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DIRICHLET SERIES SOLUTION OF MHD FLOW OVER A NONLINEAR STRETCHING SHEET

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ABSTRACT: We study the MHD boundary layer flow of an incompressible viscous fluid over a continuously stretching sheet using more suggestive schemes. The fast convergent Dirichlet series solution of governing nonlinear differential equation of MHD flow over nonlinear stretching sheet is obtained. This method has advantages over pure numerical methods in obtaining the derived quantities accurately for various values of the parameters involved at a stretch and these are valid in much larger domain compared with the classical numerical schemes.

***Keywords*:** Magnetohydrodynamics (MHD) boundary layer equations, Stretching surface, Dirichlet series, Powell's method.

1. INTRODUCTION

We consider, the magnetohydrodynamics (MHD) boundary layer flows of an incompressible viscous fluid flow over a continuously stretching sheet which are of significant interest due to their applications. The flow past a stretching sheet is often encountered in many engineering and industrial process, such as aerodynamic extrusion of plastic sheets, hot rolling, glass fibre production etc. [1-3]. These have also applications in the polymer industry, when a polymer sheet is extruded continuously from a die, with a tacit assumption that the sheet is inextensible. The real situations is that one has to encounter the boundary layer flow over the stretching sheet, i.e. melt spinning process, the extradate is stretched into a filament or sheet while it is drawn from the die. Finally, the sheet solidifies while it passes through the controlled cooling system. Sakiadis [4, 5] was the first to analyze the boundary layers on a continuous semi-infinite sheets and cylindrical rods moving steadily in an otherwise quiescent environment. Subsequently several investigators have studied various aspects of the stretching flow problem. Amongst these, Chiam [6] investigated the MHD flow of a viscous fluid bounded by a stretching surface with power law velocity and presented numerical solution of the boundary value problem by using Runge-Kutta shooting algorithm with Newton iteration.

The third order nonlinear ordinary differential equations over an infinite interval with parameters β (stretching) and M (magnetic field) are of special interest and in very few specific cases they have analytical solutions. The flow of a viscoelastic fluid over a stretching sheet was investigated by Rajagopal *et al.*, [7], Sarpkaya [8] who probably the first to consider the MHD flow of non-Newtonian fluids. Anderson [9] and Mamaloukas *et al.*, [10] have obtained similarity solution of the boundary layer equation governing the flow of a viscoelastic and a second grade fluid past a stretching sheet in the presence of an external magnetic field.

The present investigation is to analyze the magnetohydrodynamic (MHD) flow caused by a sheet with nonlinear stretching. The solution of the resulting third order nonlinear boundary value problem with infinite interval is obtained by Dirichlet series method.

We seek solution of the general equation of the type

$$f''' + Aff'' + Bf'^2 + Cf' = 0 \tag{1.1}$$

with the boundary conditions

$$f(0) = \alpha_1, \qquad f'(0) = \beta_1, \qquad f'(\infty) = 0 \tag{1.2}$$

where A, B and C are constants and prime denotes derivative with respect to the independent variable η. This equation admits a Dirichlet series solution. Necessary conditions for the existence and uniqueness of these solutions may also be found in [11, 12]. For a specific type of boundary conditions i.e. $f'(\infty) = 0$, the Dirichlet series solution is particularly useful for obtaining the derived quantities. A general discussion of the convergence of the Dirichlet series may also be found in Riesz [13] . The accuracy as well as uniqueness of the solution can be confirmed using other powerful semi-numerical schemes. Sachdev *et al*., [14] have analyzed various problems from fluid dynamics of stretching sheet using this approach and found more accurate solution compared with earlier numerical findings. Hayat *et al*., [15] has used modified decomposition method and Pade' approximants, for the solution of equation similar to (1.1) arsing in MHD. Dirichlet series solution which we present here is more attractive than this method.

The present work is structured as follows. In section 2 the mathematical formulation of the proposed problem with relevant boundary conditions is given. Section 3 is devoted to semi-numerical method for the solution of the problem using Dirichlet series. In section 4 detailed results obtained by the novel method explained here are compared with the corresponding numerical schemes.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider the magnetohydrodynamic flow of an incompressible fluid over a stretching sheet at $y = 0$. The fluid is electrically conducting under the influence of an applied magnetic filed $B(x)$ normal to the stretching sheet. By neglecting induced magnetic field, the resulting two-dimensional boundary layer equations are of the form (Hayat *et al*., [15])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)}{\rho}u \tag{2.2}$$

where u and v are the velocity components in the x and y directions, υ is the kinematic viscosity, ρ is the fluid density and σ is the electrical conductivity of the fluid. To obtain similarity solutions, we assume that the external electrical and polarization effects are negligible in equation (2.2) and the magnetic field $B(x)$ is considered in the form (see Chaim [6])

$$B(x) = B_0 x^{\frac{(n-1)}{2}}$$

where B_0 is the constant magnetic field.

The relevant boundary conditions for the nonlinear stretching of a sheet are

$$\begin{aligned} &u(x, 0) = cx^n, \qquad v(x, 0) = 0, \\ &u(x, y) \to 0 \quad \text{as} \quad y \to \infty. \end{aligned} \tag{2.3}$$

Equations (2.1) and (2.2) along with the boundary conditions (2.3) admit similarity solution. We use following similarity variables

$$\eta = \sqrt{\frac{c(n+1)}{2\upsilon}}\, x^{\frac{(n-1)}{2}} y, \qquad u = cx^n f'(\eta),$$

$$v = \sqrt{\frac{c\upsilon(n+1)}{2}}\, x^{\frac{(n-1)}{2}} \left[f(\eta) + \frac{(n-1)}{(n+1)}\eta f'(\eta) \right] \tag{2.4}$$

and substituting them into equations (2.1)-(2.3) to obtain the following nonlinear ordinary differential equation

$$f''' + ff'' - \beta f'^2 - Mf' = 0' = \frac{d}{d\eta}\ , \tag{2.5}$$

and the boundary conditions are

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0 \tag{2.6}$$

where
$$\beta = \frac{2n}{1+n}, \qquad M = \frac{2\sigma B_0^2}{\rho c(n+1)}.$$

3. DIRICHLET SERIES APPROACH TO THE BOUNDARY VALUE PROBLEMS OVER AN INFINITE INTERVAL

We seek a Dirichlet series solution of equation (1.1) satisfying $f'(\infty) = 0$ in the form (Kravchenko and Yablonskii [11, 12])

$$f = \gamma_1 + \frac{6\gamma}{A}\sum_{i=1}^{\infty} b_i a^i e^{-i\gamma\eta}, \tag{3.1}$$

where γ and a are free parameters. Substituting (3.1) into (1.1), we get

$$\sum_{i=1}^{\infty}\{-\gamma^2 i^2 + A\gamma\gamma_1 i^2 - Ci\}\, b_i a^i e^{-iy\eta} + \frac{6\gamma^2}{A}\sum_{i=1}^{\infty}\sum_{k=1}^{i-1}\{Ak^2 + Bk(i-k)\}\, b_k b_{i-k} a^i e^{-iy\eta} = 0. \tag{3.2}$$

For $i = 1$, we have
$$\gamma_1 = \frac{\gamma^2 + C}{A}. \tag{3.3}$$

Substituting (3.3) into (3.2) the recurrence relation for obtaining coefficients is given by

$$b_i = \frac{6\gamma^2}{Ai(i-1)\{\gamma^2 i - C\}}\sum_{k=1}^{i-1}\{Ak^2 + Bk(i-k)\}\, b_k b_{i-k}. \tag{3.4}$$

for $i = 2, 3, 4, \ldots\ldots$. If the series (3.1) converges absolutely when $\gamma > 0$ for some η_0, this series converges absolutely and uniformly in the half plane $\operatorname{Re}\eta \geq \operatorname{Re}\eta_0$ and represents an analytic $\frac{2\pi i}{\gamma}$ periodic function $f = f(\eta_0)$ such that $f'(\infty) = 0$ (Kravchenko and Yablonskii [12]).

The series (3.1) contains two free parameters namely a and γ. These unknown parameters are determined from the remaining boundary conditions (1.2) at $\eta = 0$

$$f(0) = \frac{\gamma^2 + C}{A\gamma} + \frac{6\gamma}{A}\sum_{i=1}^{\infty} b_i a^i - \alpha_1 \tag{3.5}$$

and

$$f'(0) = \frac{6\gamma^2}{A}\sum_{i=1}^{\infty}(-i) b_i a^i - \beta_1. \tag{3.6}$$

The solution of these transcendental equations (3.5) and (3.6) yield constants a and γ. The solution of these transcendental equations is equivalent to the unconstrained minimization of the functional

$$\left[\frac{\gamma^2 + C}{A\gamma} + \frac{6\gamma}{A}\sum_{i=1}^{\infty} b_i a^i - \alpha_1\right]^2 + \left[\frac{6\gamma^2}{A}\sum_{i=1}^{\infty}(-i) b_i a^i - \beta_1\right]^2. \tag{3.7}$$

We use Powell's method of conjugate directions (Press *et al.*, [16]) which is one of the most efficient techniques for solving unconstrained optimization problems. This helps in finding the unknown constants a and γ uniquely for different values of the parameters A, B, C, α_1 and β_1. Alternatively, Newton's method is also used to determine the unknown parameters a and γ accurately.

The shear stress at the surface is given by

$$f''(0) = \frac{6\gamma}{A} \sum_{i=1}^{\infty} b_i a^i (i\gamma)^2 . \tag{3.8}$$

For the special case $\beta = 1$, the exact analytical solution of equation (2.5) with the boundary conditions (2.6) (Pavlov [17]) is

$$f(\eta) = \frac{1}{\sqrt{1+M}} (1 - e^{-\sqrt{1+M}\,\eta}), \tag{3.9}$$

$$f''(0) = -\sqrt{1+M} . \tag{3.10}$$

4. NUMERICAL RESULTS AND CONCLUSIONS

In the present paper, the MHD flows of an incompressible viscous fluid in the presence of variable magnetic field have been considered. We have given exact analytic solution of nonlinear boundary value problem (1.1) and (1.2) in the more general form of a Dirichlet series (3.1). In this problem, one of the boundary condition is $f'(0) = \beta_1$ in addition to $f'(\infty) = 0$ and $f(0) = \alpha_1$. These are the cases that frequently occur in physical applications. Equation (1.1) is quite general and includes many problems as special cases. The computation of the Dirichlet series obtained here is accomplished quite easily. The calculated values of $f''(0)$ representing the shear stress at the surface associated with different parameters β and M for different sets of values of a and γ are given in Tables 1 and 2.

Table 1
Comparison of the Values of for Positive Values Obtained by the Dirichlet Series Method and Modified Decomposition Method

		Dirichlet Series Method			Decomposition Method [15]
	M	*A*	γ	*f''(0)*	
$\beta = 1$	0	−1.00000	1.00000	−1.00000	−1.00000
	1.0	−0.08333	1.41421	−1.41421	−1.41421
	5.0	−0.027778	2.44949	−2.44942	−2.44948
	10.0	−0.01515	3.31663	−3.31673	−3.31662
	50.0	−0.00327	7.14143	−7.14149	−7.14142
	100.0	−0.00165	10.04987	−10.04885	−10.04987
	500.0	−0.00033	22.38303	−22.38518	−22.38302
	1000.0	−0.00017	31.63858	−31.63858	−31.63858
$\beta = 1.5$	0	–	–	–	−1.1547
	1.0	−0.07732	1.39725	−1.53356	−1.5252
	5.0	−0.02704	2.44653	−2.51798	−2.5161
	10.0	−0.01493	3.31546	−3.36764	−3.3663
	50.0	−0.00326	7.14131	−7.16401	−7.1647
	100.0	−0.00165	10.04983	−10.06371	−10.0776
	500.0	−0.00033	22.38303	−22.39335	−22.3904
	1000.0	−0.00017	31.63858	−31.64908	−31.6438
$\beta = 5.0$	0	–	–	–	−1.9098
	1.0	−0.05258	1.32276	−2.18558	−2.1528
	5.0	−0.02299	2.42959	−2.94773	−2.9414
	10.0	−0.01357	3.30816	−3.69955	−3.6956
	50.0	−0.00319	7.14053	−7.32606	−7.3256
	100.0	−0.00163	10.04955	−10.18185	−10.1816
	500.0	−0.00033	22.38299	−22.44255	−22.4425
	1000.0	−0.00017	31.63858	−31.68071	−31.6806

Table 2
Comparison of the Values of for Negative Values Obtained by the Dirichlet Series Method and the Modified Decomposition Method

	M	*Dirichlet Series Method* A	γ	$f''(0)$	*Decomposition Method [15]*
β = −1.0	0	–	–	–	0.0
	1.0	−0.12119	1.50923	−0.86235	−0.8511
	5.0	−0.03123	2.46292	−2.16011	−2.1628
	10.0	−0.01613	3.32161	−3.10959	−3.1100
	50.0	−0.00331	7.14189	−7.04751	−7.0475
	100.0	−0.00166	10.05014	−9.98339	−9.9833
	500.0	−0.00033	22.38304	−22.35314	−22.3532
	1000.0	−0.00017	31.63859	−31.61540	−31.6175
β = − 1.5	0	–	–	–	–
	1.0	−0.15025	1.56205	−0.50553	−0.6532
	5.0	−0.03229	2.46689	−2.08070	−2.0852
	10.0	−0.01639	3.32297	−3.05555	−3.0562
	50.0	−0.00332	7.14201	−7.02385	−7.0238
	100.0	−0.00166	10.05008	−9.96664	−9.9666
	500.0	−0.00033	22.38305	−22.34562	−22.3457
	1000.0	−0.00017	31.63859	−31.61223	−31.6122

The problem mentioned in (2.5) corresponding to (1.1) and (1.2) describes the MHD flow of a viscous fluid over a stretching sheet. For the specific values of *A*, *B* & *C* for which exact analytic solution of the problem subject to the boundary conditions (1.2) is given by Pavlov [17]. An excellent agreement between Dirichlet series and the direct numerical solution is found. The results obtained, for different values of β and *M* are given in Table 1 and 2, which also agree formally with the Pade' approximant based Adomain decomposition method (Hayat *et al*., [15]). The solution obtained here is valid for much larger values of β and *M*.

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