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MATHEMATICAL MODELLING OF MUCUS TRANSPORT IN THE LUNG DUE TO COUGH: EFFECTS IF SEROUS FLUID VISCOSITY AND SEROUS LAYER THICKNESS

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ABSTRACT: In this paper, a biofluid dynamical model is proposed to study mucus transport by considering that moist air, mucus and serous fluid (all three are assumed to be Newtonian fluids) flow coaxially in a circular tube under time dependent pressure gradient simulating cough in an airway with immotile cilia syndrome. It is assumed that moist air and mucus flow under quasi steady state turbulent conditions while the serous fluid surrounding mucus flows under unsteady laminar condition (negligible turbulence) caused by resistance on the flow due to cilia bed.

The analysis of the model shows that as the pressure gradient caused by cough increases, the flow rates of air, mucus and serous fluid increase. It is also found that mucus transport increases as serous fluid viscosity decreases or its thickness increases for fixed mucus layer thickness, the coaxial air diameter being constant. These results are in line with the experimental observations published in literature.

1. INTRODUCTION

The mucociliary system consists of mucus layer, serous layer and cilia embedded in the epithelium. Under normal condition of the lung, contaminants of the inspired air are removed by cilia beating. However, under pathological conditions caused by diseases such as chronic bronchitis, cystic fibrosis, bronchial asthama, etc. excessive mucus is formed in the lung and mucociliary clearance is either impaired or absent. Mucus in that case is transported mainly by air motion caused by forced expiration or cough [15, 16]. Similar situation also arises when cilia in airways become immotile due to infection with influenza (cold virus) or various forms of Ciliary dyskinesia.

Mucus is secreted from goblet cell and is composed of mainly long chain glycoprotein and salts containing water 95-97%, mucin 2.5-3% and salts 1-2% [11]. Mucus is a viscoelastic fluid but behaves as a Newtonian fluid in presence of high shear rates during cough [6, 17, 22]. Its viscosity is about 10^3 poise at low shear rate (1 sec^{-1}) and 0.01 poise at high shear rate (100 sec^{-1}).

Serous fluid originates through trans-epithelial osmosis and is regulated by ion pumping. It consists of a watery solution and behaves as a Newtonian fluid [16]. The viscosity of serous fluid has not been measured but Ross and Corrsin [26] assumed that its viscosity is 0.1 poise and Silberberg [1] assumed it to be 0 .01 poise. Serous layer fluid plays an important role in the transport of mucus in the lung when cilia become immotile and in such a case during cough, these form a carpet on the wall of the airway on which serous fluid flows helping mucus layer to slide on it. This cilia carpet also causes resistance to serous fluid flow during cough and thus making it to flow under unsteady laminar condition rather than turbulent and letting it remain on the cilia bed.

In recent decades, several experiments related to two phase flow in tubes under externally applied pressure have been studied to simulate mucus transport in airways due to cough [4, 24, 25, 27]. In particular, Clarke *et al.*, [27] have shown that the resistance to air flow through a liquid lined tube is markedly increased at all flow rates in comparison to the case of a dry tube. They have noted that at all flow rates compatible with laminar flow conditions the pressure flow relationship in liquid lined tube is nonlinear and the resistance to the flow being greater than that expected from narrowing alone. They have pointed out further that after the onset of turbulence there is a considerable increase in flow resistance, which occurs simultaneously with wave formation on the surface of liquid film. These effects are more marked in case of thicker liquid layer and with lower viscosity. They have also found that the effect of gravity is negligible on mucus transport. Scherer and Burtz [24], Scherer [25] have conducted fluid mechanical experiments relevant to cough, using air and liquid blown out of a straight tube by turbulent air jet. By assuming that the turbulent flow is quasi steady and the turbulent stress in the air is equal to viscous stress in the liquid flowing under laminar condition, they have shown that the liquid transport efficiency has positive correlation with the parameter $\rho_a UT/\mu$ (where ρ_a is the density of air, μ is the viscosity of liquid, U is the air velocity, T is the cough duration) and the liquid transport decreases as this parameter decreases. They have further pointed out that for fixed values of ρ_a, U, T, transport efficiency decreases as viscosity μ increases. Kim *et al*., [4] have studied mucus transport in vertical tubes by two phase (gas, liquid) flow mechanism and noted that the elasticity of mucus does not affect its transport.

Several other experimental investigations in a cough machine (a parallel plate channel) under turbulent flow condition have also been conducted by simulating mucus transport in the trachea due to cough [10, 13, 18-21]. In particular, King *et al*., [21] in their experiments found no apparent relationship between elasticity of mucus and its transport. Zahm *et al*., [10] in their experimental studies in a cough machine pointed out that mucus transport increases due to the presence of a sol phase at the bottom surface. Agarwal *et al*., [13] have studied the mucus gel transport in a constricted simulated cough machine and found that mucus transport increases in presence of serous fluid. Agarwal *et al*., [14] have also studied, experimentally, the transport of mucus gel in a simulated cough machine where the bottom plate was grooved and, flooded with serous fluid. They found that mucus transport increases as the cross-sectional area formed by grooves saturated with serous fluid increases, suggesting the importance of cilia bed submerged in serous fluid. See also [2, 3, 22, 23 and cross references].

It may be noted here that hardly any attempt has been made to study mucus transport in the actual lung due to cough or to explain above experimental observation by using a mathematical model under turbulent flow conditions. Therefore, in this paper, we consider that the moist air, mucus and serous fluid flow in a pipe under time dependent pressure gradient simulating mucus transport in airways during cough under the following assumptions [9]:

1. The air, mucus and serous fluid flow symmetrically about the central axis.
2. The pressure gradient representing prolonged or normal cough is chosen to be a time dependent function [5].
3. Since air is saturated with water, it behaves as an incompressible Newtonian fluid in the lung during cough and flows under quasi steady state turbulent condition.
4. Mucus behaves as an incompressible Newtonian fluid due to high shear rate during cough and flows under quasi steady state turbulent condition.
5. In pathological condition cilia are considered to be immotile and during cough they form a carpet on the wall of the airway causing resistance to flow (no slip condition) making serous fluid to flow over it under unsteady laminar condition (negligible turbulence).

2. MATHEMATICAL MODEL AND SOLUTION

We consider simultaneous and co-axial layers of air, mucus and serous fluid flowing through a tube caused by time dependent pressure gradient simulating mucus transport in airways due to cough as shown in fig.1. In the central core air is assumed to flow under quasi-turbulent condition due to instantaneous pressure gradient caused by cough. The mucus layer surrounding this circular core is also assumed to flow under turbulent conditions, whereas the serous layer is assumed to flow under unsteady laminar conditions.

Since the velocities in the turbulent layers are very large due to cough, it is assumed that air and mucus flow under quasisteady state turbulent conditions while serous fluid flows under unsteady laminar condition as mentioned above. Using Prandtl mixing length theory, the means of quasi steady state equations in the turbulent layers and the unsteady state equation of serous fluid in the laminar layer, can be written in cylindrical coordinates as follows [8].

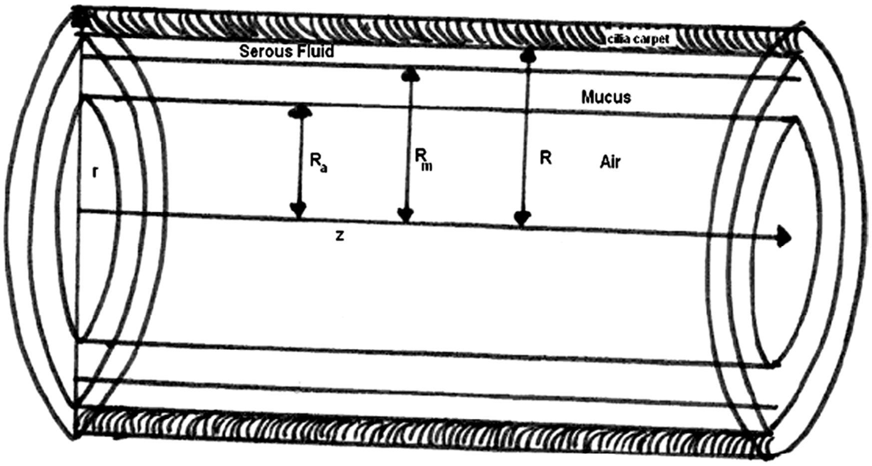


Figure 1: Mucus Transport in a Circular Tube

Region I: Quasi steady turbulent flow of air ($0 \le r \le R_a$):

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_a) = 0 \tag{1}$$

$$\tau_a = \rho_a l_a^2 \left|\frac{\partial u_a}{\partial r}\right| \frac{\partial u_a}{\partial r} = -\rho_a l_a^2 \left(-\frac{\partial u_a}{\partial r}\right)^2 \tag{2}$$

Region II: Quasi steady turblent flow of mucus ($R_a \le r \le R_m$):

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_m) = 0 \tag{3}$$

$$\tau_m = \rho_m l_m^2 \left|\frac{\partial u_m}{\partial r}\right| \frac{\partial u_m}{\partial r} = -\rho_m l_m^2 \left(-\frac{\partial u_m}{\partial r}\right)^2 \tag{4}$$

Region III: Unsteady laminar flow of serous fluid on immotile cilia bed at the wall of the airway ($R_m \le r \le R$):

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_s) = \rho_s \frac{\partial u_s}{\partial t} \tag{5}$$

$$\tau_s = \mu_s \frac{\partial u_s}{\partial r} \tag{6}$$

In the model described by (1)-(6), t is the time, z is the coordinate along the axis of the tube in the flow direction, r is the coordinate in the radial direction and perpendicular to fluid flow, R_a is the thickness up to air-mucus interface, R_m is the thickness upto the mucus and serous fluid interface, R is the radius of the outer surface of the serous layer interfacing cilia bed, p is the mean pressure which is constant across three layers, u_a, u_m, u_s are the mean velocity components of air, mucus and serous fluid in the z direction respectively, τ_a is the mean shear stress in the air, τ_m is the mean shear stress in the mucus layer and τ_s is the mean shear stress in the laminar serous layer, ρ_a, ρ_m and ρ_s are the densities of air, mucus and serous fluid respectively; μ_s is the viscosity of serous fluid. The mixing lengths l_a and l_m are assumed as follows:

$$l_a = l_0(r - r), \qquad l_m = l_1(R - r) \tag{7}$$

where l_a and l_m are constants and which are determined experimentally[8].

The initial condition needed for (5) is

$$u_s = 0 \quad \text{at} \quad t = 0 \tag{8}$$

The boundary conditions are

$$\frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{9}$$

$$u_s = 0 \quad \text{at} \quad r = R \tag{10}$$

The matching conditions are

$$u_a = u_m; \qquad \tau_a = \tau_m \qquad \text{at} \qquad r = R_a \tag{11}$$

$$u_m = u_s; \qquad \tau_m = \tau_s \qquad \text{at} \qquad r = R_m \tag{12}$$

The conditions (11) and (12) represent the continuity of velocity and the stress components at the two interface. Due to presence of cilia carpet on the airway wall, no slip condition exists as given by (10).

Since during cough the pressure gradient generated in the lung is time dependent, we assume that

$$-\frac{\partial p}{\partial z} = P = P_0 f(t) \tag{13a}$$

where P_0 is a constant, the magnitude of which depends upon the intensity of cough. The function $f(t)$ representing cough has been chosen by considering the flow rates of air in various experiments as described by Leith [5]. This function is assumed to be of the following form in the case of normal cough

$$f(t) = \begin{cases} \frac{1}{4} t \left(1 - \frac{t}{2T_m}\right) & 0 \le t \le T_m \\ \frac{9}{32} t \left(1 - \alpha \frac{t}{T}\right)^2 & T_m \le t \le \frac{T}{\alpha} \\ 0 & t \ge \frac{T}{\alpha} \end{cases} \tag{13b}$$

where $\alpha = 0.37$ and $T_m = \frac{T}{3\alpha} = 0.027\text{ sec}$ and T is the duration of cough. The graphical representation of $f(t)$ is shown in Fig. 2. The function $f(t)$ satisfies the following conditions:

I. The pressure gradient function $f(t)$ is zero at $t = 0$ and $t = \frac{T}{\alpha}$ where α is a constant such that $f(T_m)$ is continuous.

II. The maximum of pressure gradient function occurs at $t = T_m$ so that $f'(T_m) = 0$.

III. The pressure gradient function $f(t)$ is such that $f'(t) > 0$, $0 \le t \le T_m$.

IV. The pressure gradient function $f(t)$ is such that $f'(t) \le 0, T_m \le t \le \frac{T}{\alpha}$.

V. Where T is the duration of cough and α is the constant to be chosen.

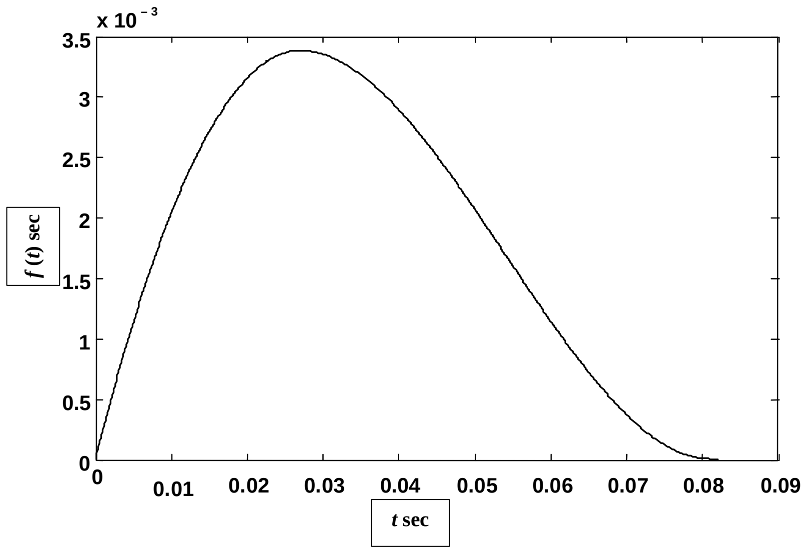


Figure 2: Graphical Representation of $f(t)$ for Various Values of t

The equations (1)-(6) have been solved by using (7)-(13) and the flow rates of air, mucus and serous fluid are determined as follows:

$$
\begin{aligned}
\frac{Q_a}{2\pi} &= \frac{PR_a^2}{4}\left[\frac{R^2-R_m^2}{2\mu_s}\right]-\frac{\rho_s\psi_s R_a^2}{4}\left[\frac{R^2-R_m^2}{2\mu_s}+\frac{R_m^2}{\mu_s}\ln\frac{R_m}{R}\right]\\
&+\frac{R^2}{2l_0}\left(\frac{PR}{2\rho_a}\right)^{\frac{1}{2}}\left[\ln\frac{R^{\frac{1}{2}}+R_a^{\frac{1}{2}}}{R^{\frac{1}{2}}-R_a^{\frac{1}{2}}}-2\left(\frac{R_a}{R}\right)^{\frac{1}{2}}\left\{1+\frac{R_a}{3R}+\frac{R_a^2}{5R^2}\right\}\right]\\
&+\frac{R_a^2}{2l_1}\left(\frac{PR}{2\rho_m}\right)^{\frac{1}{2}}\left[\ln\frac{R^{\frac{1}{2}}+R_m^{\frac{1}{2}}}{R^{\frac{1}{2}}-R_m^{\frac{1}{2}}}-\ln\frac{R^{\frac{1}{2}}+R_m^{\frac{1}{2}}}{R^{\frac{1}{2}}-R_a^{\frac{1}{2}}}-2\,\frac{R_m^{\frac{1}{2}}-R_a^{\frac{1}{2}}}{R^{\frac{1}{2}}}\right]
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\frac{Q_m}{2\pi} &= \frac{P}{4}[R_m^2-R_a^2]\left[\frac{R^2-R_m^2}{2\mu_s}\right]-\frac{\rho_s\psi_s}{4}[R_m^2-R_a^2]\left[\frac{\{R^2-R_m^2\}}{2\mu_s}+\frac{R_m^2}{\mu_s}\ln\frac{R_m}{R}\right]\\
&+\frac{1}{l_1}\left(\frac{PR}{2\rho_m}\right)^{\frac{1}{2}}\left[\begin{array}{l}\dfrac{R^2-R_a^2}{2}\left\{\ln\dfrac{R^{\frac{1}{2}}+R_m^{\frac{1}{2}}}{R^{\frac{1}{2}}-R_m^{\frac{1}{2}}}-\ln\dfrac{R^{\frac{1}{2}}+R_a^{\frac{1}{2}}}{R^{\frac{1}{2}}-R_a^{\frac{1}{2}}}\right\}\\[2ex]
+\left(\dfrac{R_a}{R}\right)^{\frac{1}{2}}\left(\dfrac{15R^2+5R_aR-12R_a^2}{15}\right)\\[2ex]
-\left(\dfrac{R_m}{R}\right)^{\frac{1}{2}}\left(\dfrac{15R^2+5R_mR+3R_m^2-12R_a^2}{15}\right)\end{array}\right]
\end{aligned}
\tag{15}
$$

$$
\frac{Q_s}{2\pi} = \frac{P}{4}[R^2-R_m^2]\left[\frac{R^2-R_m^2}{4\mu_s}\right]-\frac{\rho_s\psi_s}{4}\left[\frac{\{R^2-R_m^2\}\{R^2-3R_m^2\}}{4\mu_s}+\frac{R_m^4}{\mu_s}\ln\frac{R}{R_m}\right]
\tag{16}
$$

For detailed calculations see Appendix-A.

3. DISCUSSIONS

The flow rates Q_a, Q_m and Q_s given by equations (14), (15) and (16) have been calculated and analysed by using MatLab. Since our main aim is to study the effect of viscosity of serous fluid and its thickness on mucus transport for a given pressure gradient, we have varied μ_s, $(R - R_m)$ for calculating flow rates. In view of this we have drawn the graphs of Q_a, Q_m and Q_s with respect to time for various values of viscosity and thickness of serous fluid in Figures (3), (4), (5) and (6).

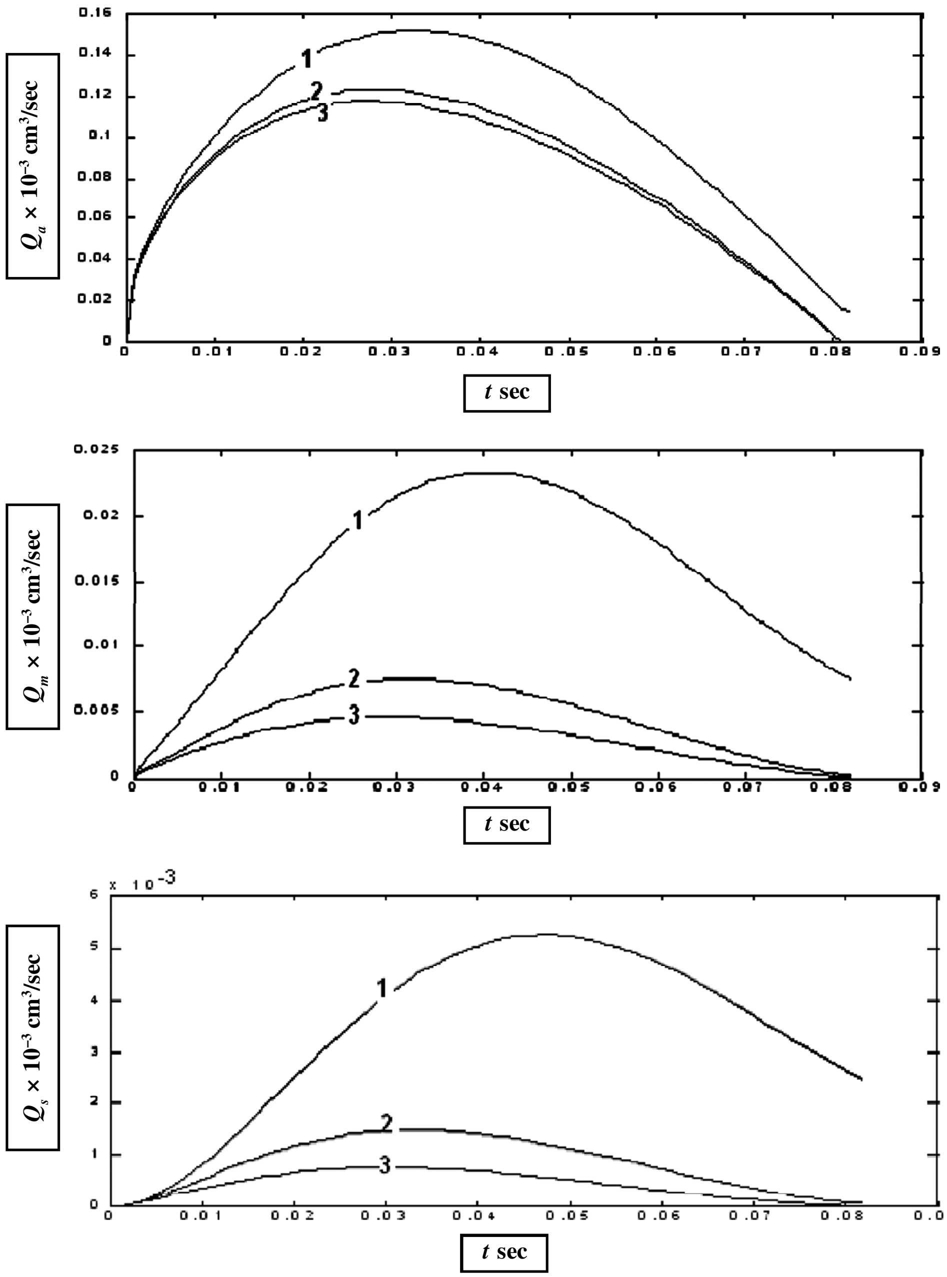


Figure 3: Variation of Q_a, Q_m and Q_s with t for Different μ_s

1 Denotes μ_s = 0.01 poise

2 Denotes μ_s = 0.05 poise

3 Denotes μ_s = 0.1 poise

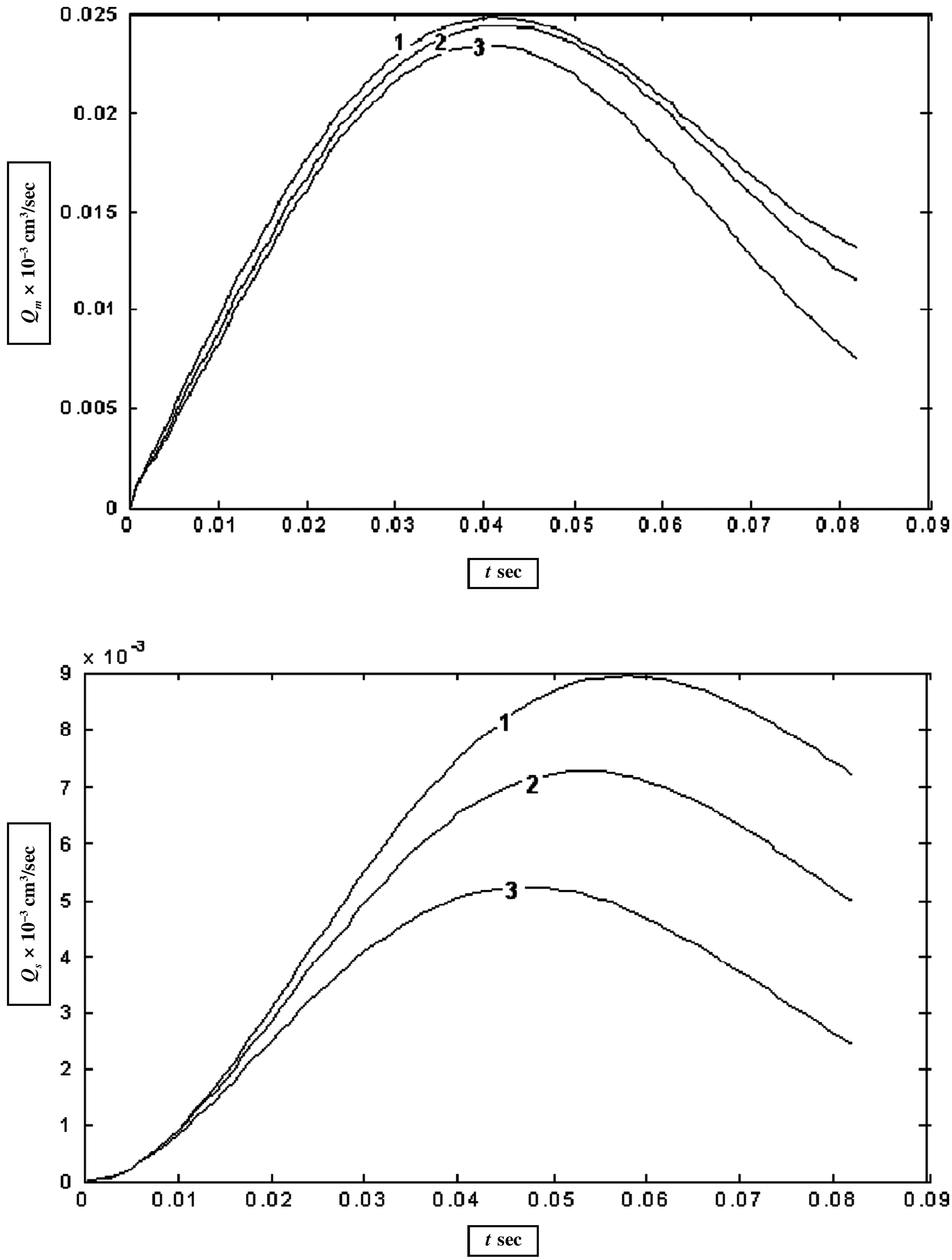


Figure 4: Variation of Q_m and Q_s with t for Different $(R - R_m)$ (μ_s = 0.01 poise)

1 Denotes $(R - R_m)$ = 0.05 cm

2 Denotes $(R - R_m)$ = 0.04 cm

3 Denotes $(R - R_m)$ = 0.03 cm

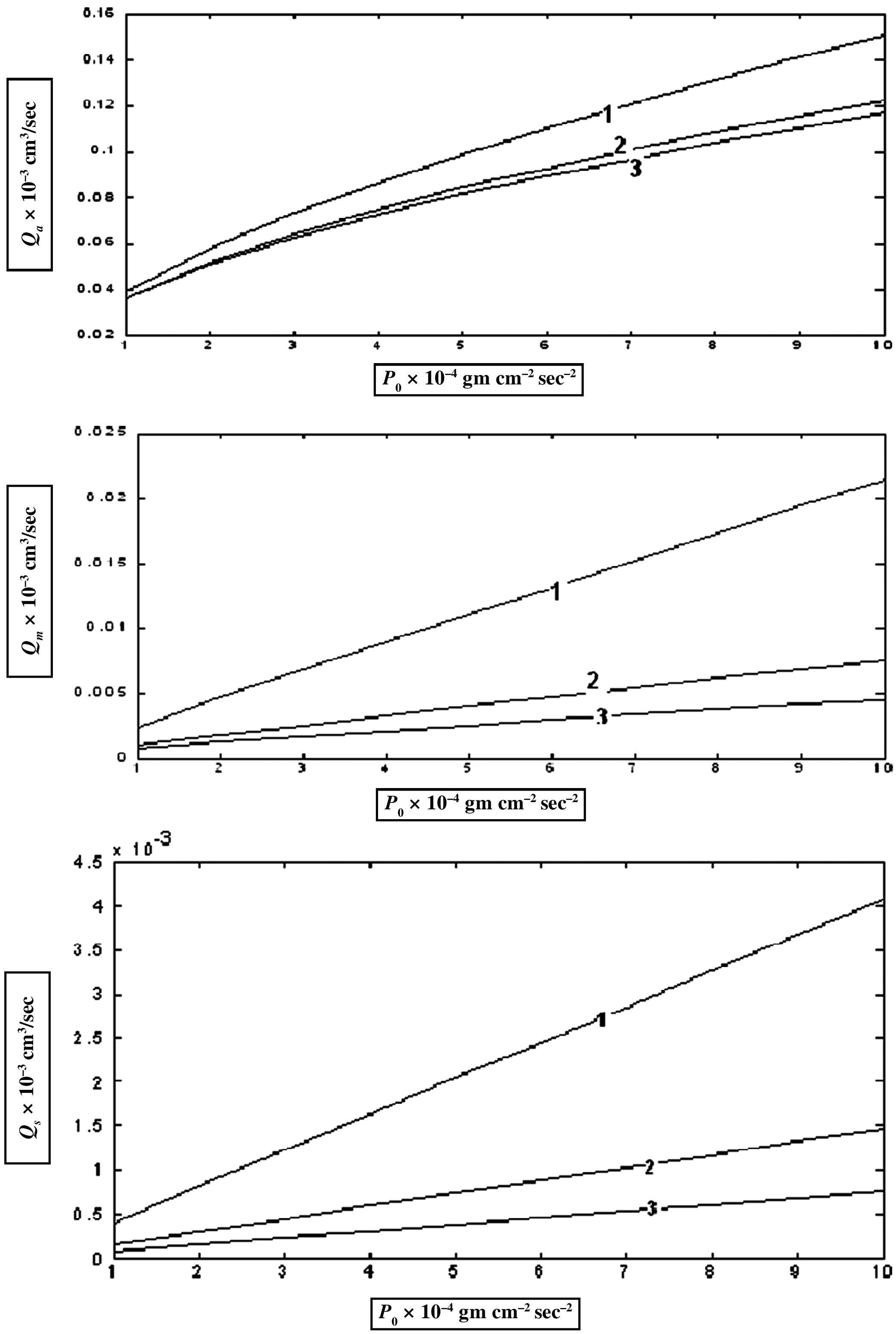
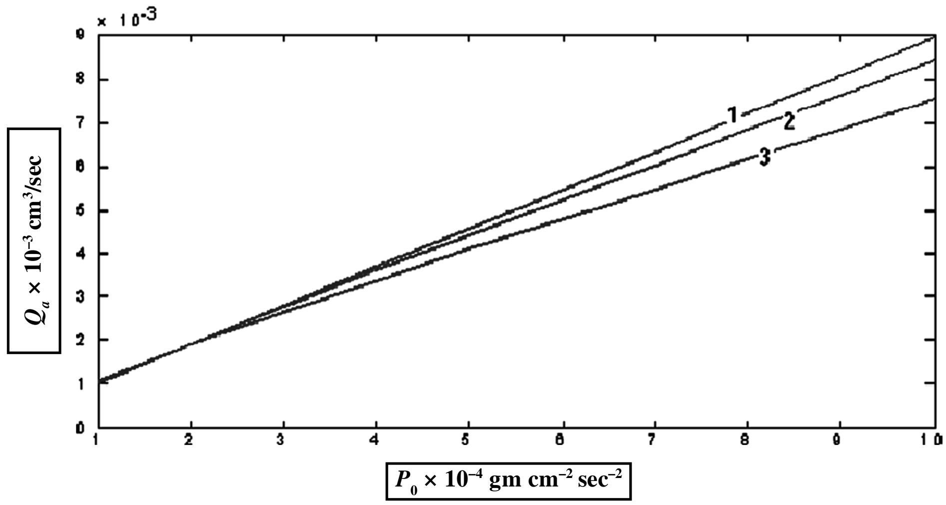


Figure 5: Variation of ,and with P_0 for different μ_s (t=.03 sec)

1 Denotes μ_s = 0.01 poise

2 Denotes μ_s = 0.05 poise

3 Denotes μ_s = 0.1 poise



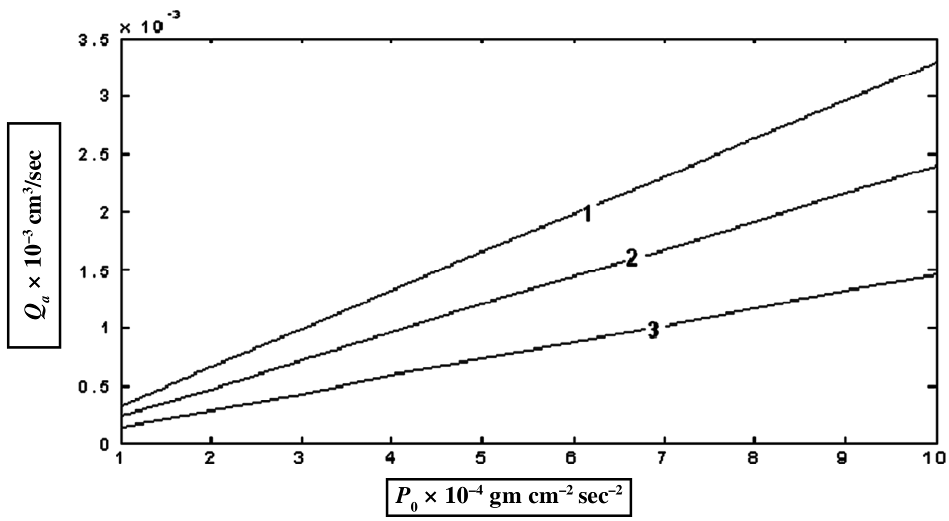


Figure 6: Variation of and with P_0 for Different ($R - R_m$) = (t = .03 sec, μ_s = 0.05 poise)
1 Denotes ($R - R_m$) = 0.05 cm
2 Denotes ($R - R_m$) = 0.04 cm
3 Denotes ($R - R_m$) = 0.03 cm

The following set of parameters has been used in calculations [7].

$T = 0.03$ sec	$t = 0 - 0.085$ sec,
$l_0 = l_1 = 0.40$	$\mu_s = (1.00 - 10.00) \times 10^{-2}$ poise.
$R_a = 31.45 \times 10^{-2}$ cm	$R = 41.45 - 43.45 \times 10^{-2}$ cm
$R_m = 38.45 \times 10^{-2}$ cm	$P_0 = (1.00 - 100.00) \times 10^5$ gm cm^{-2} sec^{-2}
$\mu_m = 1.00 - 10.00$ poise	$\rho_s = 0.90$ gm cm^{-3}
$\rho_a = 1.00 \times 10^{-3}$ gm cm^{-3}	$\rho_m = 1.00$ gm cm^{-3}

Figures (3), (4) illustrates the flow rates of moist air, mucus and serous fluid for $P_0 = 1.00 \times 10^5$ gm cm^{-2} sec^{-2} and for various values of μ_s and ($R - R_m$). Figures (5) and (6) show the effects of pressure gradient for $t = 0.03$ and for various values of μ_s and ($R - R_m$). From Fig. (3) it can be seen that as μ_s increases all the flow rates increase. From Fig. 4 it is noted that for constant air diameter as the serous layer thickness increases the flow rates of mucus and serous fluid increase but the flow rate of moist air does not change (not shown here). These results are in line with the experimental observations of Zahm *et al*., [10], Agarwal [13], [14], where the importance of serous fluid (sol phase) has been experimentally shown in a simulated cough machine for increasing the mucus transport. Also, Scherer [24] has found the same result in a tube. From Fig. (5) and (6) it is clearly noted that all the flow rates increase as the pressure gradient increases for fixed serous fluid viscosity and its thickness. This result is again qualitatively similar to the observations of Agarwal *et al*., [13], [14] found in a simulated cough machine. Further from Fig. (5) it is seen that as serous fluid viscosity decreases, the flow rates of serous fluid, mucus and moist air increase for given pressure gradient. The same result related to mucus transport have also been shown by Agarwal *et al*., [13], [14] experimentally. In Fig. 6 it is shown as serous layer thickness increases, flow rates of mucus and serous fluid increase.The result has been proved by Zahm *et al*., [10] who have shown in their experiments in a cough machine that mucus transport increases in a sol phase.

4. CONCLUSIONS

In this paper, we have studied mucus transport in an airway having immotile cilia syndrome due to cough by representing it as a circular tube. The cough has been simulated by a time dependent pressure gradient. The simultaneous and coaxial flow of air and mucus in a tube are considered to flow under quasisteady turbulent conditions while serous fluid surrounding mucus layer coaxially is assumed to flow under unsteady laminar condition.

It is assumed further that immotile cilia, during cough, form a carpet on the inner side of the wall of airway, which causes resistance to flow to serous fluid and making it to stick to the wall but it allows mucus to slide on it.

From the analysis of the model the following conclusions have been drawn in both the cases.

1. The mucus transport rate increases as the serous fluid viscosity decreases.
2. For fixed air and mucus layer thicknesses, mucus and serous fluid flow rates increase as serous layer thickness increases.
3. The flow rates of air, mucus and serous fluid increase as pressure gradient representing cough increases.

The present study demonstrates the role of serous fluid on mucus transport in normal or pathological airway during cough. The resistance (no slip condition) provided by the cilia carpet on the serous fluid is helpful in making it flow under laminar condition, even though air and mucus flow under turbulent conditions. This shows the importance of cilia that even if they are immotile, the carpet formed by them is helpful in providing resistance to serous fluid flow useful for cough dependent mucus transport.

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APPENDIX-A

ANALYSIS OF MODEL

Now we solve the system (1)-(6) under the initial condition (8), boundary and matching conditions (9), (10), (11) and (12). To solve the unsteady equation in laminar sublayer we use the method of averaging, Sestak and Charles [12]. Thus, by substituting the acceleration term on the left hand side of equation (5) by its mean value across the film thickness i.e.

$$\frac{\partial u_s}{\partial t} \approx \Psi_s = \frac{1}{R - R_m} \int_{R_m}^{R} \frac{\partial u_s}{\partial t}\, dr \tag{A1}$$

it reduces to

$$\frac{\partial}{\partial r}(r\tau_s) = (P - \rho_s \Psi_s)\, r \tag{A2}$$

where Ψ_s is a function of time only, and P is given by equation (8a). Using this and equations (1)-(12), we get

$$\tau_a = -\frac{\mathrm{Pr}}{2} \tag{A3}$$

$$\tau_m = -\frac{\mathrm{Pr}}{2} \tag{A4}$$

$$\tau_s = -\frac{\mathrm{Pr}}{2} + \frac{\rho_s \psi_s}{2}\left[r - \frac{R_m^2}{r}\right] \tag{A5}$$

$$\begin{aligned} u_a = {} & \frac{P}{2}\left[\frac{R^2 - R_m^2}{2\mu_s}\right] - \frac{\rho_s \psi_s}{2}\left[\frac{R^2 - R_m^2}{2\mu_s} + \frac{R_m^2}{\mu_s}\ln\frac{R_m}{R}\right] \\ & + \frac{1}{l_0}\left(\frac{2PR}{\rho_a}\right)^{\frac{1}{2}}\left[\ln\frac{R^{\frac{1}{2}} + R_a^{\frac{1}{2}}}{R^{\frac{1}{2}} + r^{\frac{1}{2}}} - \frac{1}{2}\ln\frac{R - R_a}{R - r} - \frac{R_a^{\frac{1}{2}} - r^{\frac{1}{2}}}{R^{\frac{1}{2}}}\right] \\ & + \frac{1}{l_1}\left(\frac{2PR}{\rho_m}\right)^{\frac{1}{2}}\left[\ln\frac{R^{\frac{1}{2}} + R_m^{\frac{1}{2}}}{R^{\frac{1}{2}} + R_a^{\frac{1}{2}}} - \frac{1}{2}\ln\frac{R - R_m}{R - R_a} - \frac{R_m^{\frac{1}{2}} - R_a^{\frac{1}{2}}}{R^{\frac{1}{2}}}\right] \end{aligned} \tag{A6}$$

$$\begin{aligned} u_m = {} & \frac{P}{2}\left[\frac{R^2 - R_m^2}{2\mu_s}\right] - \frac{\rho_s \psi_s}{2}\left[\frac{R^2 - R_m^2}{2\mu_s} + \frac{R_m^2}{\mu_s}\ln\frac{R_m}{R}\right] \\ & + \frac{1}{l_1}\left(\frac{2PR}{\rho_m}\right)^{\frac{1}{2}}\left[\ln\frac{R^{\frac{1}{2}} + R_m^{\frac{1}{2}}}{R^{\frac{1}{2}} + r^{\frac{1}{2}}} - \frac{1}{2}\ln\frac{R - R_m}{R - r} - \frac{R_m^{\frac{1}{2}} - r^{\frac{1}{2}}}{R^{\frac{1}{2}}}\right] \end{aligned} \tag{A7}$$

$$u_s = \frac{P}{2}\left[\frac{R^2 - r^2}{2\mu_s}\right] - \frac{\rho_s \psi_s}{2}\left[\frac{R^2 - r^2}{2\mu_s} + \frac{R_m^2}{\mu_s}\ln\frac{r}{R}\right] \tag{A8}$$

To determine ψ_s we differentiate Equation (A8) with respect to t to get

$$\frac{\partial u_s}{\partial t} = \frac{P'}{2}\left[\frac{R^2 - r^2}{2\mu_s}\right] - \frac{\rho_s \psi_s'}{2}\left[\frac{R^2 - r^2}{2\mu_s} + \frac{R_m^2}{\mu_s}\ln\frac{r}{R}\right] \tag{A9}$$

where ($'$) denotes the derivative with respect to t.

Using equation (A1) and (A9) we get

$$\psi_s' + \frac{\psi_s}{a_2} = \frac{a_1}{a_2} P' = \frac{a_1}{a_2} P_0 f'(t) \tag{A10}$$

where,

$$a_1 = \left[\frac{(R - R_m)(2R + R_m)}{12\mu_s} \right]$$

$$a_2 = \left[\frac{(R - R_m)(2R + R_m)}{12\mu_s} - \frac{R_m^2}{2\mu_s} \left\{ 1 + \frac{R_m}{R - R_m} \ln \frac{R_m}{R} \right\} \right].$$

Since $P = 0$ and $u_s = 0$ at $t = 0$, from equation (A8) we have $\psi_s = 0$ at $t = 0$. From Equation (A10) the expression for can then be obtained as follows:

$$\psi_s = -a_1 p_0 \exp\left(-\frac{t}{a_2}\right) \begin{cases} \dfrac{-1}{4T_m}\left[-a_2 + a_2 \exp\left(\dfrac{t}{a_2}\right) - t\exp\left(\dfrac{t}{a_2}\right) - T_m + T_m \exp\left(\dfrac{t}{a_2}\right) \right] & 0 \le t \le T_m \\ \dfrac{1}{864T^2T_m} \left[\begin{array}{l} -216 a_2 \left(-1 + \exp\left(\dfrac{T_m}{a_2}\right)\right) T^2 \\ +T_m \left(\begin{array}{l} -200 a_2^2 \left(\exp\left(\dfrac{t}{a_2}\right) - \exp\left(\dfrac{T_m}{a_2}\right)\right) \\ -\exp\left(\dfrac{t}{a_2}\right)(10t - 27T)(10t - 9T) \\ +27\left(8 + 9\exp\left(\dfrac{T_m}{a_2}\right)\right) T^2 \\ +40 a_2 \left(\exp\left(\dfrac{t}{a_2}\right)(5t - 9T) + 9\exp\left(\dfrac{T_m}{a_2}\right) T \right) \\ +20\exp\left(\dfrac{T_m}{a_2}\right) T_m \left(-2(5a_2 + 9T) + 5T_m\right) \end{array} \right) \end{array} \right] & T_m \le t \le \dfrac{T}{\alpha} \end{cases} \tag{A11}$$

The volumetric flow rates in each layer can be defined as

$$Q_a = \int_0^{R_a} 2\pi r u_a \, dr, \qquad Q_m = \int_{R_a}^{R_m} 2\pi r u_m \, dr, \qquad Q_s = \int_{R_m}^{R} 2\pi r u_s \, dr$$

which after using Equations (A6)-(A8) can be found as given in equations (14), (15) and (16) respectively.