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HOMOTOPY ANALYSIS METHOD FOR SOLVING FRACTIONAL DIFFUSION EQUATION

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ABSTRACT: In this letter, the powerful, easy-to-use and effective mathematical tool like Homotopy Analysis Method is used to solve the diffusion equation with fractional time derivative $\alpha\,(0 < \alpha \leq 1)$. Using the initial condition, the approximate analytical solution of the problem is obtained. Choosing proper values of auxiliary and homotopy parameters, the convergence of the approximate series solution is illustrated for different particular cases.

Keywords: Homotopy analysis method, Fractional diffusion equation, Fractional Brownian motion, Approximate analytical solution.

1. INTRODUCTION

Fractional diffusion equation is obtained from the classical diffusion equation in mathematical physics by replacing the first order time derivative by a fractional derivative of order α where $0 < \alpha < 1$; of late this being a field of growing interest as evident from literature survey. An important phenomenon of these evolution equations is that it generates the fractional Brownian motion (FBM) which is a generalization of Brownian motion (BM). In 1995, Sebastian [1] has given the correct path integral representation whose measurement shows that the process FBM is Gaussian but in general non-Markovian though BM is Markovian. Due to this, Jumarie [2] has correctly made the statement that fractional calculus which was in earlier stage considered as a mathematical curiosity, has now become the object of extensive development of fractional partial differential equations for its engineering applications. Considerable work on fractional diffusion equations has already been been done by Angulo *et al*., [3], Pezat and Zabczyk [4], Schneider and Wyss [5], Yu and Zhang [6], Mainardi [7], Mainardi *et al*., [8], Anh and Leonenko [9] etc., using numerical techniques. In view of the restricted applications of prevailing analytical methods and the difficulties posed due to the rounding off errors involved in numerical techniques used by the researchers, the authors have made a sincere effort to explore the solution of the fractional diffusion equation by using a powerful analytical method called Homotopy Analysis Method (HAM).

In 1992, Liao [10] proposed a mathematical tool based on homotopy, a fundamental concept in topology and differential geometry known as Homotopy Analysis Method. It is an analytical approach to get the series solution of linear and nonlinear partial differential equations (PDEs). The difference with the other perturbation methods is that this method is independent of small/large physical parameters. It also provides a simple way to ensure the convergence of series solution. Moreover the method provides great freedom to choose base function to approximate the linear and nonlinear problems ([11], [12]). Another advantage of the method is that one can construct a continuous mapping of an initial guess approximation to the exact solution of the given problem through an auxiliary linear operator. To ensure the convergence of the series solution an auxiliary parameter is used. Liao and Tan [13] have shown that with the help of this method, even complicated nonlinear problems are reduced to the simple linear problems. Recently Liao [14] has substantiated the fact that the difference of this method with the other analytical ones is that one can ensure the convergence of the series solution by choosing a proper value of convergence-control parameter. Another important advantage as compared to the other existing perturbation and non-perturbation methods is the freedom to choose proper base function to get the better approximations of the solution to the problems.

The fractional diffusion equation considered here is

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} \frac{\partial^{2} u(x,t)}{\partial x^{2}} + \frac{\partial}{\partial x}(xu(x,t)), \quad 0 < \alpha < 1, \quad x > 0, \quad t > 0 \tag{1}$$

with the initial condition

$$u(x, 0) = x^n, n \quad \text{is a positive integer.} \tag{2}$$

Here $\frac{\partial^\alpha}{\partial t^\alpha}(\cdot)$ is the Caputo derivative of order α, $u(x, t)\,\alpha$, represents the probability density function at a position x in time t. This type of problem is recently solved by Das [15] by using another mathematical tool Variational Iteration Method (VIM). Although the method is simple, concise, accurate and effective for linear fractional problems but there are certain drawbacks of the method, namely the restrictions on the order of the nonlinearity term or even the form of the boundary conditions (Tari [16]) and uncontrollability of nonzero end conditions. In this article, a more accurate, flexible and very powerful analytical method HAM is used to solve the equation (1). Using the initial condition (2), the approximate analytical solution of $u(x, t)$ is obtained. As a particular case of the problem for $n = 1$, it is shown through Table 1, the convergence of the series solution with the proper choice of auxiliary parameter and homotopy parameter, which shows validity and potential of the method.

Table 1
Comparison of VIM and HAM Results of $u(x, t)$ for Different Values of t and α at $n = 1$ and $x = 1$

			u_{HAM}		
t	a	u_{VIM}	$q = 0.99$ $h = -1.0318$	$q = 0.98$ $h = -1.065$	$q = 0.97$ $h = -1.0997$
0.2	1/3	8.45162	8.45119	8.45110	8.45141
	1/2	3.83977	3.80305	3.76661	3.73051
	2/3	2.38133	2.36102	2.34153	2.32291
	1	1.49182	1.48605	1.48065	1.47559
0.4	1/3	16.8347	17.0075	17.1882	17.3778
	1/2	7.80412	7.76338	7.72154	7.67859
	2/3	4.34221	4.29472	4.24768	4.20119
	1	2.22513	2.20825	2.19232	2.17737
0.6	1/3	26.7196	27.1926	27.6910	28.2172
	1/2	13.5100	13.5289	13.5472	13.5649
	2/3	7.36696	7.29904	7.22991	7.15962
	1	3.31514	3.27955	3.24547	3.21301
0.8	1/3	37.9404	38.8256	39.7628	40.7574
	1/2	21.1565	21.3229	21.4940	21.6700
	2/3	11.8060	11.7442	11.6792	11.6108
	1	4.92311	4.86004	4.79852	4.73877
1.0	1/3	50.3772	51.7769	53.2643	54.8488
	1/2	30.9036	31.3258	31.7649	32.2222
	2/3	18.0184	18.0127	18.0030	17.9890
	1	7.26667	7.16835	7.07059	6.97370

2. THE BASIC IDEA OF HAM

In this paper, we apply the HAM to the solution of the fractional diffusion problem to be discussed. In order to show the basic idea of HAM, consider the following differential equation

$$N[u(x, t)] = 0 \tag{3}$$

where N is a non-linear operator, x and t are independent variables, $u(x, t)$ is the unknown function. By means of the HAM, we first construct the so-called zeroth-order deformation equation

$$(1-q)L[\phi(x, t; q) - u_0(x, t)] = hH(x, t)N[\phi(x, t; q)] \tag{4}$$

where $q \in [0, 1]$ is the embedding parameter, $h \neq 0$, is a nonzero auxiliary parameter, $H(x, t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x, t)$ is the initial guess of $u(x, t)$, It is obvious that when the embedding parameter $q = 0$ and $q = 1$, Eq. (4) becomes $\phi(x, t; 0) = u_0(x, t)$ and $\phi(x, t; 1) - u_0(x, t)$ respectively. Thus, as q increases from 0 to 1, the solution $\phi(x, t; q)$ varies from the initial guess $u_0(x, t)$ to the exact solution $u(x, t)$. Expanding $\phi(x, t; q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t)q^k \tag{5}$$

where

$$u_k(x,t) = \frac{1}{k!}\frac{\partial^k \phi}{\partial q^k}\bigg|_{q=0}. \tag{6}$$

The convergence of the series (5) depends upon the auxiliary parameter $\hbar$. If it is convergent at $q = 1$, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t)$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao [17]. Now we define the vector

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}, \tag{7}$$

So the mth-order deformation equations are

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\, R_m(\vec{u}_{m-1}(x,t)), \tag{8}$$

with the initial conditions

$$u_m(x, 0) = 0, \tag{9}$$

where

$$R_m(\vec{u}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Now, the solution of the mth-order deformation equation (8) for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar J_t^{\alpha}[R_m(\vec{u}_{m-1}(x,t))] + c, \tag{10}$$

where c is the integration constants determined by the initial condition (9) and $J_t^{\alpha}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} f(\xi)d\xi$. In this way, it is easy to obtain $u_m(x, t)$ for $m \geq 1$, at mth-order and finally get the solution as

$$u(x,t) = \sum_{m=0}^{N-1} u_m(x,t). \tag{11}$$

3. SOLUTION OF THE PROBLEM BY HAM

To solve equation (1) by HAM, we choose the initial approximation

$$u_0(x, t) = x^n, \tag{12}$$

and the linear operator,

$$L[\phi(x,t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha}, \tag{13}$$

with the property

$$L[c] = 0, \tag{14}$$

where c is integral constant. Furthermore, equation (1) suggests that we define an equation of nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} - \frac{\partial}{\partial x}(x\phi(x,t;q)), \tag{15}$$

Now, we construct the zero-th order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)], \tag{16}$$

Now proceeding is the previous section, we successively obtain

$$u_1(x, t) = -\hbar[(n+1)x^n + n(n-1)x^{n-2}]\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{17}$$

$$\begin{aligned} u_2(x, t) = &-\hbar(\hbar+1)[(n+1)x^n + n(n-1)x^{n-2}]\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ \hbar^2[(n+1)^2 x^n + 2n^2(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4}]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{18}$$

$$\begin{aligned} u_3(x, t) = &-\hbar(\hbar+1)^2[(n+1)x^n + n(n-1)x^{n-2}]\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ 2\hbar^2(\hbar+1)[(n+1)^2 x^n + 2n^2(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4}]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &- \hbar^3[(n+1)^3 x^n + n(n-1)(3n^2+1)x^{n-2} + 3n(n-1)^2(n-2)(n-3)x^{n-4} \\ &+ n(n-1)(n-2)(n-3)(n-4)(n-5)x^{n-6}]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{19}$$

$$\begin{aligned} u_4(x, t) = &-\hbar(\hbar+1)^3[(n+1)x^n + n(n-1)x^{n-2}]\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ 3\hbar^2(\hbar+1)^2[(n+1)^2 x^n + 2n^2(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4}]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &- 3\hbar^2(\hbar+1)^2[(n+1)^3 x^n + 2n^2(n-1)x^{n-2} + 3n(n-1)^2(n-2)(n-3)x^{n-4} \\ &+ n(n-1)(n-2)(n-3)(n-4)(n-5)x^{n-6}]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \hbar^4[(n+1)^4 x^n + 4n^2(n-1)(n^2+1)x^{n-2} \\ &+ n(n-1)(n-2)(n-3)(6n^2-12n+10)x^{n-4} + 4n(n-1)(n-2)^2(n-3)(n+4)(n-5)x^{n-6} \\ &+ n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)x^{n-8}]\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \end{aligned} \tag{20}$$

In this manner the rest of the components u_n, $n \geq 1$ of the HAM can be completely obtained and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x, t)$ by the truncated series

$$u(x,t) = \lim_{N \to \infty} \Phi_N(x,t) \tag{21}$$

where

$$\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t), \quad N \geq 1,$$

The above series solutions generally converge very rapidly. The rapid convergence means that few terms are required.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, the values of $u(x, t)$ for the initial condition $u(x, 0) = x$ for various values of t and $x = 1$ with the proper choices of $\hbar$, q are obtained and the results are depicted in the Table 1. Here sixth order term approximation is taken of the series solution during the numerical computation. As stated by Liao [12], it is seen that when $\hbar = -1$, $q = 1$ the result resonates with the result obtained by another mathematical tool Homotopy Perturbation Method [18, 19]. It is seen that for this particular case the result is similar to the result obtained by VIM (Das [15]). It is also seen for the table that for HAM the convergence of the values of $u(x, t)$ are found quite similar even for $q > 1$, only by controlling the values of auxiliary parameter $\hbar$. This conforms with the statement provided by Liao ([10], [12], [20]) that the auxiliary parameter $\hbar$ provides a simple way to ensure the convergence of the series solution given by HAM.

It is also seen from the table that $u(x, t)$ increases with the increase in t for every α and decreases with the increase of the fractional time derivative α, which is in complete agreement with the results predicted by Das [15] and Giona and Roman [21].

In Figs. 1 and 2, the physical quantity $u_t(x, t)$which is a function of auxiliary parameter $\hbar$ are plotted for different values of $\hbar$ for $\alpha = 0.75$, $\alpha = 1$ and $t = 0.3$, 0.7. Since $u_t(x, t)$ converges to the exact values for different values of $\hbar$, there exists horizontal line segments shown in the figures, which are usually called $\hbar$-curves and this shows the validity of the region of convergence of the series solution (21).

Figs. 1 and 2 represent the plots of the physical quantity $u_t(x, t)$ which is a function of auxiliary parameter $\hbar$, against $\hbar$ for $\alpha = 0.75$, $\alpha = 1$ and $t = 0.3$, 0.7. Since the convergent series of $u_t(x, t)$ for different values

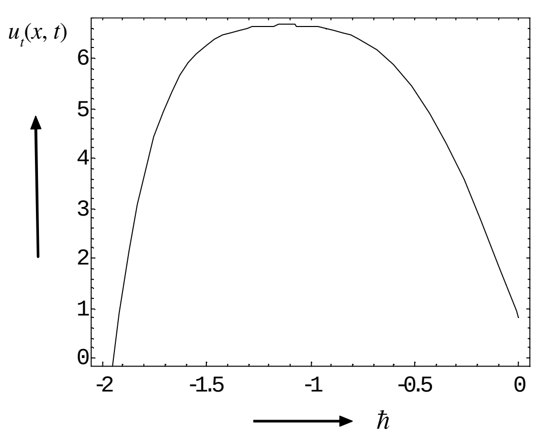

Figure 1(a): Plot of $u_t(x, t)$ vs. $\hbar$ at $\alpha = 0.75$ and $t = 0.3$

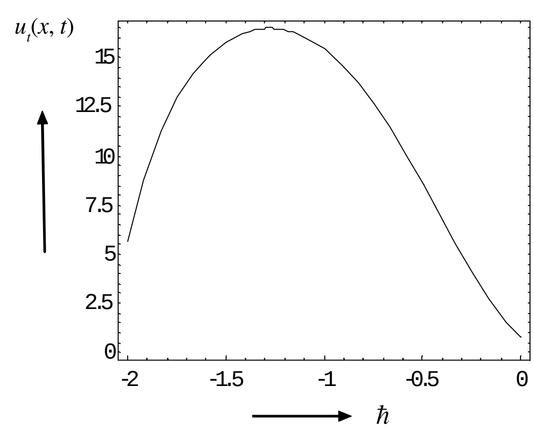

Figure 1(b): Plot of $u_t(x, t)$ vs. $\hbar$ at $\alpha = 0.75$ and $t = 0.7$

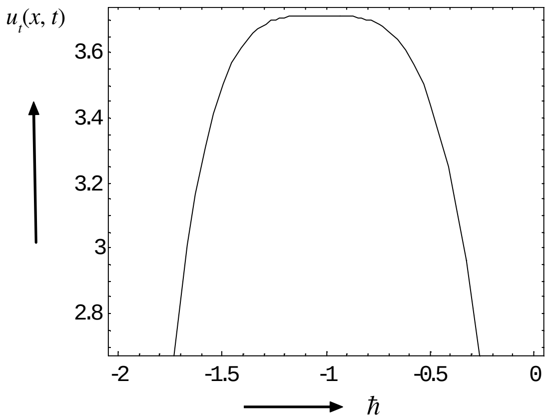

Figure 2(a): Plot of $u_t(x, t)$ vs. $\hbar$ at $\alpha = 1$ and $t = 0.3$

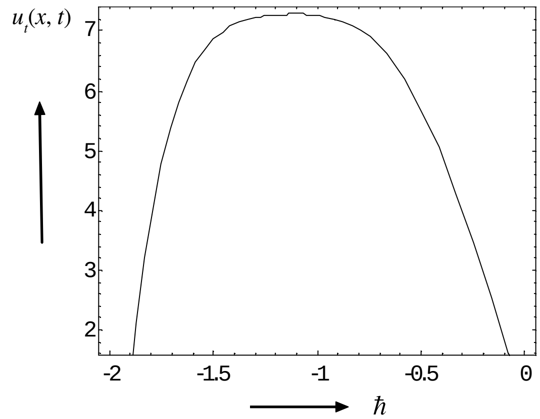

Figure 2(b): Plot of $u_t(x, t)$ vs. $\hbar$ at $\alpha = 1$ and $t = 0.7$

converges to the exact value $\hbar$, there exists horizontal line segments shown in the figures, which are usually called $\hbar$-curves, establishing the validity of the region of convergence of series solution (21).

It is also observed from the figures that for smaller time the region of the convergence will be better. It is also observed that in comparison to the standard motion, the convergence region will be less for fractional Brownian motion.

5. CONCLUSION

The main interest of this study is successful implementation of the powerful mathematical tool HAM to investigate the solution of the evolution equation and to show its efficacy in contrast to the other reliable mathematical tools like HPM and VIM. This method provides us a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. Thus it may be concluded that HAM is a simple and a powerful analytical approach for handling fractional related PDEs.

As in agreement with the previous work (Das [15]), the decrease of the values of with the increase of fractional time derivatives is observed. However faster computation procedure of the present method and the convergence criterion of the series solution with the proper choices of auxiliary and homotopy parameters render the article a different dimension and the authors strongly believe that the present article will be highly acceptable by the researchers working in the field of fractional calculus.

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