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EQUIVALENCES IN MAX-MIN DETERMINISTIC GENERAL FUZZY AUTOMATA OF ORDER *n*

ABSTRACT: *In this paper, we define the notions of max-min deterministic general fuzzy automata of order* n *of a max-min general fuzzy automaton, the overall transition function of a max-min deterministic general fuzzy automaton of order* n*, the initialized max-min deterministic general fuzzy automaton of a max-min general fuzzy automaton of order* n *and the response number of an initialized max-min deterministic general fuzzy automaton. Then by using these notions, three types of equivalence relations are considered, namely, statewise, compositewise, distributionwise equivalence. We show that the last two are equivalent. Finally, we define the notions of the max-min (statewise irreducible, compositewise irreducible, distributionwise irreducible, effective, statewise minimal, compositewise minimal) deterministic general fuzzy automaton of order* n *of a max-min general fuzzy automaton and find the relationship between them.*

Keywords: *(General) Fuzzy automata; Equivalence; Irreducibility; Convex max-min combinations; Set of vertices; Basis*

1. INTRODUCTION AND PRELIMINARIES

The theory of fuzzy sets was introduced by Zadeh [9]. Wee [8] introduced the idea of fuzzy automata. Automata have a long history both in theory and application [1, 2]. Automata are the prime example of general computational systems over discrete spaces [4].

A fuzzy finite-state automaton (FFA) is a six-tuple denoted as $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval [0, 1] and $\omega : Q \to Z$ is the output function. In an FFA, as can be seen, associated with each fuzzy transition, there is a membership value in [0, 1]. We call this membership value the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted as $\delta(q_i, a_k, q_j)$. We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition. Otherwise, it specifies the transition itself. Also, the set of all transitions of $\tilde{F}$ is denoted as Δ.

The above definition is generally accepted as a formal definition for FFA [5,6,7]. There is the important problem which should be clarified in the definition of FFA. It is the assignment of membership values to the next states. There are two issues within state membership assignment. The first one is how to assign *a* membership value to a next state upon the completion of a transition. Secondly, how should we deal with the cases where *a* state is forced to take several membership values simultaneously via overlapping transitions?

In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and gave the notion of general fuzzy automata [3]. Now, we follow [3] and give some new notions and results as mentioned in the abstract.

Let X be a set. A word of X is the product of a finite sequence of elements in X, Λ is empty word and X^* is the set of all words on X. In fact, X^* is the free monoid on X. The length $\ell(x)$ of word $x \in X^*$ is the number of its letters; so $\ell(\Lambda) = 0$. For a nonempty set X, $\tilde{P}(X)$ denoted the set of all fuzzy sets on X and $P(X)$ denoted the set of all subsets on X.

Definition 1.1: [3] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted as $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,

(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,

(iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subset \tilde{P}(Q)$,

(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,

(v) $\omega : Q \to Z$ is the output function,

(vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$ is the augmented transition function,

(vii) $F_1 : [0, 1]) \times [0, 1] \to [0, 1]$ is called membership assignment function.

Function $F_1(\mu, \delta)$ as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition.

In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

Which means that the membership value (mv) of the state q_j at time $t + 1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

There are many options which can be used for the function $F_1(\mu, \delta)$, for example $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ or $(\mu + \delta)/2$.

(viii) $F_2 : [0, 1]^* \to [0, 1]$ is called multi-membership resolution function.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

We let $Q_{act}(t_i)$ be the set of all active states at time t_i, $\forall_i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$,

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall_i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$.

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm 1.2: [3] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t + 1$, the following algorithm will assign a unified membership value to that:

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call

this v_i,

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta\,(q_i, a_k, q_j)).$$

(2) These membership value's (υ_i's) are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j ,

$$\mu^{t+1}(q_j) = F_2{}_{i=1}^{n}[v_i] = F_2{}_{i=1}^{n}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

Where

- n : is the number of simultaneous transitions to the active state q_j at time $t + 1$.
- $\delta(q_i, a_k, q_j)$: is the weight of the transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$: is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t + 1$.

Definition 1.3: Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton, which is defined in Definition 1.1. We define max-min general fuzzy automata of the form:

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

such that :

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0, 1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \ldots\}$ and let for every i, $i \geq 0$

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & \text{otherwise} \end{cases}$$

and for every i, $i \geq 1$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p),$$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p)),$$

and recursively

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1u_2 \ldots u_n, p) = \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \ldots \wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \ldots, p_{n-1} \in Q_{act}(t_{n-1})\},$$

in which $u_i \in \Sigma$, $\forall 1 \le i \le n$ and assuming that the entered input at time t_i be u_i, $\forall 1 \le i \le n-1$.

Definition 1.4: [5] Let $A = [a_{ij}]$ be an $n \times p$ matrix and $B = [b_{ij}]$ be a $p \times m$ matrix of nonnegative real numbers. Let $A \otimes B$ be the $n \times m$ matrix $[c_{ij}]$, where

$$c_{ij} = \vee\{a_{ik} \wedge b_{kj} : k = 1, 2, \ldots, p\}.$$

Note: Let A be a matrix. Then $\rho(A)$ denotes the set of distinct rows of A. In the rest of this section, X and Y denote collections of sequences of real numbers.

Definition 1.5: [5] (i) Let $X = \{x_1, x_2, \ldots, x_n\}$. A max-min combination of X is an expression of the form

$$\bigvee_{i=1}^{n} (a_i \wedge x_i), \tag{1}$$

where a_i is a nonnegative real number, $i = 1, 2, \ldots, n$. If $0 \le a_i \le 1$, for $i = 1, 2, \ldots, n$, then (1) is called a convex max-min combination of X.

(ii) The (convex) max-min span of X is the collection of all (convex) max-min combinations of finite subsets of X. Let $C(X)$ denote the convex max-min span of X.

(iii) Y is called a convex max-min set if for every $y_1, y_2 \in Y$, all convex max-min combinations of $\{y_1, y_2\}$ are also in Y.

(iv) Let $x \in X$ and T_x be the set of all distinct terms of x. Then x is called admissible if T_x is finite and T_x can be effectively constructed from x. X is called admissible if every x in X is admissible.

Theorem 1.6: [5] (i) $X \subseteq C(X)$,

(ii) If $X_1 \subseteq X_2$, then $C(X_1) \subseteq C(X_2)$,

(iii) $C(C(X)) = C(X)$.

Definition 1.7: [5] Let Y be a convex max-min set and $X \subseteq Y$.

(i) X is called a set of generators of Y if $Y = C(X)$,

(ii) If X does not contain any proper subset which is itself a set of generators of Y, then X is called a set of vertices of Y.

Theorem 1.8: [5] Let $X \subseteq Y$. Then X is a set of vertices of Y if and only if

(i) $Y = C(X)$ and

(ii) If $x \in X$, then $x \notin C(X \backslash \{x\})$.

Definition 1.9: [5] Let $X_1, X_2 \subseteq Y$. Then X_1 is called a basis of X_2 if every $x \in X_2$ can be expressed uniquely as a convex max-min combination of a unique finite subset of X_1.

Definition 1.10: [5] Let Y be a convex max-min set. Y is called finitary if it contains a set of generators that is finite.

Theorem 1.11: [5] Let A be a matrix. Then $C(\rho(A))$ is admissible and finitary.

Theorem 1.12: [5] Let Y be a convex max-min set, admissible and finitary and let X_1, X_2 be sets of generators of Y. If $|X_1| > |X_2|$, then there exists $x \in X_1$ such that $x \in C(X_1 \backslash \{x\})$.

2. EQUIVALENCES IN MAX-MIN DETERMINISTIC GENERAL FUZZY AUTOMATA OF ORDER *n*

Definition 2.1: A max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n is a nine-tuple $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$, where $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ is a max-min general fuzzy automaton, $w = u_1 u_2 u_3 \ldots u_n$ is a fixed element of Σ^*, $\ell(w) = n$ and assuming that the entered input at time t_i be u_i, for every $i = 1, 2, \ldots, n$. The overall transition function $q^{\tilde{F}_n^*}$ of $\tilde{F}_n^*$ is a function from $Q_{actn} \times \Sigma'$ into [0, 1] defined as follows:

$$q^{\tilde{F}_n^*}((q, \mu^{t_{i-1}}(q)), u_i u_{i+1} \ldots u_n) = \bigvee_{q' \in Q_{act}(t_n)} \tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1} \ldots u_n, q')$$

for every $i = 1, 2, \ldots, n$, $q \in Q_{act}(t_{i-1})$, where $\Sigma' = \{u_i u_{i+1} \ldots u_n : i = 1, 2, \ldots, n\}$ and $Q_{actn} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \ldots, Q_{act}(t_{n-1})\}$.

Furthermore, let $Q^{\tilde{F}_n^*}(u_1 u_2 \ldots u_n) = [a_i]_{n \times 1}$ be the column matrix, where a_i the i-th row is defined by

$$a_i = \bigvee_{q \in Q_{act}(t_{i-1})} q^{\tilde{F}^*_n}((q, \mu^{t_{i-1}}(q)), u_i u_{i+1} \ldots u_n), \; i = 1, 2, \ldots, n.$$

Example 2.2: Consider the following GFA with several transition overlaps. It is specified as: $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where $Q = \{q_0, q_1, q_2, q_3, q_4\}$ is the set of states, $\Sigma = \{a, b\}$ is the set of input symbols, $\tilde{R} = \{(q_0, 1)\}$, $Z = \emptyset$, ω is not applicable and

$$\delta(q_0, a, q_1) = 0.4, \delta(q_0, a, q_4) = 0.5, \delta(q_0, b, q_3) = 0.3,$$
$$\delta(q_1, a, q_2) = 0.8, \delta(q_1, a, q_4) = 0.35, \delta(q_1, b, q_3) = 0.3,$$
$$\delta(q_2, a, q_1) = 0.75, \delta(q_2, b, q_2) = 0.6, \delta(q_2, a, q_3) = 0.2,$$
$$\delta(q_2, b, q_3) = 0.45, \delta(q_3, a, q_1) = 0.4, \delta(q_3, b, q_4) = 0.9,$$
$$\delta(q_4, a, q_1) = 0.4, \delta(q_4, b, q_2) = 0.1, \delta(q_4, b, q_3) = 0.7,$$

and

$$^1F_1(\mu, \delta) = \delta, F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^2F_1(\mu, \delta) = \min(\mu, \delta), F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^3F_1(\mu, \delta) = \min(\mu, \delta), F_2() = \mu^{t+1}(q_m) = \bigvee_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^4F_1(\mu, \delta) = \max(\mu, \delta), F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^5F_1(\mu, \delta) = \max(\mu, \delta), F_2() = \mu^{t+1}(q_m) = \bigvee_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^6F_1(\mu, \delta) = \min(\mu, \delta), F_2() = \mu^{i+1}(q_m) = \sum_{i=1}^{n} F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))/n,$$

$$^7F_1(\mu, \delta) = \frac{\mu + \delta}{2}, F_2() = \mu^{t+1}(q_m) = \bigvee_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

where n is the number of simultaneous transitions to the active state q_m at time $t + 1$.

If we choose $^1F_1(\mu, \delta) = \delta$, $F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$, then we have :

$\mu^{t_0}(q_0) = 1$, $\mu^{t_1}(q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = F_1(1, 0.4) = 0.4$,
$\mu^{t_1}(q_4) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_4)) = F_1(1, 0.5) = 0.5$,
$\mu^{t_2}(q_1) = F_1(\mu^{t_1}(q_4), \delta(q_4, a, q_1)) = F_1(0.5, 0.4) = 0.4$,
$\mu^{t_2}(q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = F_1(0.4, 0.8) = 0.8$,
$\mu^{t_2}(q_4) = F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_4)) = F_1(0.4, 0.35) = 0.35$,
$\mu^{t_3}(q_2) = F_1(\mu^{t_2}(q_4), \delta(q_4, b, q_2)) \wedge F_1(\mu^{t_2}(q_2), \delta(q_2, b, q_2))$
$= F_1(0.4, 0.1) \wedge F_1(0.8, 0.6) = 0.1 \wedge 0.6 = 0.1$,
$\mu^{t_3}(q_3) = F_1(\mu^{t_2}(q_1), \delta(q_1, b, q_3)) \wedge F_1(\mu^{t_2}(q_2), \delta(q_2, b, q_3)) \wedge F_1(\mu^{t_2}(q_4), \delta(q_4, b, q_3))$
$= F_1(0.4, 0.3) \wedge F_1(0.8, 0.45) \wedge F_1(0.35, 0.7) = 0.3 \wedge 0.45 \wedge 0.7 = 0.3$,

which there are two simultaneous transitions to the active state q_2 at time t_3 and there are three simultaneous transitions to the active state q_3 at time t_3. So, we can draw the following table:

Table 1

Active States and Their Membership Values (mv) at Different Times in Example 2.2

time	t_0	t_1		t_2			t_3		t_4		
input	Λ	a		a			b		b		
$Q_{act}(t_i)$	q_0	q_1	q_4	q_1	q_2	q_4	q_2	q_3	q_2	q_3	q_4
mv^1	1.0	0.4	0.5	0.4	0.8	0.35	0.1	0.3	0.6	0.45	0.9
mv^2	1.0	0.4	0.5	0.4	0.4	0.35	0.1	0.3	0.1	0.1	0.3
mv^3	1.0	0.4	0.5	0.4	0.4	0.35	0.4	0.4	0.4	0.4	0.4
mv^4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^6	1.0	0.4	0.5	0.4	0.4	0.35	0.25	0.35	0.25	0.25	0.35
mv^7	1.0	0.7	0.75	0.575	0.75	0.525	0.763	0.613	0.682	0.607	0.756

The operation of this fuzzy automaton upon input string a^2b^2 is shown in Table 1 for different membership assignment functions and multi-membership resolution strategies. In this table, we have considered different cases for combining functions F_1 and F_2.

Now, we consider the max-min deterministic general fuzzy automaton

$$\tilde{F}_2^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, aa)$$

of $\tilde{F}^*$ of order 2, where ${}^1F_1(\mu, \delta) = \delta$, $F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$.

Then we have $Q^{\tilde{F}_2^*}(aa) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that

$$a_1 = \bigvee_{q \in Q_{act}(t_0)} q^{\tilde{F}_2^*}((q, \mu^{t_0}(q)), aa) = q^{\tilde{F}_2^*}((q_0, \mu^{t_0}(q_0)), aa)$$

$$= \bigvee_{q' \in Q_{act}(t_2)} \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q')$$

$$= \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q_1) \wedge \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q_2) \vee \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q_4)$$

$$= ([\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1) \wedge \tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_1)] \vee [\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_4)$$
$$\wedge \tilde{\delta}((q_4, \mu^{t_1}(q_4)), a, q_1)]) \vee ([\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1) \wedge \tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2)]$$
$$\vee [\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_4) \wedge \tilde{\delta}((q_4, \mu^{t_1}(q_4)), a, q_2)]) \vee ([\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1)$$
$$\wedge \tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_4)] \vee [\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_4) \wedge \tilde{\delta}((q_4, \mu^{t_1}(q_4)), a, q_4)])$$

$$= ([F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) \wedge F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_1))] \vee [F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_4))$$
$$\wedge F_1(\mu^{t_1}(q_4), \delta(q_4, a, q_1))]) \vee ([F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) \wedge F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2))]$$
$$\vee [F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_4)) \wedge F_1(\mu^{t_1}(q_4), \delta(q_4, a, q_2))]) \vee ([F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1))$$
$$\wedge F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_4))] \vee [F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_4)) \wedge F_1(\mu^{t_1}(q_4), \delta(q_4, a, q_4))])$$

$$= [(0.4 \wedge 0) \vee (0.5 \wedge 0.4)] \vee [(0.4 \wedge 0.8) \vee (0.5 \wedge 0)] \vee [(0.4 \wedge 0.35) \vee (0.5 \wedge 0)]$$
$$= 0.4 \vee 0.4 \vee 0.35 = 0.4,$$

$$a_2 = \bigvee_{q \in Q_{act}(t_1)} q^{\tilde{F}_2^*}((q, \mu^{t_1}(q)), a) = q^{\tilde{F}_2^*}((q_1, \mu^{t_1}(q_1)), a) \vee q^{\tilde{F}_2^*}((q_4, \mu^{t_1}(q_4)), a)$$

$$= [\bigvee_{q' \in Q_{act}(t_2)} \tilde{\delta}^*((q_1, \mu^{t_1}(q_1)), a, q')] \vee [\bigvee_{q' \in Q_{act}(t_2)} \tilde{\delta}^*((q_4, \mu^{t_1}(q_4)), a, q')]$$

$$= [\tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_1) \vee \tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2) \vee \tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_4)]$$
$$\vee [\tilde{\delta}((q_4, \mu^{t_1}(q_4)), a, q_1) \vee \tilde{\delta}((q_4, \mu^{t_1}(q_4)), a, q_2) \vee \tilde{\delta}((q_4, \mu^{t_1}(q_4)), a, q_4)]$$

$= (0 \vee 0.8 \vee 0.35) \vee (0.4 \vee 0 \vee 0) = 0.8.$

So we have

$$Q^{\tilde{F}_2^*}(aa) = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix}$$

Definition 2.3: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order and $\bar{q}_i = \{q' : q' \in Q_{act}(t_i)\}$, $i = 0, 1, \ldots,$ $n - 1$. An active state distribution of is $\tilde{F}_n^*$ a function η from $\bar{Q} = \{\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_{n-1}\}$ into [0, 1]. η is said to be concentrated at $\bar{q}_i \in \bar{Q}$ if $\eta(\bar{q}_i) = 1$ and $\eta(\bar{q}_j) = 0$, $\forall \bar{q}_j \in \bar{Q} \setminus \{q_i\}$.

Definition 2.4: An initialized max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n is an ordered pair $(\tilde{F}_n^*, \eta)$, where $\tilde{F}_n^*$ is a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n and η is an active state distribution of $\tilde{F}_n^*$. If η is concentrated at $\bar{q}_i \in \bar{Q}$, we write $(\tilde{F}_n^*, \bar{q}_i)$ for $(\tilde{F}_n^*, \eta)$.

Definition 2.5: Let $I = (\tilde{F}_n^*, \eta)$ be an initialized max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n. Then the response number $r_I(w)$ of I is defined by

$$r_I(w) = r_I(u_1 u_2 \ldots u_n) = \bigvee_{i=1}^{n} \{\eta(\bar{q}_{i-1}) \wedge a_i\}$$

where a_i is the i-th row of $Q^{\tilde{F}_n^*}(u_1 u_2 \ldots u_n)$.

Definition 2.6: Let η_1 be an active state distribution of $\tilde{F}_{n_1}^*$ and η_2 be an active state distribution of $\tilde{F}_{n_2}^*$, where $\tilde{F}_{n_i}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w_i)$ is a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, for $i = 1, 2$, $w_1 = u_1 u_2 u_3 \ldots u_n$, $w_2 = u'_1 u'_2 u'_3 \ldots u'_n$, $I_1 = (\tilde{F}_{n_1}^*, \eta_1)$ and $I_2 = (\tilde{F}_{n_2}^*, \eta_2)$. Then I_1 and I_2 are called equivalent, denoted by $I_1 \sim I_2$, if $r_{I_1}(w_1) = r_{I_2}(w_2)$.

Definition 2.7: Let η_1 and η_2 be two active state distributions of $\tilde{F}_n^*$. Then η_1 and η_2 are called equivalent, denoted by $\eta_1 \sim \eta_2$, if $(\tilde{F}_n^*, \eta_1) \sim (\tilde{F}_n^*, \eta_2)$.

If η_1 is concentrated at $\bar{q}_i \in \bar{Q}$, then we write $\bar{q}_i \sim \eta_2$ for $\eta_1 \sim \eta_2$.

Example 2.8: In $\tilde{F}_2^*$ introduced in Example 2.2, we have $\bar{q}_0 = \{q_0\}$, $\bar{q}_1 = \{q_1, q_4\}$, $\bar{Q} = \{\bar{q}_0, \bar{q}_1\}$. Let $\eta_1(\bar{q}_0) = 0.5$, $\eta_1(\bar{q}_1) = 0.9$, $\eta_2(\bar{q}_0) = 0.6$, $\eta_2(\bar{q}_1) = 1$, $I_1 = (\tilde{F}_2^*, \eta_1)$ and $I_2 = (\tilde{F}_2^*, \eta_2)$. Then we have

$$r_{I_1}(aa) = (\eta_1(\overline{q}_0) \wedge a_1) \vee (\eta_1(\overline{q}_1) \wedge a_2) = (0.5 \wedge 0.4) \vee (0.9 \wedge 0.8) = 0.8,$$
$$r_{I_2}(aa) = (\eta_2(\overline{q}_0) \wedge a_1) \vee (\eta_2(\overline{q}_1) \wedge a_2) = (0.6 \wedge 0.4) \vee (1 \wedge 0.8) = 0.8.$$

So we get that $\eta_1 \sim \eta_2$.

Theorem 2.9: Let $I_1 = (\tilde{F}^*_{n_1}, \eta_1)$ and $I_2 = (\tilde{F}^*_{n_2}, \eta_2)$, where η_i is an active state distribution of $\tilde{F}^*_{n_i}$, $\tilde{F}^*_{n_i} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w_i)$ is a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $\overline{Q}_i = \{(\overline{q}_0)_i, (q_1)_i, \dots, (\overline{q}_{n-1})_i\}$, for $i = 1, 2$, $w_1 = u_1u_2u_3 \dots u_n$, $w_2 = u'_1u'_2u'_3 \dots u'_n$. Then $I_1 \sim I_2$ if and only if

$$[\eta_1((\overline{q}_0)_1)\ \ \eta_1((\overline{q}_1)_1)\ \ \dots\ \ \eta_1((\overline{q}_{n-1})_1)] \otimes Q^{\tilde{F}^*_{n_1}}(u_1u_2 \dots u_n)$$
$$= [\eta_2((\overline{q}_0)_2)\ \ \eta_2((\overline{q}_1)_2)\ \ \dots\ \ \eta_2((\overline{q}_{n-1})_2)] \otimes Q^{\tilde{F}^*_{n_2}}(u'_1u'_2 \dots u'_n).$$

Proof: Let $I_1 \sim I_2$. Then $r_{I_1}(w_1) = r_{I_2}(w_2)$. So we have

$$\bigvee_{i=1}^{n} \{\eta_1((\overline{q}_{i-1})_1) \wedge a_i\} = \bigvee_{i=1}^{n} \{\eta_2((\overline{q}_{i-1})_2) \wedge a'_i\}$$

where a_i is the i-th row of $Q^{\tilde{F}^*_{n_1}}(u_1u_2 \dots u_n)$ and a'_i is the i-th row of $Q^{\tilde{F}^*_{n_2}}(u'_1u'_2 \dots u'_n)$. By Definition 1.4, we get that

$$[\eta_1((\overline{q}_0)_1) \dots \eta_1((\overline{q}_0)_1) \dots \eta_1((\overline{q}_{n-1})_1)] \otimes Q^{\tilde{F}^*_{n_1}}(u_1u_2 \dots u_n)$$
$$= [\eta_2((\overline{q}_0)_2) \dots \eta_2((\overline{q}_1)_2) \dots \eta_2((\overline{q}_{n-1})_2)] \otimes Q^{\tilde{F}^*_{n_2}}(u'_1u'_2 \dots u'_n).$$

The converse of proof is similarly.

Theorem 2.10: Let $\tilde{F}^*_n = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $w = u_1u_2u_3 \dots u_n$, $\overline{Q} = \{\overline{q}_0, \overline{q}_1, \dots, q_{n-1}\}$, $\overline{q}_i = \overline{q}_j$, for some $i \neq j$ and $i < j$. Also let $\tilde{\delta}^*((q, \mu^{t_i}(q)), u_{i+1}u_{i+2} \dots u_n, q') = \tilde{\delta}^*((q, \mu^{t_j}(q)), u_{j+1}u_{j+2} \dots u_n, q')$, $\forall q \in \overline{q}_i = \overline{q}_j$, $\forall q' \in Q_{act}(t_n)$. Then $\overline{q}_i \sim \overline{q}_j$.

Proof: Let $Q^{\tilde{F}^*_n}(u_1u_2 \dots u_n) = [a_i]_{n\times 1}$. Then we have

$$a_{i+1} = \bigvee_{q \in \overline{q}_i} q^{\tilde{F}^*_n}((q, \mu^{t_i}(q)), u_{i+1}u_{i+2} \dots u_n)$$

$$= \bigvee_{q \in \overline{q}_i} \bigvee_{q' \in Q_{act}(t_n)} \tilde{\delta}^*((q, \mu^{t_i}(q)), u_{i+1}u_{i+2} \ldots u_n, q'),$$

$$a_{j+1} = \bigvee_{q \in \overline{q}_i} q^{\tilde{F}_n^*}((q, \mu^{t_j}(q)), u_{j+1}u_{j+2} \ldots u_n)$$

$$= \bigvee_{q \in \overline{q}_j} \bigvee_{q' \in Q_{act}(t_n)} \tilde{\delta}^*((q, \mu^{t_j}(q)), u_{j+1}u_{j+2} \ldots u_n, q').$$

By hypothesis, we get that $a_{i+1} = a_{j+1}$. Let η_1 be concentrated at $\overline{q}_i$ and η_2 be concentrated at $\overline{q}_j$. Now, since $a_{i+1} = a_{j+1}$, then we have

$$[\eta_1(\overline{q}_0)\ \eta_1(\overline{q}_1) \ldots \eta_1(\overline{q}_{n-1})] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)$$

$$= [\eta_2(\overline{q}_0)\ \eta_2(\overline{q}_1) \ldots \eta_2(\overline{q}_j) \ldots \eta_2(\overline{q}_{n-1})] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n).$$

By Theorem 2.9, we have $\eta_1 \sim \eta_2$. Therefore $\overline{q}_i \sim \overline{q}_j$.

Definition 2.11: Let $\tilde{F}_{n_1}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w_i)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, for i = 1, 2, $\overline{Q}_1 = \{(\overline{q}_0)_1, (\overline{q}_1)_1, \ldots, (\overline{q}_{n-1})_1\}$ and $\overline{Q}_2 = \{(\overline{q}_0)_2, (\overline{q}_1)_2, \ldots, (\overline{q}_{n-1})_2\}$, where $w_1 = u_1u_2u_3 \ldots u_n$, $w_2 = u'_1u'_2u'_3 \ldots u'_n$. Then

(i) $\tilde{F}_{n_1}^*$ and $\tilde{F}_{n_2}^*$ are called statewise equivalent, denoted by $\tilde{F}_{n_1}^* \sim \tilde{F}_{n_2}^*$, if for every $(\overline{q}_i)_1 \in \overline{Q}_1$, there exists $(\overline{q}_j)_2 \in \overline{Q}_2$ such that $(\tilde{F}_{n_1}^*, (\overline{q}_i)_1) \sim (\tilde{F}_{n_2}^*, (\overline{q}_j)_2)$ and vice versa.

(ii) $\tilde{F}_{n_1}^*$ and $\tilde{F}_{n_2}^*$ are called compositewise equivalent, denoted by $\tilde{F}_{n_1}^* \cong \tilde{F}_{n_2}^*$, if for every $(\overline{q}_i)_1 \in \overline{Q}_1$, there exists an active state distribution η of $\tilde{F}_{n_2}^*$ such that $(\tilde{F}_{n_1}^*, (\overline{q}_i)_1) \sim (\tilde{F}_{n_2}^*, \eta)$ and vice versa.

(iii) $\tilde{F}_{n_1}^*$ and $\tilde{F}_{n_2}^*$ are called distributionwise equivalent, denoted by $\tilde{F}_{n_1}^* \approx \tilde{F}_{n_2}^*$ if for every active state distribution η_1 of $\tilde{F}_{n_1}^*$, there exists an active state distribution η_2 of $\tilde{F}_{n_2}^*$ such that $(\tilde{F}_{n_1}^*, \eta_1) \sim (\tilde{F}_{n_2}^*, \eta_2)$ and vice versa.

Theorem 2.12: Let $\tilde{F}_{n_i}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w_i)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, for i = 1, 2, where $w_1 = u_1u_2u_3 \ldots u_n$ and $w_2 = u'_1u'_2u'_3 \ldots u'_n$.

(i) $\tilde{F}^*_{n_1} \sim \tilde{F}^*_{n_2}$ if and only if $\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)) = \rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n))$,

(ii) $\tilde{F}^*_{n_1} \cong \tilde{F}^*_{n_2}$ if and only if $\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)) \subseteq C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)))$ and

$$\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)) \subseteq C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n))),$$

(iii) $\tilde{F}^*_{n_1} \approx \tilde{F}^*_{n_2}$ if and only if $C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n))) = C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)))$.

Proof: We prove the part (i), the other parts are proved similarly.

$\Rightarrow$) Let $\tilde{F}^*_{n_1} \sim \tilde{F}^*_{n_2}$, $(\overline{q}_0)_1 \in \overline{Q}_1$, $Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n) = [a_i]_{n\times 1}$ and $Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)$ $=[a'_i]_{n\times 1}$. Then there exists $(\overline{q}_j)_2 \in \overline{Q}_2$ such that $(\tilde{F}^*_{n_1}, (\overline{q}_0)_1) \sim (\tilde{F}^*_{n_2}, (\overline{q}_j)_2)$. Thus by Theorem 2.9, we have

$$[1\ \ 0\ \ 0\ \ \ldots\ \ 0] \otimes Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n) = [0\ \ \ldots\ \ 0\ \ 1\ \ 0\ \ \ldots\ \ 0] \otimes Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n).$$

So $a_1 = a'_{j+1}$. By replacing $(\overline{q}_0)_1$ with $(\overline{q}_1)_1, \ldots, (\overline{q}_{n-1})_1$, we have

$$\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)) \subseteq \rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)).$$

Similarly, we get that

$$\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)) \subseteq \rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)).$$

$\Leftarrow$) The proof is easy.

Corollary 2.13: Let $\tilde{F}^*_{n_i} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w_i)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, for $i = 1, 2$, where $w_1 = u_1u_2u_3 \ldots u_n$ and $w_2 = u'_1u'_2u'_3 \ldots u'_n$. If $\tilde{F}^*_{n_1} \sim \tilde{F}^*_{n_2}$, then $\tilde{F}^*_{n_1} \cong \tilde{F}^*_{n_2}$.

Proof: Let $\tilde{F}^*_{n_1} \sim \tilde{F}^*_{n_2}$. Then $\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)) = \rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n))$. By Theorem 1.6 (i), $\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)) \subseteq C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)))$. Thus we have

$$\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)) \subseteq C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n))).$$

Similarly, we have

$$\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)) \subseteq C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n))).$$

Therefore $\tilde{F}^*_{n_1} \cong \tilde{F}^*_{n_2}$.

Corollary 2.14: Let $\tilde{F}^*_{n_i} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w_i)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, for $i = 1, 2$, where $w_1 = u_1u_2u_3 \ldots u_n$ and $w_2 = u'_1u'_2u'_3 \ldots u'_n$. Then $\tilde{F}^*_{n_1} \cong \tilde{F}^*_{n_2}$ if and only if $\tilde{F}^*_{n_1} \approx \tilde{F}^*_{n_2}$.

Proof: Let $\tilde{F}^*_{n_1} \cong \tilde{F}^*_{n_2}$. Then $\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)) \subseteq C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)))$ and $\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)) \subseteq C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)))$. By Theorem 1.6 (ii), we get that

$$C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n))) \subseteq C(C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n)))),$$

$$C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n))) \subseteq C(C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n)))).$$

By Theorem 1.6 (iii), we have

$$C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n))) \subseteq C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n))),$$

$$C(\rho(Q^{\tilde{F}^*_{n2}}(u'_1u'_2 \ldots u'_n))) \subseteq C(\rho(Q^{\tilde{F}^*_{n1}}(u_1u_2 \ldots u_n))).$$

So $\tilde{F}^*_{n_1} \approx \tilde{F}^*_{n_2}$. The converse of this corollary is obvious.

Definition 2.15: Let $\tilde{F}^*_n = (\mathrm{Q}, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n. Then

(i) $\tilde{F}^*_n$ is called statewise irreducible if for every $\bar{q}_i, \bar{q}_j \in \bar{Q}$, $\bar{q}_i \sim \bar{q}_j$ implies $\bar{q}_i = \bar{q}_j$,

(ii) $\tilde{F}^*_n$ is called compositewise irreducible if for every $\bar{q}_i \in \bar{Q}$ and active state distribution η of $\tilde{F}^*_n$, $\bar{q}_i \sim \eta$ implies $\eta(\bar{q}_i) > 0$,

(iii) $\tilde{F}^*_n$ is called distributionwise irreducible if for every active state distributions η_1 and η_2 of $\tilde{F}^*_n$, $\eta_1 \sim \eta_2$ implies $\eta_1 = \eta_2$.

Theorem 2.16: Let $\tilde{F}^*_n = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n and $|\bar{Q}| = n$. Then $\tilde{F}^*_n$ is statewise irreducible if and only if no two rows of $Q\tilde{F}^*_n(u_1u_2 \ldots u_n)$ are identical.

Proof: Let $\tilde{F}_n^*$ be statewise irreducible and two rows of $Q^{\tilde{F}_n^*}(u_1u_2 \dots u_n)$ be identical. Without loss of generality, assume that the first row and the second row be identical. Then we have

$$[1 \;\; 0 \;\; 0 \;\; \dots \;\; 0] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \dots u_n) = [0 \;\; 1 \;\; 0 \;\; \dots \;\; 0] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \dots u_n).$$

By Theorem 2.9, $\overline{q}_0 \sim \overline{q}_1$. Since $|\overline{Q}| = n$, then $\overline{q}_0 \neq \overline{q}_1$, which is a contradiction to the fact that $\tilde{F}_n^*$ is statewise irreducible. Conversely, let no two rows of $Q^{\tilde{F}_n^*}(u_1u_2 \dots u_n)$ be identical and $\tilde{F}_n^*$ does not be statewise irreducible. Then there exist $\overline{q}_i$, $\overline{q}_j \in \overline{Q}$ such that $\overline{q}_i \sim \overline{q}_j$ and $\overline{q}_i \neq \overline{q}_j$. Thus we can conclude that the $(i + 1)$-th row and the $(j + 1)$-th row of $Q^{\tilde{F}_n^*}(u_1u_2 \dots u_n)$ are identical, which is a contradiction.

Definition 2.17: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $w = u_1u_2u_3 \dots u_n$ and $\overline{Q} = \{\overline{q}_0, \overline{q}_1, \dots, \overline{q}_{n-1}\}$. Then $\tilde{F}_n^*$ is called effective if $\overline{q}_i = \overline{q}_j$, for some $i \neq j$, then $\overline{q}_i \sim \overline{q}_j$.

Example 2.18: In $\tilde{F}_2^*$ introduced in Example 2.2, we have $\overline{q}_0 = \{q_0\}$, $\overline{q}_1 = \{q_1, q_4\}$, and

$$Q^{\tilde{F}_2^*}(aa) = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix}.$$

By Theorem 2.9, since $0.4 \neq 0.8$, then $\overline{q}_0$ is not equivalent with $\overline{q}_1$, and since $\overline{q}_0 \neq \overline{q}_1$, so $\tilde{F}_2^*$ is effective.

Definition 2.19: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n and $\overline{Q} = \{\overline{q}_0, \overline{q}_1, \dots, \overline{q}_{n-1}\}$. Then $\tilde{F}_n^*$ is called statewise minimal if there is not an $\tilde{F}_n^{*\prime}$ such that $\tilde{F}_n^{*\prime} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w')$ be a max-min effective deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $\tilde{F}_n^*$ be statewise equivalent to $\tilde{F}_n^{*\prime}$ and $|\overline{Q}_1| < |\overline{Q}|$, where $Q_1 = \{(\overline{q}_0)', (\overline{q}_1)', \dots, (\overline{q}_{n-1})'\}$.

Theorem 2.20: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $\overline{Q} = \{\overline{q}_0, \overline{q}_1, \dots, \overline{q}_{n-1}\}$, $|\overline{Q}| = n$ and $w = u_1u_2u_3 \dots u_n$. If $\tilde{F}_n^*$ is statewise irreducible, then it is statewise minimal.

Proof: Suppose that $\tilde{F}_n^*$ is not statewise minimal. Then there exists an $\tilde{F}_n^{*\prime}$ such that $\tilde{F}_n^{*\prime} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w')$ is a max-min effective deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $\tilde{F}_n^*$ is statewise equivalent to $\tilde{F}_n^{*\prime}$ and $|\overline{Q}_1| < n$, where $w' = u'_1u'_2u'_3 \ldots u'_n$ and $\overline{Q}_1 = \{(\overline{q}_0)', (\overline{q}_1)', \ldots, (\overline{q}_{n-1})'\}$. By Theorem 2.12, since $\tilde{F}_n^* \sim \tilde{F}_n^{*\prime}$, then

$$\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)) = \rho(Q^{\tilde{F}_n^{*\prime}}(u'_1u'_2 \ldots u'_n)).$$

On the other hand, since $|\overline{Q}_1| < n$, then there exist $(\overline{q}_i)', (\overline{q}_j)' \in \overline{Q}_1$ such that $(\overline{q}_i)' = (\overline{q}_j)'$. Since $\tilde{F}_n^{*\prime}$ is effective, then $(\overline{q}_i)' \sim (\overline{q}_j)'$. So two rows of $Q^{\tilde{F}_n^{*\prime}}(u'_1u'_2 \ldots u'_n)$ are identical. Since

$$\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)) = \rho(Q^{\tilde{F}_n^{*\prime}}(u'_1u'_2 \ldots u'_n)),$$

then two rows of $Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)$ are identical. So by Theorem 2.16, $\tilde{F}_n^*$ is not statewise irreducible.

Theorem 2.21: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n and $\overline{Q} = \{\overline{q}_0, \overline{q}_1, \ldots, \overline{q}_{n-1}\}$. Then $\tilde{F}_n^*$ is compositewise irreducible if and only if $\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ is a set of vertices of $C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$.

Proof: Let $\tilde{F}_n^*$ be compositewise irreducible and $\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ does not be a set of vertices of $C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$. Then one row of $Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)$ is a convex max-min combination of the other rows of $Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)$. Suppose that, this row be the first row. Thus $a_1 = \bigvee_{i=2}^{n}(c_i \wedge a_i)$, where $0 \le c_i \le 1$ and $Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n) = [a_i]_{n\times 1}$. So we have

$$[1\ 0\ 0 \ldots 0] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n) = [0\ c_2\ c_3 \ldots c_n] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n).$$

Now, we define the active state distribution η of $\tilde{F}_n^*$ by $\eta(\overline{q}_i) = c_{i+1}$ for every $i = 0, 1, \ldots, n-1$, where $c_1 = 0$. By Theorem 2.9, $\overline{q}_0 \sim \eta$. Since $\eta(\overline{q}_0) = 0$, we get a contradiction to the fact that $\tilde{F}_n^*$ is compositewise irreducible.

Conversely, let $\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ be a set of vertices of $C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$ and $\tilde{F}_n^*$ does not be compositewise irreducible. Then there exist $\bar{q}_i \in \bar{Q}$ and an active state distribution η of $\tilde{F}_n^*$ such that $\bar{q}_i \sim \eta$ and $\eta(\bar{q}_i) = 0$. By Theorem 2.9, we have

$$[0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)$$
$$= [\eta(\bar{q}_0) \ \eta(\bar{q}_1) \ \ldots \ \eta(\bar{q}_i) \ \ldots \eta(\bar{q}_{n-1})] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n),$$

where 1 is the $(i + 1)$-th column of the matrix $[0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0]$.

Then $a_{i+1} \in C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ \ldots \ u_n))\backslash\{a_{i+1}\})$, which is a contradiction to Theorem 1.8.

Definition 2.22: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n and $\bar{Q} = \{\bar{q}_0, \ \bar{q}_1, \ \ldots , \ \bar{q}_{n-1}\}$. Then $\tilde{F}_n^*$ is called compositewise minimal if there is not an $\tilde{F}_n^{*\prime}$ such that $\tilde{F}_n^{*\prime} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w')$ be a max-min effective deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $\tilde{F}_n^*$ be compositewise equivalent to $\tilde{F}_n^{*\prime}$ and $|\bar{Q}_1| < |\bar{Q}|$, where $\bar{Q}_1 = \{(\bar{q}_0)', (\bar{q}_1)', \ldots , (\bar{q}_{n-1})'\}$.

Theorem 2.23: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $\bar{Q} = \{\bar{q}_0, \ \bar{q}_1, \ \ldots , \ \bar{q}_{n-1}\}$, $|\bar{Q}| = n$, $|\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))| = n$ and $w = u_1u_2u_3 \ldots u_n$. If $\tilde{F}_n^*$ is compositewise irreducible, then it is compositewise minimal.

Proof: Suppose that $\tilde{F}_n^*$ is not compositewise minimal. Then there exists an $\tilde{F}_n^{*\prime}$ such that $\tilde{F}_n^{*\prime} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w')$ is a max-min effective deterministic general fuzzy automaton of $\tilde{F}^*$ of order n, $\tilde{F}_n^*$ is compositewise equivalent to $\tilde{F}_n^{*\prime}$ and $|\bar{Q}_1| < n$, where $w' = u'_1u'_2u'_3 \ldots u'_n$ and $\bar{Q}_1 = \{(\bar{q}_0)', (\bar{q}_1)', \ldots , (\bar{q}_{n-1})'\}$. By Corollary 2.14, since $\tilde{F}_n^* \cong \tilde{F}_n^{*\prime}$, then $\tilde{F}_n^* \approx \tilde{F}_n^{*\prime}$. By Theorem 2.12 (iii), since $\tilde{F}_n^* \approx \tilde{F}_n^{*\prime}$, then

$$C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))) = C(\rho(Q^{\tilde{F}_n^{*\prime}}(u'_1u'_2 \ldots u'_n))).$$

On the other hand, since $|\bar{Q}_1| < n$, then there exist $(\bar{q}_i)', (\bar{q}_j)' \in \bar{Q}_1$ such that $(\bar{q}_i)' = (\bar{q}_j)'$. Since $\tilde{F}_n^{*\prime}$ is effective, then $(\bar{q}_i)' \sim (\bar{q}_j)'$. So two rows of $Q^{\tilde{F}_n^{*\prime}}(u'_1u'_2 \ldots u'_n)$ are identical. Thus we get that

$$|\rho((Q^{\tilde{F}_n^{*\prime}}(u'_1u'_2 \ldots u'_n))| < n = |\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))|.$$

By Theorems 1.11 and 1.12, there exists $x \in \rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ such that

$$x \in C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)).\backslash\{x\}).$$

Therefore, by Theorem 1.8, $\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ is not a set of vertices of

$$C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))).$$

Consequently, by Theorem 2.21, $\tilde{F}_n^*$ is not compositewise irreducible.

Theorem 2.24: Let $\tilde{F}_n^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, w)$ be a max-min deterministic general fuzzy automaton of $\tilde{F}^*$ of order n and $\bar{Q} = \{\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_{n-1}\}$. Then $\tilde{F}_n^*$ is distributionwise irreducible if and only if $\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ is a basis of $C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$.

Proof: Let $\tilde{F}_n^*$ be distributionwise irreducible and $\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ does not be a basis of $C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$. Then there exists $x \in C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$ such that $x = \bigvee_{i=1}^{n}(c_i \wedge a_i) = \bigvee_{i=1}^{n}(d_i \wedge a_i)$, where $a_i \in \rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$, $0 \le c_i \le 1$, $0 \le d_i \le 1$ and for some i, $c_i \ne d_i$. Thus we have

$$[c_1 \;\; c_2 \;\; c_3 \;\; \ldots \;\; c_n] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n) = [d_1 \;\; d_2 \;\; d_3 \ldots d_n] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n).$$

Now, we define two active state distributions η_1 and η_2 of $\tilde{F}_n^*$ by $\eta_1(\bar{q}_i) = c_{i+1}$, $\eta_2(\bar{q}_i) = d_{i+1}$ for every $i = 0, 1, \ldots, n-1$. By Theorem 2.9, $\eta_1 \sim \eta_2$. Since $\eta_1 \ne \eta_2$, we get a contradiction to the fact that $\tilde{F}_n^*$ is distributionwise irreducible.

Conversely, let $\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$ be a basis of $C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$ and $\tilde{F}_n^*$ does not be distributionwise irreducible. Then there exist two active state distributions η_1 and η_2 of $\tilde{F}_n^*$ such that $\eta_1 \ne \eta_2, \ne \eta_2$. By Theorem 2.9, we have

$$\begin{aligned}&[\eta_1(\bar{q}_0) \;\; \eta_1(\bar{q}_1) \ldots \eta_1(\bar{q}_{n-1})] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)\\ = \;&[\eta_2(\bar{q}_0)\; \eta_2(\bar{q}_1) \ldots \eta_2(\bar{q}_{n-1})] \otimes Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n).\end{aligned}$$

Then $x = \bigvee_{i=1}^{n}(\eta_1(\bar{q}_{i-1}) \wedge a_i) = \bigvee_{i=1}^{n}(\eta_2(\bar{q}_{i-1}) \wedge a_i)$, where $a_i \in \rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n))$.

Thus $x \in C(\rho(Q^{\tilde{F}_n^*}(u_1u_2 \ldots u_n)))$, which is a contradiction to Definition 1.9.

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