unified Approach, Kluwer Academic Publishers, Dordrecht, 1998.

[18] G. Stampacchia, Formes Bilineaires Coercivites sur les Ensembles Convexes, C. R. l'Acad. Sci., Paris, **258** (1964), 4413-4416.

[19] W. Takahashi and M. Toyoda, Weak Convergence Theorems for Nonexpansive Mappings and Monotone Mappings, J. Optim. Theory Appl. **118** (2003), 417-428.

[20] X. L. Weng, Fixed point iteration for local strictly pseudocontractive mappings, Proc. Amer. Math. Soc. **113** (1991), 727-731.

Muhammad Aslam Noor
Mathematics Department, COMSATS Institute of Information Technology
Islamabad, Pakistan
e-mail: *noormaslam@gmail.com and aslamnoor@comsats.edu.pk*

Zhenyu Huang
Department of mathematics, Nanjing University
Nanjing 210093, P. R. China
e-mail: *zhenyu@nju.edu.cn*