Journal of Advance Computational Research
Vol. 2 No. 2 (July-December, 2017)

A Ten-mode Social Network of Afghan Society

Walid K. Sharabati, Edward J. Wegman and Yasmin H. Said
Department of Statistics, The Volgenau School of Information Technology and Engineering
George Mason University, 4400 University Drive, MSC 4A7, Fairfax, VA 22030

Abstract: *Network science has become an important tool in the analysis of societies and entities. There has been discussion in the literature about dichotomous two-mode networks and methods to derive the one- mode network, but no research has yet contributed to the expansion of this problem and its applications. In this paper, we present a novel advanced mathematical methodology to manipulate multi-mode networks. A network may be represented by a matrix or graph. We focus our study on the matrix approach; we incorporate multi-dimensional tensor analysis to expand on vertex attributes. We utilize linear algebra to derive relational networks from the multi-mode networks. We next discuss a patrimonial/tribal model for the social system of Afghanistan, and finally, we illustrate the application of a ten-mode network to the Afghan social system with methods for reduced mode networks to illustrate special characteristics of that social system.*

1. INTRODUCTION

In recent years, network theory; also known as social network analysis, has grown to have many uses and applications in science. It is concerned with understanding the linkages among social entities and the implications of these linkages. The social entities are referred to as actors that are represented by the vertices of the graph. Most social network applications consider a collection of vertices that are all of the same type; these are known as one-mode networks [6, 4, 5]. One-mode network consists of one set of vertices. However, if the network is set up so that there are two different sets of actors or entities mutually exclusive, the network is considered a two-mode. The two-mode network, sometimes called a bipartite network, has advantages over the one-mode network. Two-mode networks offer insight on the relationship among the different types of entities; they reveal more network features and can help understand its structure. One-mode networks may be obtained from two-mode networks through matrix multiplication, but converse is not possible. Similarly, three-mode or tripartite networks have three different sets of vertices all of which are distinct. Very few researchers have tackled tripartite networks [9] or multi-mode networks [11]. The concept of manipulating multi-mode networks is analogous to projecting from higher dimensions to lower dimensions.

Graphs and matrices are mathematical tools often used to express a network. In network theory terminology, entities are called vertices and linkages are called edges. A graph G (V, E), is a collection of vertices V and edges E used to visualize the network. This definition applies to the one-mode network setup; if however, the network is constructed of two different types of vertices, say type A and type B, then the graph representing that network is $G\,(V^a, V^b, E)$; this graph resembles a two-mode network. The adjacency or proximity matrix A is a mathematical tool used to quantify the network. Usually the adjacency matrix is binary (dichotomous); i.e. the entries of the matrix are only ones and zeros, but they could also be valued as well.

2. TWO- MODE NETWORKS

The analysis of two-mode binary networks have been presented and studied in [12, 13, 8, 7]. Our goal in this section is to build on the theory of two-mode networks. To begin with, we present some of the two-mode networks features.

Consider a two-mode network having vertices of types A and B, let AB be the binary adjacency matrix associated with the network. The relationship defined on A and B is "entity of class A is affiliated with entity of class B". The matrix AB is the matrix relating vertices of type A to vertices of type B; i.e. there is a path of size 1 between the two-mode vertices. The matrix AB represents the graph $G\,(V^{a,} V^{b,} E)$ of the two-mode network. Without loss of generality, suppose that there are n vertices of type A and m vertices of type B; that is to say, $|V| = n$ and $V = m$. Thus, AB is of size $n \times m$. is a path of size 1 between the two-mode vertices. The matrix AB represents the graph $G\,(V^{a,} V^{b,} E)$ of the two-mode network. Without loss of generality, suppose that there are n vertices of type A and m vertices of type B; that is to say, $|V| = n$ and $V = m$. Thus, AB is of size $n \times m$.

Definition 1: For a bipartite graph $G\ (V^{a,}\ V^{b,}\ E)$ with sets of vertices

$v^a = \left\{v_1^a, v_2^a, \ldots. v_i^a, \ldots. v_{|v^a|}^a\right\}$ of type A; $|V^a| = n$,

$v^b = \left\{v_1^b, v_2^b, \ldots. v_i^b, \ldots. v_{|v^b|}^b\right\}$ of type B; $|V^b| = m$, and a set of edges $E = \left\{e_1, e_2, \ldots\ldots. e_{|E|}\right\}$ connecting types A and B vertices, the adjacency matrix $AB_{n\times m}$, also known as Edmonds matrix, is defined by

$$AB_{nxm} = \begin{cases} ab_{ij}, & (v_i^a, v_j^b) \in E \\ 0, & (v_i^a, v_j^b) \notin E \end{cases}$$

$1 \le i \le n$, $1 \le j \le m$, and the indeterminate $ab_{ij} \in \mathbb{R}$ If AB is binary then $ab_{ij} = 1$

Definition 2: The multiplicity of an edge, denoted $m(E(v_i^a, v_j^b)) = ab_{ij}$, is the number of multiple edges sharing the same end-vertices; i.e. the edge weight.

We can perform matrix operations on the two-mode binary matrix such as the product of matrices to obtain interesting results. The two-mode network can be decomposed into two one-mode networks. Let $AB_{n\times m}$ be the two-mode binary adjacency matrix of vertices of types A and B. Then, we have

$$AB_{nxm}.AB_{nxm}^T = AB_{nxm}.BA_{mxn} = AA_{nxn},$$

The elements of the main diagonal are

$$aa_{ij} = \sum_{k=1}^{m} ab_{ik}.ab_{jk},$$

And along the main diagonal where $i = j$ we have,

$$aa_{ii} = \sum_{k=1}^{m} ab_{ik}.ab_{ik}$$

$$= \sum_{k=1}^{m} ab_{ik} = \text{ vertex i count of interactions,}$$

with

$$ab = \begin{cases} 1, & v_i \leftrightarrow v_k \\ 0, & v_i \leftrightarrow v_k \end{cases}$$

First, we note AA_{nxn} is diagonally dominant. The element aa_{ii} along the main diagonal is the total count of interactions of type B for vertex i of type A. The elements aa_{ii}, $1 \le i \le n$ are the marginal distribution of total interactions for vertices of type A. The off-diagonal elements $aa_{ii,}$ $i \ne j$ of AA_{nxn} represent the sum of edge weights for a vertex i. aa_{ii} is the edge weight (tie-strength) between vertices i and j. The multiplicity of the one-node network represented by AA_{nxn} is max(aa_{ii}); the highest edge weight.

Another interesting feature, may be derived from the matrix $AA_{n\times n}$. Consider $AA_{n\times n} = AA_{n\times n}^2$. a_{ii} is the degree of vertex i of type A.

In this manner, we proceed to obtain the one-node matrix for vertices of type B

$$AB_{nxm}^T.AB_{nxm} = BA_{mxn}.AB_{nxm} = BB_{mxm}$$

The elements of the main diagonal are

$$bb_{ij} = \sum_{k=1}^{n} ab_{ki}.ab_{kj}$$

and along the main diagonal where $i = j$ we have,

$$bb_{jj} = \sum_{k=1}^{n} ab_{kj}.ab_{kj}$$

$$= \sum_{k=1}^{m} ab_{kj} = \textit{vertex j count of interactions}$$

Again, BB_{mxm} is diagonally dominant. The element bb_{jj} along the main diagonal is the total count of interactions of type A for vertex j of type B. The elements bb_{jj}, $1 \le j \le m$ are the marginal distribution of total interactions for vertices of type B. The off-diagonal elements bb_{jj}, $i \ne j$ of BB_{mxm} represent the sum of edge weights for a vertex j. bb_{jj} is the edge weight (tie-strength) between vertices i and j. The multiplicity of the one-mode network represented by BB_{mxm} is max(bb_{jj}); the highest edge weight.

Consider $B_{m\times m} = BB_{m\times m}^2$. b_{jj} is the degree of vertex j of type of type B.

The mechanism of converting a two-mode matrix to one-mode is similar to the effect of projecting from 2-D to 1-D; in the sense that one detailed feature about the network is being lost and the one-mode setup provides only one dimensionality encompassing the one-type marginal relationship. Moreover, this process is irreversible; once the one-mode network is obtained, it is possible to retrieve the original two-mode network.

3. THREE-MODE NETWORKS

In order to indicate the method for scaling to higher mode networks, in this section we consider in more detail the three-mode case. Continuing with the analogy to a two mode network; a third network feature may be introduced to add another dimensionality to the problem; the result is a 3-D cuboid (rectangular parallel piped) matrix, in which 3-mode matrix manipulations can be explored. The cuboid matrix resembles a tripartite network. A cuboid matrix is a tensor of rank 3; however, for the purposes of this present discussion we will use the term cuboid instead. An example of a three-mode network might be author-by-papers-by-universities. Throughout this section, we make the assumption that the graph is finite to understand how the mathematics work for three-mode matrices, which implies that the matrices have finite dimensions. Finally, the graphs are assumed to represent dichotomous relations, i.e. the multiplicity of an edge is one.

Definitation 3: Let $v^a = \left\{v_1^a, v_2^a, \ldots. v_i^a, \ldots. v_{|v^a|}^a\right\}$ be the set of vertices of type a, $v^b = \left\{v_1^b, v_2^b, \ldots. v_i^b, \ldots. v_{|v^b|}^b\right\}$ be the set of vertices of type *b*, and $v^c = \left\{v_1^c, v_2^c, \ldots. v_k^c, \ldots. v_{|v^c|}^a\right\}$ be the set of vertices of type c. Furthermore, let $E = \left\{e_1, e_2, \ldots\ldots. e_{|E|}\right\}$ be the set of edges connecting types *a*, *b*, *c* vertices; in this sense, e_i is a hyper edge. Assume $|V^a| = n$, $|V^b| = m$, and $|V^c| = p$. The binary adjacency matrix $AB_{n\times m\times p,}$ corresponding to the finite graph $G\ (V^{a,}\ V^{b,}\ V^c, E)$ is defined by

$$abc_{ijk} = \begin{cases} 1, & (v_i^a, v_j^b, v_k^c) \in E \\ 0, & otherwise. \end{cases}$$

$1 \le i \le n$, $1 \le j \le m$, $1 \le k \le p$.

We start by computing the two-mode weighted matrix *AB*, the method is presented below

$$ab_{ij} = \sum_{k=1}^{p} abc_{ijk}, \quad 1 \le k \le p.$$

This is equivalent to projecting from 3D cuboid matrix onto the 2D planar matrix giving the marginal bi-partite distribution for vertices of types *A* and *B*.

Because the cuboid is a three-dimensional object, there are several matrix arithmetic operations to perform on a cuboid matrix, some of which result in a rectangular matrix, while others result in a cuboid matrix. Here, we explore few meaningful operators related to networks, but before we explain how these operations are performed, we would like to discuss how a cuboid is being transposed in 3D. Unlike the 2D rectangular matrix, which only has two faces, the 3D cuboid has six faces leading to six different ways to view the block in terms of size, namely, $n \times m \times p$, $n \times p \times m$, $m \times n \times p$, $m \times p \times n$, $p \times n \times m$, and $p \times m \times n$, As a result, the transpose can be performed in six different ways.

Suppose $ABC_{n\times m\times p}$ a dichotomous tripartite matrix. Then,

$$ABC_{nxmxp}^{T_{cba}} = CBA_{pxmxn}, \text{ with } cba_{ijk} = abc_{kji},$$

and

$$ABC_{nxmxp}^{T_{cba}} = BAC_{mxnxp}, \text{ with } bac_{ijk} = abc_{jik}$$

The following cuboid matrix multiplication definition is the traditional 3D matrix product; it results in another 3D matrix.

Definition 4: Given the matrix $ABC_{n\times m\times p}$. The weighted 3D matrix

$$\begin{aligned} AAC_{n\times n\times p} &= ABC_{nxmxp} \cdot ABC_{nxmxp}^{T_{bac}} \\ &= ABC_{n\times m\times p} \cdot BAC_{m\times n\times p} \end{aligned}$$

so that the product of the sub-matrices $AB_{n\times m}$. $BA_{m\times n}$ is well defined, is computed as follows

$$aac_k = ABC_k \cdot BAC_k, \qquad 1 \le k \le p.$$

$AAC_{n\times n\times p}$ is a two-mode graph (network) represented with a 3D matrix.

Definition 5: Given the matrix. $ABC_{n\times n\times p}$. The weighted 3D matrix

$$\begin{aligned} AAA_{n\times n\times n} &= AAC_{n\times n\times p} \cdot AAC_{n\times n\times p}^{Tcaa} \\ &= AAC_{n\times n\times p} \cdot CAA_{p\times n\times n}, \end{aligned}$$

so that the product of the sub-matrices $AC_{n\times p}$. $BA_{p\times n}$ is well-defined, is computed as follows

$$aaa_i = AAC_i \cdot CAA_i$$

$AAA_{n\times n\times n}$ is the one- mode 3D matrix of triadic vertices (triplets) of type A related through vertices of types B and C. In a graph terminology, a non zero entry in $AAA_{n\times n\times n}$ indicates that the hyper edge is connecting three vertices altogether as opposed to two vertices in the traditional graph context. $AAA_{n\times n\times n}$ represents triadic relations, a nonzero value aaa_{ijk} indicates that all three vertices are connected through a hyperedge.

The one- mode 2D matrix $AA_{n\times n}$ of pairwise vertices of type A related through vertices of types B and C is found by summing up over one of the dimensions. In this regard, the 3D matrix is assumed to be symmetric

$$aa_{\mathrm{ij}} = \sum_{k=1}^{n} aaa_{ijk}.$$

Definition 6: Given a 3D binary cuboid matrix *ABC* of size $n \times m \times p$ of relation types *A*, *B* and *C* respectively. The hyper product, denoted $A \circ B$, is define by

$$ABC_{n\times m\times p} \circ ABC^{T_{cba}}_{n\times m\times p} = ABC_{n\times m\times p} \circ CBA_{p\times m\times n}$$
$$= ABBA_{n\times m\times m\times n},$$

where $ABBA_{n\times m\times m\times n}$ is the hyper-cuboid two-mode proximity matrix of pair of vertices (v_i, v_j) of types A and B related through the set of vertices v_k of type C. Let the product of the sub-matrices $BC_{m\times p} \cdot CB_{p\times m}$ be defined. Then, the elements of $ABBA_{n\times m\times m\times n}$ can be found as follows,

$$abba_{kl} = ABC_{kl} \cdot CBA_{kl}, \quad 1 \le k, l \le p.$$

The 4D hyper matrix can be represented using 3D matrix by stacking n cuboid matrices each of size $m\times m\times n$, which results in a 3D matrix of size $m\times m\times n^2$. The 4D hyper matrix is a tensor of rank 4.

The following 3D matrix multiplication definition results in a one-mode 2D matrix

Definition 7: Given the matrix $ABC_{n\times m\times p}$. The weighted one-mode 2D matrix

$$CC_{p\times p} = ABC_{n\times m\times p} \odot ABC^{T_{bac}}_{n\times m\times p}$$
$$= ABC_{n\times m\times p} \odot BAC_{m\times n\times p}$$

So that the product of the sub-matrices $ABC_{n\times m}$. $BAC_{m\times n}$ is well defined, is computed as follows

$$cc_{ij} = \sum_{q=1}^{n}\sum_{r=1}^{n} ABC_{(n\times m)_i} \cdot BAC_{(m\times n)_j}$$

$$= \sum_{q=1}^{n}\sum_{r=1}^{n} AA_{(qr)_{ij}}, \;\; 1 \le i, j \le p.$$

$CC_{p\times p}$ is the one-mode 2D matrix of pairwise vertices of type C related through vertices of types A and B. $CC_{p\times p}$ represents dyadic relations, a nonzero CC_{ij} value indicates that the two vertices are connected through an edge.

4. MODELING THE SOCIAL SYSTEM OF AFGHANISTAN

Social systems may be characterized as closed, open, or tribal/patrimonial. Examples of closed regimes include Burma, Iran, Libya, North Korea and Zimbabwe to name a few. These are governments with a strong, near dictatorial central government. Most western nations can be characterized as open in the sense that although there may be relatively strong central governments, the society is democratic and the government governs at the will of the citizens. In addition, there are countries such as Iraq, Afghanistan, and Somalia, which could accurately be characterized as having relatively weak or non-existent central governments, but having a strong tribal or patrimonial regional governmental structure. In the first and third social systems, a statistical picture of the population is hard to develop either because statistical population surveys and censuses are an exceptionally low priority of those countries or because data are not released or are often released with deliberate fabrications.

Because of the mixture of patrimonial society, the extensive drug trade, the cross border intrusions from Pakistan, and the continuingly growing strength of the Taliban, Afghanistan is a particularly complex society to model. Geller and Moss [1, 2] and Moss and Geller [10] introduce several important notions associated with the patrimonical society in Afghanistan. In a conflict-torn society such as Afghanistan, virtually anything goes. That is, the normative boundaries for the aspirations of members of the society are discarded. This situation is called *anomie*. This situation emerges when the means to obtain a specific goal such as accumulation of power or wealth run out of social control. Because of the lack of social forms, the society organizes itself in a patrimonial fashion. Stakeholders interested in gaining power in a conflict situation act in a patrimonial manner to accumulate and re-distribute material and social resources.

A related concept is that of *qawm*, which a fluid social solidarity network is. A qawm network is defined by such social dimensions as family, kinship, tribe, ethnicity, occupation as well as abstract concepts such as solidarity, rivalry, cooperation, and conflict. The emphasis is on the fluid nature of a qawm network in which alliances and relationships are able to change depending on perceptions of self-interest of an individual or subgroup. A qawm network is a complex network that can have political, social, economic, military, and cultural components to it. A qawm network can face competition with other qawm networks. There may be internal competition with other members of the qawm and a particular individual may

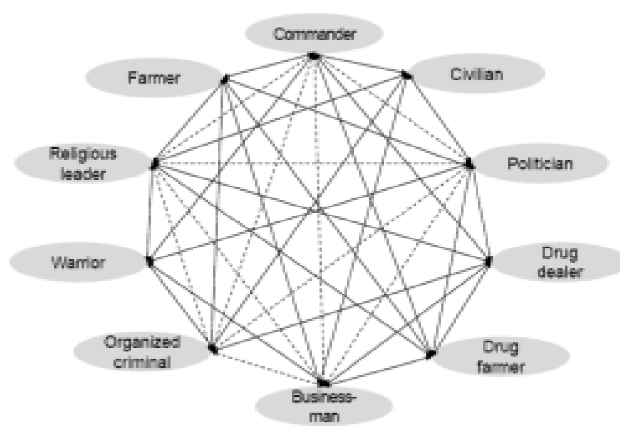

Figure 1: Network Model of the the Qawm-based Afghan Social System. Dashed lines Represent Affiliations, Solid Lines Represent Patron-client Relationships. Source: Moss and Geller [10]

be a member of several different qawm networks. The qawm network needs to be sustained and it is an Afghan leader's ability to accumulate and redistribute resources that makes him powerful and thus sustains the qawm.

Moss and Geller [10] identify ten types of agents active within a prototypical Afghan qawm: politicians, religious leaders, military commanders, businessmen, warriors, civilians, farmers, drug farmers, organized criminals, and drug dealers.

The model represents social stratification insofar as strongmen, i.e. politicians, religious leaders, military commanders, businessmen and organized criminals, form affiliations amongst each other, while ordinary agents, i.e. civilians, warriors, farmers and drug farmers, form patron-client relationships with strong- men [10]. Although these are conceptually distinct roles, in fact, one individual may occupy a number of the roles simultaneously. Figure 1 illustrates the social network of the qawm. As indicated earlier, a society such as found in Afghanistan is an anomic society in which many of the normative social boundaries of have been discarded. Thus to Western eyes, the linkages between a religious leader and a drug dealer or an organized criminal would seem taboo, but in an anomic society where many social norms have been discarded, it is quite feasible for these linkages to exist, especially say if there are familial, ethnic, and geographic connections. What is perhaps less well appreciated is that by trying to break connections within the qawm among organized criminals, drug dealers, and drug farmers with the other sectors of the qawm is in e_ect an attempt to disrupt a complex functional social system. The Taliban do not try to disrupt the functional qawm, but rather become an integral part of it; becoming simultaneously the politicians, religious leaders, and military commanders. In the following section we develop a mathematical formulation of the Afghan qawm system.

5. INTRODUCING TEN- MODE NETWORKS

The ten-mode network of Afghan society we are investigating has the features: commander, warrior, civilian, politician, drug dealer, drug farmer, business man, religious leader, farmer, organized criminal, which we represent with a tensor (hyper-matrix) of rank 10 and ten-partite graph. In this section, we reduce the ten-mode social network to lower interesting modes, to do this we define several matrix multiplications in higher dimensions. We make the assumption that the network and its graph are finite; therefore, the tensor has finite dimensions. We also assume the multiplicity of an edge is one.

Let $V^1 = \left\{v_1^1, v_2^1, ...v_a^1, ...v_{|v^1|}^1\right\}$ be the set of vertices of type commander, $V^2 = \left\{v_1^2, v_2^2, ...v_b^2, ...v_{|v^2|}^2\right\}$ be the set of vertices of type warrior, $V^3 = \left\{v_1^3, v_2^3, ...v_c^3, ...v_{|v^3|}^3\right\}$ be the set of vertices of type civilian, $v^4 = \left\{v_1^4, v_2^4, ...v_d^4, ...v_{|v^4|}^4\right\}$ be the set of vertices of type politician, $V^5 = \left\{v_1^5, v_2^5, ...v_e^5, ...v_{|v^5|}^5\right\}$ be the set of vertices of type drug dealer, $V^6 = \left\{v_1^6, v_2^6, ...v_f^6, ...v_{|v^6|}^6\right\}$ be the set of vertices of type drug former, $V^7 = \left\{v_1^7, v_2^7, ...v_g^7, ...v_{|v^7|}^7\right\}$ be the set of vertices of type business man, $V^8 = \left\{v_1^8, v_2^8, ...v_h^8, ...v_{|v^4|}^8\right\}$ be the set of vertices of type religious leader, $V^9 = \left\{v_1^9, v_2^9, ...v_i^9, ...v_{|v^4|}^9\right\}$ be the set of vertices of type farmer, and $V^{10} = \left\{v_1^{10}, v_2^{10}, ...v_j^{10}, ...v_{|v^{10}|}^{10}\right\}$ be the set of vertices of type organized criminal. Furthermore, let $E = \left\{e_1, e_2, ...e_{|E|}\right\}$ be the set of edges connecting vertices of type commander, warrior, civilian, politician, drug dealer, drug farmer, business man, religious leader, farmer, organized criminal. Assume $|V^1| = m$, $|V^2| = n$, $|V^3| = o$, $|V^4| = p$, $|V^5| = q$, $|V^6| = r$, $|V^7| = s$, $|V^8| = t$, $|V^9| = u$, $|V^{10}| = v$. Thus, there are m – commanders, n-warriors, o-civilians, p-politicians, q- drug dealers, r- drug farmers, s-business men, t-religious leaders, u-farmers, and v- organized criminals.

Definition 8: A path graph P_n of size n – vertices is a graph that contains vertices of degree two and one. All vertices have degree 2 except the end vertices, which have degree 1. A path graph is a broken cycle graph. The path – plus graph P_n^+ has $(n + 1)$ – vertices.

Definition 9: The diameter of the path graph P_n is diam $(P_n) = n–1$, and the diameter of the path- plus graph P_n^+ is diam $(P_n^+) = n$.

The complexity of this social network suggests that individuals of the aforementioned features may interact with actors of different types as well as actors of the same type; for example, a business man may interact with other business men and/ or civilians, religious leaders, drug dealers, or farmers. Therefore, it is possible for an edge to connect two vertices of the same type. Although, it may be the case that actors may have multiple roles, we assume that actors have only one role; for example, a religious leader may not play the role of a business man. Networks that have actors who play multiple roles are modeled with hyper- graphs. In hyper- graphs, hyper-edges are expressed in terms of subsets of vertices in which a vertex may be present in two different subsets. For instance, a drug dealer may also be an organized criminal.

Before we discuss the arithmetic's, we ought to explain how the ten- mode network is expressed with a tensor in a meaningful way. Because there are ten different features; unlike the two- mode network, we cannot just express the relationship among vertices with a single edge connecting only two actors, nor can we express the relationship in terms of a clique. This requires all ten actors of the different modes to know each other, which is less likely to happen. The adjacency matrix will be very sparse. Therefore, to have a meaningful representation we are using paths of size 10 or paths – plus of size 11 to construct the proximity matrix.

The binary proximity matrix $Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$ associated with the ten-mode social network represented by the graph $G(V_1, V\exists_2, \ldots\ldots V_{10}; E)$ is defined by

$$Z_{abcdefghij} = \begin{cases} 1, \text{ if } \exists\ P_{10} \text{ or } P_{10}^{+} \text{ connecting vertices } v_a^1, v_b^2, v_c^3, v_d^4, v_e^5, v_f^6, v_g^7, v_h^8, v_i^9, v_j^{10} \\ 0, \quad \text{otherwise} \end{cases}$$

$1 \le a \le m$, $1 \le b \le n$, $1 \le c \le 0$, $1 \le d \le p$, $1 \le e \le q$, $1 \le f \le r$, $1 \le g \le s$, $1 \le h \le t$, $1 \le i \le u$, $1 \le j \le v$.

A 1 entry of the Z tensor of rank 10 indicates that there is a path sub-graph connecting ten vertices; one from each mode, or eleven vertices; one from each type plus one repeated type, if there are 10 or 11 actors from ten categories; 1 implies that these actors are accessible to each other.

Figure 2 shows an example of a three mode social network of types *A*, *B*, and *C*. Table 1 represents the 3-mode proximity tensor of paths P_3 or paths-plus P_3^+ associated with the network in Figure 2.

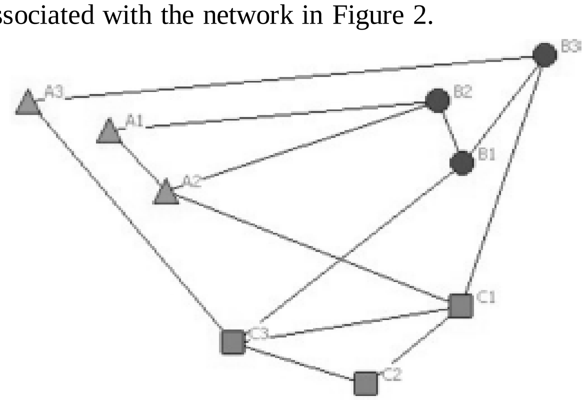

Figure 2: Three-mode Social Network

There are several matrix arithmetic operations to perform on the matrix Z, some of which result in angular matrix, while others result in a tensor. To understand how to manipulate and reduce the ten-mode tensor to lower ranks, we discuss how the transpose is handled on the ten- mode tensor. Unlike the 2–D rectangular matrix, which only has two faces, the 10-D hyper-matrix has $\binom{10}{2} 2^8 = 11520$ faces leading to 11520 different ways to view 2-dimensional matrix, this means the transpose can be performed in that many different ways.

Table 1
Three Mode Proximity Tensor of P_3 or P_3^+ Associated with the Network in Figure 2

Vertices	P_3 *or* P_3^+	*Vertices*	P_3 *or* P_3^+	*Vertices*	P_3 *or* P_3^+
$A_1 B_1 C_1$	0	$A_2 B_1 C_1$	0	$A_3 B_1 C_1$	1
$A_1 B_1 C_2$	0	$A_2 B_1 C_2$	1	$A_3 B_1 C_2$	1
$A_1 B_1 C_3$	0	$A_2 B_1 C_3$	1	$A_3 B_1 C_3$	1
$A_1 B_2 C_1$	0	$A_2 B_2 C_1$	0	$A_3 B_2 C_1$	1
$A_1 B_2 C_2$	0	$A_2 B_2 C_2$	1	$A_3 B_2 C_2$	0
$A_1 B_2 C_3$	0	$A_2 B_2 C_3$	1	$A_3 B_2 C_3$	1
$A_1 B_3 C_1$	0	$A_2 B_3 C_1$	1	$A_3 B_3 C_1$	1
$A_1 B_3 C_2$	1	$A_2 B_3 C_2$	1	$A_3 B_3 C_2$	0
$A_1 B_3 C_3$	1	$A_2 B_3 C_3$	1	$A_3 B_3 C_3$	1

Suppose $Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$ a dichotomous 10 – partite tensor. Then, $Z^{T_{rsmnopqtuv}}_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v} = Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$, with $Z^{T}_{abcdefghij} = Z_{abcdefghij}$,

We are ready to explore the ten-mode network, the following ten- mode tensor multiplication results in other ten-mode tensor

Definition 10: Given the binary ten-mode proximity tensor $Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$ of paths of size 10.

The weighted ten-mode tensor

$$Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v} = Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v} \cdot Z^{T_{rsmnopqtuv}}_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$$

$$Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v} \cdot Z_{n\times m\times o\times p\times q\times r\times s\times t\times u\times v}$$

So that the product of the 2-D matrices $Z_{m\times n}.Z_{n\times m}$ is well defined. $Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$ is a nine-mode network represented by a ten-mode tensor.

Continuing in the same manner with the n mode being eliminated, the one-mode weighted tensor $\underbrace{Z_{m\times\ldots\times m}}_{10 times}$

for a mode *m* is obtained in nine tensor multiplications and transpose.

$\underbrace{Z_{m\times\ldots\times m}}_{10times}$ is the one-mode ten-dimensional tensor of paths P_{10} of type m related through all other types. A nonzero entry $\underbrace{Z_{m\times\ldots\times m}}_{10times}$ indicates that there is a path connecting ten vertices.

The one-mode 2-D matrix Z_{mxn} of pairwise vertices of type *m* related through paths of all other types may be found by summing up over nine dimensions.

$$Z_{xy}=\sum_{k=1}^{m}\sum_{k=1}^{m}\sum_{k=1}^{m}\sum_{k=1}^{m}\sum_{k=1}^{m}\sum_{k=1}^{m}\sum_{k=1}^{m}\sum_{k=1}^{m}Z_{xykkkkkkkk}$$

The following 10-D tensor multiplication definition results in a one-mode 2-D matrix.

Definition 11: Given the matrix $Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$. The weighted eight-mode 9-D matrix

$$Z_{o\times o\times p\times q\times r\times s\times t\times u\times v}=Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}\cdot Z^{T_{rsmnopqtuv}}_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$$
$$=Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}\cdot Z_{n\times m\times o\times p\times q\times r\times s\times t\times u\times v}$$

So that the product of the sub-matrices Zmxn. Znxm is well defined is computed as follows

$$Z_{abcdefghij}=\sum_{k=1}^{m}\sum_{l=1}^{m}Z_{(m\times n)_z}.Z_{(n\times m)_y}=\sum_{k=1}^{m}\sum_{l=1}^{m}Z_{(kl)xy},1\le x,y\le 0$$

6. EXAMPLES

There are many relationships to develop from this network. In this section, we show two examples of lower mode networks. We first demonstrate how actors of the three-mode sub network of commander, drug dealers, and the religious leaders are related, and then we show how to develop the relationship between organized criminals and politicians.

Consider the social network of commander- by-warrior- by-civilian –by-politician-by-drug dealer-by-drug farmer-by-business man-by-business man-by-religious leader-by-farmer-by-organized criminal represented by the tensor(hyper-matrix) $Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$. We adopt the following procedure: Start by eliminating the n mode

$$Z_{m\times m\times o\times p\times q\times r\times s\times t\times u\times v}=Z_{m\times o\times p\times q\times r\times s\times t\times u\times v}\cdot Z_{n\times m\times o\times p\times q\times r\times s\times t\times u\times v}$$

Reduce the nine-mode 10-D tensor $Z_{m\times m\times o\times p\times q\times r\times s\times t\times u\times v}$ to a nine-mode 9-D tensor $Z_{m\times p\times q\times r\times s\times t\times u\times v}$

$$Z_{abefghij}=\sum_{k=1}^{m}Z_{kadefghij}$$

Eliminate the 0 mode.

$$Z_{m\times m\times q\times r\times s\times t\times u\times v}=Z_{m\times p\times q\times r\times s\times t\times u\times v}\cdot Z_{p\times m\times q\times r\times s\times t\times u\times v}$$

Reduce the eight-mode 9-D tensor $Z_{m\times m\times q\times r\times s\times t\times u\times v}$ to a eight-mode 8-D tensor $Z_{m\times p\times q\times r\times s\times t\times u\times v}$

$$Z_{aefghij}=\sum_{k=1}^{m}Z_{kadefghij}$$

Eliminate the p mode.

$$Z_{m\times q\times q\times s\times t\times u\times v}=Z_{m\times q\times r\times s\times t\times u\times v}\cdot Z_{m\times r\times q\times s\times t\times u\times v}$$

Reduce the seven-mode 8-D tensor $Z_{m\times q\times q\times r\times t\times u\times v}$ to a seven-mode 7-D tensor $Z_{m\times q\times s\times t\times u\times v}$

$$Z_{aeghij}=\sum_{k=1}^{m}Z_{akeghij}$$

Notice that the *m* and *q* modes are now related directly, Proceed by eliminating the *r* and *s* modes.

$$Z_{m\times q\times q\times s\times t\times u\times v}=Z_{m\times q\times r\times s\times t\times u\times v}\cdot Z_{m\times r\times q\times s\times t\times u\times v}$$

Reduce the six-mode 7-D tensor $Z_{m\times q\times q\times r\times t\times u\times v}$ to a six-mode 6-D tensor $Z_{m\times q\times s\times t\times u\times v}$

$$Z_{aeghij}=\sum_{k=1}^{m}Z_{akeghij}$$

$$Z_{m\times q\times q\times t\times u\times v}=Z_{m\times q\times s\times t\times u\times v}\cdot Z_{m\times s\times q\times t\times u\times v}$$

Reduce the five-mode 6-D tensor $Z_{m\times q\times q\times t\times u\times v}$ to a five-mode 5-D tensor $Z_{m\times q\times s\times t\times u\times v}$

$$Z_{aehij}=\sum_{k=1}^{m}Z_{akeghij}$$

Finally, eliminate the u and v modes.

$$Z_{m\times q\times t\times t\times v}=Z_{m\times q\times t\times u\times v}\cdot Z_{m\times q\times u\times t\times v}$$

Reduce the four-mode 5-D tensor $Z_{m\times q\times q\times t\times t\times v}$ to a four-mode 4-D tensor $Z_{m\times q\times t\times v}$

$$Z_{aehj}=\sum_{k=1}^{t}Z_{aekhj}$$

$$Z_{m\times q\times t\times t}=Z_{m\times q\times t\times v}\cdot Z_{m\times q\times v\times t}$$

Reduce the three-mode 4-D tensor $Z_{m\times q\times t\times t}$ to a three-mode 3-D tensor $Z_{m\times q\times t}$

$$Z_{aeh} = \sum_{k=1}^{t} Z_{aekh}$$

$Z_{m\times q\times t}$ is the three mode social network of commanders-by-drug dealers-by-religious leaders related through paths of size 10.

The second example we provide involves only two features; politicians and organized criminals. We are utilizing definition 11 to derive the two-mode social network.

Consider $Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v}$. Start by eliminating the commander and warrior modes; m and n, to obtain the weighted eight- mode 9-D hyper- matrix

$$Z_{o\times o\times p\times q\times r\times s\times t\times u\times v} = Z_{m\times n\times o\times p\times q\times r\times s\times t\times u\times v} \odot Z_{n\times m\times o\times p\times q\times r\times s\times t\times u\times v}$$

Reduce the eight-mode 9-D tensor $Z_{o\times o\times p\times q\times r\times s\times t\times u\times v}$ to an eight –mode 8-D tensor $Z_{o\times p\times q\times r\times s\times t\times u\times v}$

$$Z_{cdefghij} = \sum_{k=1}^{o} Z_{kcdefghij}$$

Eliminate the civilian and drug dealer modes; o and q, to obtain the weighted six mode 7-D hyper-matrix

$$Z_{p\times p\times q\times r\times s\times t\times u\times v} = Z_{o\times q\times p\times r\times s\times t\times u\times v} \odot Z_{q\times o\times p\times r\times s\times t\times u\times v}$$

Reduce the six- mode 7-D tensor $Z_{p\times q\times r\times s\times t\times u\times v}$ to a six –mode 6-D tensor $Z_{p\times q\times r\times s\times t\times u\times v}$

$$Z_{dfghij} = \sum_{k=1}^{p} Z_{kdfghij}$$

Eliminate the drug farmer and business man modes; *r* and *s*, to obtain the weighted four-mode 5-D hyper- matrix

$$Z_{p\times p\times t\times u\times v} = Z_{r\times s\times p\times t\times u\times v} \odot Z_{s\times r\times p\times t\times u\times v}$$

Reduce the four- mode 5-D tensor $Z_{p\times p\times t\times u\times v}$ to a four-mode 4-D tensor $Z_{p\times t\times u\times v}$

$$Z_{dhij} = \sum_{k=1}^{p} Z_{kdhij}$$

Eliminate the religious leader and farmer modes; *t* and *u*, to obtain the weighted two- mode 3-d hyper matrix $Z_{p\times p\times v} = Z_{t\times u\times p\times v} \;\; Z_{u\times t\times p\times v}$

Reduce the two- mode 3-D tensor $Z_{p\times p\times v}$ to a two-mode 2-D matrix $Z_{p\times v}$

$$Z_{dj} = \sum_{k=1}^{p} Z_{kdj}$$

$Z_{p\times v}$ is the two-mode social network of politician –by-organized criminal related through paths of size 10.

CONCLUSION

One-mode and two-mode social networks have been studied by researchers, but the analysis of multi-mode networks have not yet been explored. In this paper, we extended the theory of two-mode networks and presented an example of a ten-mode social network. We have built the mathematical foundations for the multi-mode networks using graph and matrix theory. Indeed, the ten-mode network example we use to illustrate the theory has direct application to understanding the patrimonial/tribal society in Afghanistan and allows for teasing out subtle relationships among sectors within this social system.

ACKNOWLEDGEMENT

We are grateful for support provided through the Center for Social Complexity and the Department of Computational Social Sciences at George Mason University.

REFERENCES

[1] Geller A. and Moss S., \The Afghan nexus: Anomie, neo-patrimonialism and the emergence of small-world networks," CPM Report No.: 07-179, Manchester Metropolitan University Business School, 2007.

[2] Geller A. and Moss S., \Growing qawm: An Evidence-Driven Declarative Model of Afghan Power Structures," *Advances in Complex Systems*. **11**(2), 321-335, 2008.

[3] De Malafosse B., "An Application of the in_nite Matrix Theory to Mathieu Equation," *Elsevier: Computers and Mathematics with Applications*. **52**, 1439-1452, 2006.

[4] Iacobucci D., "Modeling Multivariate Sequential Dyadic Interactions," *Social Networks*. **11**, 315-362, 1989.

[5] Iacobucci D., "Derivation of Subgroups from Dyadic Interactions," *Psychological Bulletin*. **107**, 114-132, 1990.

[6] Iacobucci D. and Hopkins N., "Modeling Dyadic Interactions and Networks in Marketing," *Journal of Marketing Research*. **29**, 5-17, 1992.

[7] Iacobucci D. and Wasserman S., "Social Networks with Two Sets of Actors," *Psychometrika*. **55**, 707-720, 1990.

[8] Krackhardt D. and Carley K., "PCANS Model of Structure in Organizations," Pp. 113-119 in Proceedings of the 1998 International Symposium on Command and Control Research and Technology. Conference held in June. Monterey, CA.Evidence Based Research, Vienna, VA, 1998.

[9] Lambiotte R. and Ausloos M., "Collaborative Tagging as a Tripartite Network," arXiv, PACS Numbers: 89.75.Fb, 89.65.Ef,64.60.Ak, 2005.

[10] Moss S. and Geller A., "International Contemporary Conict as Small-world Phenomenon and the Hermeneutic net," Pre-sented at the ISA's 49th Annual Convention, Bridging Multiple Divides, San Francisco, CA, USA, Mar 26, 2008. 2009-05-23, *www:allacademic:com= meta=p252387 index:html.*

[11] Tang L., Liu H., Zhang J. and Nazeri Z., "Community Evolution in Dynamic Multi-Mode Networks," Proceeding of the14th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, ACM 978-1-60558-193, 2008.

[12] Wasserman S., and Iacobucci D., "Statistical Modeling of One-mode and Two-mode Networks: Simultaneous Analysis of Graphs and Bipartite Graphs," *British Journal of Mathematical and Statistical Psychology*. **44**, 13-44, 1990.

[13] Wasserman S. and Faust K., Social Network Analysis: Methods and Applications, Cambridge University Press, New York, 1994.