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# A NOTE ON A SINGLE SERVER QUEUEING MODEL WITH SINGLE AND GROUP SERVICES

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**ABSTRACT:** A single server queueing model having two types of services is considered in this paper with a restriction on the size of arriving customers to obtain the transient state, steady state probabilities, expected queue length, busy period distribution and the expected busy period and variance of busy period. Some examples are provided for different values of the parameters and levels.

## 1. INTRODUCTION

We consider a single server queueing model with single and group or lot service. Customers arrive according to a Poisson Process with parameter $\lambda$ and are served by a single server. The server serves the customers either one at a time(manual service) or in group (lot) according to the levels '$a$' and '$c$' for the group size (For example in beauty parlor or in such places like barber shop where a single service will be done to some customers and group or collected services will be done to the customers according to their demand. In particular, after a service completion epoch, if the system size ($n$) is less than or equal to the level '$c$' then the server serves the single customer according to FCFS rule and the service time is exponentially distributed with parameter $\mu_1$ and $n > c$, then he serves all the units in a lot or group where the service time is exponentially distributed with parameter $\mu_2$ independent of batch size. But once the group or lot service (machine mode) has been initiated it is changed to single service (manual mode) not when the group or lot size is '$c$' but at a level '$a$' below '$c$' . this will avoid the problem of never ending transfer from one type of service to the other.

Several authors considered Poisson arrival queues with single as well as group service. For example see Klienrock (1975), Chaudhry and Templeton (1984), Gross and Harris (1998), Bhat (2000) and Naoto (2007). There are many real life queueing situations in which services are rendered with certain limit or level criterion. For example it may be possible to process jobs manually or by machine. When the number of jobs to be processed is not more than a fixed number it will be advantageous to do them manually and if the number of jobs exceeds that fixed number, processing by machine is turns out to be economical.

A single and lot service with single control limit '$c$' on the group service are found to be vital in many real life situations, but when the level '$c$' is very large the model suffers from the risk of never ending change-over from group or lot service (machine ) to single service (manual). This problem may be overcome by the introduction of a level '$a$' ($< c$) called the secondary level, so that when a group or lot service has been on, the server may continue the group service even with a size less than '$c$' but up to the limit $a + 1$ and when the number of customers drops below $a + 1$ manual service commences (See Revus (1975), Kingman (1993), Atencia and Moreno (2000), Kemeny (2001) and Boxma *et al*., (2004)).

A single and group service queueing model has been considered in this paper with a secondary level on lot or group size and obtain the transient state and steady state probabilities, expected queue length, busy period distribution and hence expected busy period and variance of busy period. A numerical illustration is given in the last section.

## 2. MARKOVIAN MODEL AND ITS ANALYSIS

Let $X(t)$ denote the number of customers in the waiting line at time $t$ and $Y(t)$ denote the type of service given by the server at time $t$. Specifically $Y(t) = 0, 1, 2$, according as the server is idle busy with a simple service or busy with a batch. Then $\{Y(t), X(t)\}$ constitute a $MC$ with state space $S = S_1 \cup S_2 \cup S_3$

where $S_1 = \{(0, 0)\}$,

$S_2 = \{(1, n); n = 0, 1, 2, ..., c-1\}$

and $S_3 = \{(2, n); n \geq 0\}$.

We shall denote the probability of the occurrence of two events $[Y(t) = i]$ and $[X(t) = j]$ by $P(i, j, t)$. Thus, $P(i, j, t) = P\{[Y(t) = i]) \cap [X(t) = j]\}$. Then, the transient distribution of the system $P(i, j, t)$, then satisfy the following system of difference differential equations.

$$P'(0, 0, t) = -\lambda P(0, 0, t) + \mu_1 P(1, 0, t) + \mu_2 P(2, 0, t) \tag{2.1}$$

$$P'(1, 0, t) = -(\lambda + \mu_1)P(1, 0, t) + \lambda P(0, 0, \mathrm{t}) + \mu_1 P(1, 1, t) + \mu_2 P(2, 1, t) \tag{2.2}$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n-1, t) + \mu_1 P(1, n+1, t) + \mu_2 P(2, n+1, t); \quad \text{where} \quad 1 \leq n \leq (a-1) \tag{2.3}$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n-1, t) + \mu_1 P(1, n+1, t) \text{ where } a \leq n \leq c-2 \tag{2.4}$$

$$P'(1, c-1, t) = (-\lambda + \mu_1)P(1, c-1, t) + \lambda P(1, c-2, t) \tag{2.5}$$

$$P'(2, 0, t) = (-\lambda + \mu_2)P(2, 0, t) + \lambda P(1, c-1, t) + \mathrm{m}_2 \sum_{n=a+1}^{\infty} P(2, n, t) \tag{2.6}$$

$$P'(2, n, t) = (-\lambda + \mu_2)P(2, n, t) + \lambda P(2, n-1, t); \quad n > 0 \tag{2.7}$$

Let $P^*(i, j, s)$ denote the Laplace transform of $P(i, j, t)$. We shall assume that $P(0, 0, 0) = 1$. Hence, we have

$$(s + \lambda) P^*(0, 0, s) - 1 = \mu_1 P^*(1, 0, s) + \mu_2 P^*(2, 0, s) \tag{2.8}$$

$$(s + \lambda + \mu_1) P^*(1, 0, s) = \lambda P^*(0, 0, s) + \mu_1 P^*(1, 1, s) + \mu_2 P^*(2, 1, s) \tag{2.9}$$

$$(s + \lambda + \mu_1) P^*(1, \mathrm{n}, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s) + \mu_2 P^*(2, n+1, s); \quad \text{where} \quad 1 \leq n \leq (a-1) \tag{2.10}$$

$$(s + \lambda + \mu_1) P^*(1, n, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s) \text{ where a £ n £ (c-2)} \tag{2.11}$$

$$(s + \lambda + \mu_1) P^*(1, c-1, s) = \lambda P^*(1, c-2, s) \tag{2.12}$$

$$(s + \lambda + \mu_2) P^*(2, 0, s) = \lambda P^*(1, c-1, s) + \mu_2 \sum_{n=a+1}^{\infty} P(2, n, s) \tag{2.13}$$

$$(s + \lambda + \mu_2) P^*(2, n, s) = \lambda P^*(1, n-1, s); \; n > 0. \tag{2.14}$$

Solving (2.14) as a difference equation in $P^*(2, n, s)$ to get,

$$P^*(2, n, s) = A(s) R^n \text{ where } n \geq 0 \tag{2.15}$$

$$A(s) = P^*(2, 0, s) \quad \text{and} \quad R = \lambda\,(s + \lambda + , \mu_2)^{-1}$$

From (2.13) ,

$$P^*(1, c-1, s) = \frac{A(s)}{\lambda}\left( a + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1-R} \right). \tag{2.16}$$

From ( 2.11),

$$P^*(1, n, s) = P^*(1, a, s)\, \theta^{n-a} \quad \text{where} \quad a \geq n \geq (c-2) \tag{2.17}$$

where $\theta$ is the positive root less than unity of the equation

$$\mu_1 z^2 = (s + \lambda + \mu_1) z + \lambda = 0.$$

Hence, by using (2.12) we get,

$$P^*(1, a, s) = \frac{A(s)}{\lambda^2 \theta^{c-2a}}\left( s + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1-R} \right)$$

$$= A(s)\ \mathrm{K}_1(\mathrm{s}).\ \text{(Say)} \tag{2.18}$$

From (2.10),

$$P^*(1, n, s) = P^*(1, 0, s)\,\theta^n - \mu_2\,[A(s)\,R^{n+1}/K(R)]; \quad 1 \le n \le a-1 \tag{2.19}$$

where $K(z) = \mu z^2 - (s + \lambda + \mu_1)\,z + \lambda.$

Thus, we have

$$P^*(1, 0, s) = A(s)\frac{1}{\theta^{a-1}}\left(K_1(s)\left(\frac{s+\lambda+\mu_1}{\lambda} - \mu_1\theta\right) + \frac{\mu_2 R^a}{K(R)}\right)$$

$$= A(s)\,K_2(s). \quad \text{(Say)} \tag{2.20}$$

$$P^*(0, 0, s) = A(s)\,(s + \lambda)^{-1}\,(\mu_1 K_2(s) + \mu_2) + (s + \lambda)^{-1}. \tag{2.21}$$

Now, using the condition,

$$P^*(0,0,s) + \sum_{n=0}^{c-1} P^*(1,n,s) + \sum_{n=0}^{\infty} P^*(2,n,s) = s^{-1}.$$

we get,

$$\text{A(s)} = \frac{\lambda}{s(s+\lambda)}\left[\begin{array}{l} K_2(s)\left[\dfrac{\mu_1}{s+\lambda} + \dfrac{1-\theta^{\text{a}}}{1-\theta}\right] + K_1(s)\left[\dfrac{1-\theta^{c-a-1}}{1-\theta}\right] \\ +\mu_2\left[\dfrac{1}{\text{s}+\lambda} - \dfrac{R^2(1-R^{a-1})}{K(R)(1-R)} - \dfrac{R^{a+1}}{\lambda(1-R)}\right] + \dfrac{s+\lambda+\mu_2}{\lambda} + \dfrac{1}{1-R} \end{array}\right]^{-1}. \tag{2.22}$$

## 3. STEADY STATE DISTRIBUTION, AVERAGE QUEUE LENGTH AND BUSY PERIOD DISTRIBUTION

Appeal to final value theorem on Laplace transforms the steady state distribution of the queue size can be obtained as

$$P(i, j) = \lim_{t \to \infty} P(i, j, t) = \lim_{s \to \infty} P^*(i, j, s).$$

Hence from (2.15) to (2.22) the steady state distribution can be obtained as

$$P(0, 0) = B\lambda^{-1}[\mu_1 L_2 + \mu_2] \tag{3.1}$$

$$P(1, 0) = BL_2 \tag{3.2}$$

$$P(1, n) = \left[L_2\theta_1^n - \mu_2\frac{r^{n+1}}{K'(r)}\right]; \quad 1 \le n \le a-1 \tag{3.3}$$

$$P(1, n) = BL_1\theta_1^{n-a}; \quad a \le n \le c-1 \tag{3.4}$$

$$P(1, c-1) = B\frac{1}{\lambda}\left(\lambda + \mu_2 - \mu_2\frac{r^{a+1}}{1-r}\right) \tag{3.5}$$

$$P(2, n) = Br^n; \quad n \ge 0 \tag{3.6}$$

where

$$L_1 = \lim_{s \to \infty} K_1(s) = \frac{1}{\theta_1^{c-2-a^2}\lambda}\left(\lambda + \mu_2 - \mu_2\frac{r^{a+1}}{1-r}\right)$$

$$L_2 = \lim_{s \to \infty} K_2(s) = \frac{1}{\theta_1^{a-1}}\left[L_1\left(\frac{\lambda+\mu_1}{\lambda} - \mu_1\theta_1\right) - \mu_2\frac{r^a}{K'(r)}\right]$$

$$B = \lim_{s \to \infty} sA(s) = \begin{bmatrix} L_2\left[\dfrac{\mu_1}{\lambda} + \dfrac{1-\theta^a}{1-\theta}\right] + L_1(s)\left[\dfrac{1-\theta^{c-a-1}}{1-\theta}\right] \\ + \mu_2\left[\dfrac{1}{\lambda} - \dfrac{r^2(1-r^{a-1})}{K(r)(1-r)} - \dfrac{r^{a+1}}{\lambda(1-r)}\right] + \dfrac{\lambda+\mu_2}{\lambda} + \dfrac{1}{1-r} \end{bmatrix}^{-1}$$

where $K'(r) = \mu_1 r^2 - (\lambda + \mu_1) r + \lambda$,

$$\theta_1 = \lambda/\mu_1 \quad \text{and} \quad r = \lim_{s \to \infty} R = \lambda/(\lambda + \mu_2).$$

We now proceed to obtain the expected queue length.

The expected queue length $L_q$ given by

$$\begin{aligned} L_q &= \sum_{n=1}^{c-1} nP(1, n) + \sum_{n=1}^{\infty} nP(2, n) \\ &= \sum_{n=1}^{a-1} n.B\left[L_2\theta_1^n - \mu_2 \frac{r^{n+1}}{K'(r)}\right] + \sum_{n=a}^{c-2} n.B.L_1\theta_1^{n-a} \\ &\quad + (c-1)B\frac{1}{\lambda}\left[\lambda + \mu_2 - \mu_2\frac{r^{a+1}}{1-r}\right] + \sum_{n=1}^{\infty} nBr^n \\ &= B\begin{Bmatrix} L_2\theta_1[(1-\theta_1)^{-2}(1-\theta_1^{a-1}) - (a-1)\theta_1^{a-1}(1-\theta_1)^{-1} \\ + \dfrac{L_1}{\theta_1^{a-1}}[(1-\theta_1)^{-2}(\theta_1^{a-1} - \theta_1^{c-2}) + (1-\theta_1)^{-1}[(a-1)\theta_1^{a-1}(c-2)\theta_1^{c-2} \\ - \dfrac{\mu_2}{K'(r)} r^2\,[(1-r)^{-2}(1-r^{a-1}) - (a-1)r^{a-1}(1-r)^{-1}] \\ + \dfrac{c-1}{\lambda}\left[\lambda + \mu_2 - \mu_2\dfrac{r^{a+1}}{1-r}\right] + r(1-r)^{-2} \end{Bmatrix}, \end{aligned} \tag{3.7}$$

We now proceed to obtain the busy period distribution. In this model the server is idle only when the system is empty. Thus, the busy period $T$ commences with the arrival of a unit to the empty system and lasts till system is empty again.

Let $\quad b(t) = P\{t \le T < t + dt, X(t + dt) = 0\}$.

Then, $\quad b(t) = P(0, 0, t)$.

Hence, the Laplace transform $b^*(s)$ is given by

$$b^*(s) = s\,.\,P^*(0, 0, s).$$

In order to find the distribution of the busy period we consider the system avoiding the zero state. Here we assume that $P(1, 0, 0) = 1$.

Hence we get the following difference differential equations.

$$P'(0, 0, t) = -\mu_1 P(1, 0, t) + \mu_2 P(2, 0, t) \tag{3.8}$$

$$P'(1, 0, t) = -\mu_1 P(1, 0, t) + \mu_1 P(1, 1, t) + \mu_2 P(2, 1, t) \tag{3.9}$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n-1, t) + \mu_1 P(1, n+1, t) + \mu_2 P(2, n+1, t); \quad \text{where} \quad 1 \le n \le (a-1) \tag{3.10}$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n-1, t) + \mu_1 P(1, n+1, t) \quad \text{where} \quad a \le n \le (c-2) \tag{3.11}$$

$$P'(1, c-1, t) = -(\lambda + \mu_1) P(1, c-1, t) + \lambda P(1, c-2, t) \quad (3.12)$$

$$P'(2, 0, t) = -(\lambda + \mu_2) P(2, 0, \mathrm{t}) + \lambda P(1, c-1, t) + \mu_2 \quad (3.13)$$

$$P'(2, n, t) = -(\lambda + \mu_2) P(2, n, t) + \lambda P(2, n-1, t) \quad \text{where} \quad n > 0. \quad (3.14)$$

Taking Laplace Transform on both sides of (3.1) to (3.7) we have the following set of equations.

$$sP^*(0, 0, s) = \mu_1 P^*(1, 0, s) + \mu_2 P^*(2, 0, s) \quad (3.15)$$

$$(s + \mu_1) P^*(1, 0, s) - 1 = + \mu_1 P^*(1, 1, s) + \mu_2 P^*(2, 1, s) \quad (3.16)$$

$$(s + \lambda + \mu_1) P^*(1, n, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s) + \mu_2 P^*(2, n+1, s);$$

$$\text{where } 1 \le n \le (a-1) \quad (3.17)$$

$$(s + \lambda + \mu_1) P^*(1, n, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s); \quad a \le n \le c-2 \quad (3.18)$$

$$(s + \lambda + \mu_1) P^*(1, c-1, s) = \lambda P^*(1, c-2, s) \quad (3.19)$$

$$(s + \lambda + \mu_2) P^*(2, 0, s) = \lambda P^*(1, c-1, s) + \mu_2 \quad (3.20)$$

$$(s + \lambda + \mu_2) P^*(2, n, s) = \lambda P^*(1, n-1, s) \quad \text{where} \quad n > 0. \quad (3.21)$$

Solving (3.21) as a difference equation in $P^*(2, n, s)$ we get,

$$P^*(2, n, s) = A_1(s) R^n; \quad n \ge 0 \quad (3.22)$$

where $\quad A_1(s) = P^*(2, 0, s) \quad$ and $\quad R = 1(s + \lambda + \mu_2)^{-1}$

From (3.20),

$$P^*(1, c-1, s) = \frac{A_1(s)}{\lambda}\left(s + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1-R}\right). \quad (3.23)$$

From ( 3.18),

$$P^*(1, n, s) = P^*(1, a, s)\, \theta^{n-a} \quad \text{where} \quad a \le n \le (c-2) \quad (3.24)$$

where $\theta$ is the positive root less than unity of the equation

$$\mu_1 z^2 = (s + \lambda + \mu_1) z + \lambda = 0.$$

Hence, by using (3.19) we get,

$$P^*(1, a, s) = \frac{A_1(s)}{\lambda^2 \theta^{c-2a}}\left(s + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1-R}\right)$$

$$= A_1(s) K_1(s). \quad \text{(Say)} \quad (3.25)$$

From (3.16),

$$P^*(1, n, s) = \mathrm{P}^*(1, 0, s)\, \theta^n - \mu_2 [A_1(s) R^{n+1}/K(R)]; \quad 1 \le n \le a-1 \quad (3.26)$$

where $\quad K(z) = \mu z^2 - (s + \lambda + \mu_1) z + \lambda.$

Therefore,

$$P^*(1, 0, s) = A_1(s) \frac{1}{\theta^{a-1}}\left(K_1(s)\left(\frac{s + \lambda + \mu_1}{\lambda} - \mu_1 \theta\right) + \frac{\mu_2 R^a}{K(R)}\right)$$

$$= A_1(s) K_2(s). \quad \text{(Say)} \quad (3.27)$$

From (3.15),

$$(s + \mu_1) A_1(\mathrm{s}) K_2(s) - 1 = -\mu_1 A_1(s) K_2(s)\, \theta + \mu_2 A_1(s) [R^2/K(R)] + \mu_2 A_1(s) R$$

$$A_1(\mathrm{s}) = [(s + \mu_1) K_2(s) + \mu_1 K_2(s)\, \theta - \mu_2 [R^2/K(R)] - \mu_2 R]^{-1}. \quad (3.28)$$

From (3.14)

$$sP^*(0, 0, s) = \mu_1 A_1(s) K_2(s)\, \theta + \mu_2 A_1(s)$$

$$= A_1(s) [\mu_1 K_2(s)\, \theta + \mu_2].$$

Thus, the Laplace transform $b^*(s)$ of $b(t)$ is given by

$$b^*(s) = A_1(s)\,[\mu_1 K_2(s)\,\theta + \mu_2]. \tag{3.29}$$

From the above expression one may compute the expectation and variance of the busy period using the following formula.

$$E(T) = \lim_{s \to \infty}\left[-\frac{d}{ds}[b^*(s)]\right]$$

$$V(T) = \lim_{s \to \infty}\left[-\frac{d^2}{ds}[b^*(s)]\right] - [E(T)]^2.$$

The expected busy period in the steady state is obtained as follows:In this model the busy periods $T$ and idle periods $I$ alternate and form a busy cycle. From the theory of renewal process we have

$$\begin{aligned} P(0, 0) &= \lim_{t \to \infty} P[X(t) = 0,\ Y(t) = 0] \\ &= E(I)\,[E(I) + E(T)]^{-1} \\ &= (\lambda - B[\mu_1 L_2 + \mu_2])\,(\lambda B[\mu_1 L_2 + \mu_2])^{-1}. \end{aligned} \tag{3.30}$$

## 4. OBSERVATIONS AND REMARKS

For different values of $\lambda$, $\mu_1$, $\mu_2$, $a$ and $c$. The steady state probabilities $\{P(i; n), i = 0, 1, 2, ...\}$ are calculated by using the equations (3.1.) to (3.6.) on a computer for $P(0, 0)$. For the same values of $\lambda$, $\mu_1$, $\mu_2$ but with different values of $a$ and $c$ we may obtain the expected busy periods in steady state by using (2.5). It is observed that that the expected busy period is monotonically increasing, but the rate of increase is very small for larger values of $a$ and $c$. It is remarked here that the average queue length also monotonically increasing, but the rate of increase is very small for larger values of $a$ and $c$.

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