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(L, M)-TOPOLOGIES AND (2, M)-FUZZIFYING TOPOLOGIES

ABSTRACT: *In this paper, we introduce notions of (*L,M*)-topological spaces and (2,***M***)-fuzzifying topological spaces. We prove that the category (L,M)-**TOP** of (L,M)-topological spaces is a topological category over Set. We investigate the relation between (L,M)-topological spaces and (2,M)-fuzzifying topological spaces.*

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1. INTRODUCTION AND PRELIMINARIES

Since Chang [1] introduced a fuzzy topology, many authors have discussed various aspects of fuzzy topology. However, in a completly different direction, Höhle [2] created the notion of a topology being viewed as an L-subset of a powerset. Kubiak [6] and Sostak [11] independently exteneded Höhle's notion to L-subsets of LX. Kotzé [5] introduced an (L,M)-topological space as a general approach where L and M are frames with 0 and 1.

In this paper, we introduce notions of (L, M)-topological spaces as an extension of that of Kotzé [5]. Here, L is a completely distributive lattice with with 0 and 1 and M is a strictly two-sided, commutative quantale as an extension of a frame. We investigate the relation between (L, M)-topological spaces and (2,M)-fuzzifying topological spaces. We show the existence of initial (L, M)-topological structures. From this fact, the category (L, M)-**TOP** is a topological category over **Set**.

In this paper, let X be a nonempty set and $L = (L, \leq, \vee, \wedge, ')$ a completely distributive lattice with the least element 0 and the greatest element 1 in L with an order reversing involution $'$. The family L^X denotes the set of all fuzzy subsets of a given set X. For each $\alpha \in L$, let $\overline{\alpha}$ denote the constant fuzzy sets of X. We denote the characteristic function of a subset A of X by 1_A.

Definition 1.1: [5] Let L and M be frames. A map $\mathcal{T}: L^X \to M$ is called an (L, M)-*topology* on X if it satisfies the following conditions:

(1) $\mathcal{T}(\overline{0}) = \mathcal{T}(\overline{1}) = \top$,

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,

(3) $\mathcal{T}(\vee_{i\in\Lambda} \mu_i) \geq \wedge_{i\in\Lambda} \mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i\in\Lambda} \subset L^X$.

The pair $(X, \mathcal{T})$ is called an (L, M)-*topological space.*

Let $M = (M, \leq, \vee, \wedge, \perp, \top)$ be a completely distributive lattice with the least element $\perp$ and the greatest element $\top$ in M.

Definition 1.2: [10] A triple $(M, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following properties:

(M1) $(M,\odot)$ is a commutative semigroup,

(M2) $a = a \odot \top$, for each $a \in M$,

(M3) $\odot$ is distributive over arbitrary joins, i.e.,

$$\left(\bigvee_{i\in\Gamma} a_i\right) \odot b = \bigvee_{i\in\Gamma}(a_i \odot b).$$

Remark 1.3: [10](1) Each frame is a stsc-quantale. In particular, the unit interval $([0, 1], \leq, \wedge, 0, 1)$ is a stsc-quantale.

(2) Every left continuous t-norm t on $([0, 1], \leq, t)$ with $\odot = t$ is a stsc-quantale.

(3) Every GL-monoid is a stsc-quantale.

Lemma 1.4: [10] Let $(M, \leq, \odot)$ be a stsc-quantale. For each $x, y, z \in M$, $\{y_i \mid i \in \Gamma\} \subset M$, we have the following properties.

(1) *If* $y \leq z$, *then* $(x \odot y) \leq (x \odot z)$.

(2) $x \odot y \leq x \wedge y$.

(3) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w)$.

2. (*L*, *M*)-TOPOLOGICAL SPACES

Definition 2.1: A map $T : L^X \to M$ is called an (L, M)-*topology* on X if it satisfies the following conditions:

(LO1) $\mathcal{T}(\bar{0}) = \mathcal{T}(\bar{1}) = \top$,

(LO2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \odot \mathcal{T}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,

(LO3) $\mathcal{T}(\vee_{i\in\Lambda} \mu_i) \geq \wedge_{i\in\Lambda} \mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i\in\Lambda} \subset L^X$.

The pair $(X, \mathcal{T})$ is called an (L, M)-*topological space.*

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be (L, M)-topologies on X. We say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ ($\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$), denoted by $\mathcal{T}_2 \leq \mathcal{T}_1$, if $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\lambda)$ for all $\lambda \in L^X$.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be (L, M)-topological spaces. A map $\phi : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *LF-continuous* iα $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\phi^{\leftarrow}(\lambda))$, for all $\lambda \in L^Y$. The category of (L, M)-topological spaces and *LF*-continuous maps is denoted by (L,M)-**TOP.**

Remark 2.2: Let $L = \{0, 1\}$ be given and $2^X \cong P(X)$ in a sense $1_A \in 2^X$ iff $A \in P(X)$. A map $\tau : P(X) \to M$ is called a $(2, M)$-fuzzifying topology on X if it satisfies the following conditions:

(O1) $\tau(X) = \tau(\emptyset) = \top$,
(O2) $\tau(A \cap B) \geq \tau(A) \odot \tau(B)$, for all $A, B \in P(X)$,
(O3) $\tau(\bigcup_{i\in\Lambda} A_i) \geq \wedge_{i\in\Lambda} \tau(A_i)$, for any $\{A_i\}_{i\in\Lambda} \subset P(X)$.

The pair (X, τ) is called a $(2, M)$-fuzzifying topological space.

Let (X, τ_1) and (Y, τ_2) be $(2, M)$-fuzzifying topological spaces. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is called *fuzzifying continuous* iff

$$\tau_2(A) \leq \tau_1(f^{-1}(A)), \forall A \in P(Y).$$

$(2, M)$-**TOP** denotes the category of $(2, M)$-fuzzifying topological spaces and fuzzifying continuous functions.

Remark 2.3: (1) If $(L = [0, 1], \wedge)$ and $M = \{0, 1\}$, (L, M)-topological space is the concept of Chang [1].

(2) If $(L = M = [0, 1], \odot = \wedge)$, (L, M)-topological space is the concept of Kubiak [6] and Ŝostak [11].

(3) If $L = \{0, 1\}$ and ($M = [0, 1]$, $\odot = \wedge$), (L, M)-topological space is the concept of Ying [12,13].

(4) If L and M are frames with 0 and 1, (L, M)-topological space is the concept of Kotzé [5] in Definition 1.1.

Theorem 2.4: *Let* (X, τ) *be a* $(2, M)$*-fuzzifying topological space. We define a function* $\mathcal{T}_\tau : L^X \to M$ *as follows:*

$$\mathcal{T}_\tau(\lambda) \bigwedge_{r \in L} \tau(\lambda_r)$$

where $\lambda_r = \{x \in X : \lambda(x) > r\}$*: Then* $\mathcal{T}_\tau$ *is an* (L, M)*-topology.*

Proof: (LO1) Clear.

(LO2) For each $\lambda, \mu \in L^X$, we have

$$\mathcal{T}_\tau(\lambda \wedge \mu) = \bigwedge_{r \in L} \tau\big((\lambda \wedge \mu)_r\big) = \bigwedge_{r \in L} \tau(\lambda_r \cap \mu_r)$$

$$\geq \bigwedge_{r \in L} \big(\tau(\lambda_r) \odot \tau(\mu_r)\big) \geq \bigwedge_{r \in L} \tau(\lambda_r \odot \bigwedge_{r \in L} \tau(\mu_r)$$

$$= T_\tau(\lambda) \odot T_\tau(\mu).$$

(LO3) Since $(\vee_{j \in J} \mu_j)_r = \bigcup_{j \in J} (\mu_j)_r$, we have

$$T_\tau(\bigvee_{j \in J} \mu_j) = \bigwedge_{r \in L} \tau\left(\bigcup_{j \in J} (\mu_j)_r\right) \geq \bigwedge_{r \in L} \bigwedge_{j \in J} \tau((\mu_j)_r)$$

$$\bigwedge_{j \in J} \bigwedge_{r \in L} \tau((\mu_j)_r) = \bigwedge_{j \in J} \mathcal{T}_\tau(\mu_j).$$

Lemma 2.5: Let $A \in P(X)$ and $\alpha \in L - \{0\}$. Then $T_\tau(\alpha 1_A) = \tau(A)$.

Theorem 2.6: *Let* (X, τ_1) *and* (Y, τ_2) *be* $(2, M)$*-fuzzifying topological spaces. A mapping* $f : (X, \tau_1) \to (Y, \tau_2)$ *is fuzzifying continuous iff* $f : (X, \mathcal{T}_{\tau 1}) \to (Y, \mathcal{T}_{\tau 2})$ *is* LF-continuous.

Proof: For each $\mu \in L^Y$, we have

$$\mathcal{T}_{\tau 1}(f^{\leftarrow}(\mu)) = \bigwedge_{r\in L} \tau_1((f^{\leftarrow}(\mu))_r) \geq \bigwedge_{r\in L} \tau_2(\mu_r) = \mathcal{T}_{\tau 2}(\mu).$$

Conversely, suppose there exists $A \in P(X)$ such that $\tau_1(f^{-1}(A)) \ngeq \tau_2(A)$. It implies

$$\mathcal{T}_{\tau_1}(1_{f^{-1}(A)}) = \tau_1(f^{-1}(A)) \ngeq \tau_2(A) = \mathcal{T}_{\tau_2}(1_A).$$

Example 2.7: Let $X = \{x, y, z\}$ be a set. Define a binary operation $\otimes$ on $M = [0, 1]$ by $x \otimes y = \max\{0, x + y - 1\}$. Then $(M = [0, 1], \leq, \otimes)$ is a stsc-quantale. Define a $(2, M)$-fuzzifying topology $\tau : P(X) \to [0, 1]$ as follows:

$$\tau(A) = \begin{cases} 1, & \text{if } A \in \{\emptyset, X\} \\ 0.8, & \text{if } A = \{x, y\} \\ 0.6, & \text{if } A = \{y\} \\ 0.7, & \text{if } B = \{y, z\} \\ 0, & \text{otherwise} \end{cases}$$

For $\lambda, \mu \in [0, 1]^X$ with

$\lambda(x) = 0.3, \lambda(y) = 0.7, \lambda(z) = 0.5, \mu(x) = 0.7, \mu(y) = 0.2, \mu(z) = 0.5,$

we have

$$(\lambda)_r \in \{\{y\}, \{y, z\}, \emptyset, X\}, (\mu)r \in \{\{x\}, \{x, z\}, \emptyset, X\}.$$

Hence $\mathcal{T}_\tau(\lambda) = 0.6$ and $\mathcal{T}_\tau(\mu) = 0$.

3. PRODUCTS AND SUBSPACES OF (*L*, *M*)-TOPOLOGICAL SPACES

Definition 3.1: A map $\mathcal{B} : L^X \to M$ is called an (L, M)-fuzzy base on X if it satisfies the following conditions:

(LB1) $\mathcal{B}(\overline{1}) = \mathcal{B}(\overline{0}) = \top$.

(LB2) $\mathcal{B}(\mu_1 \wedge \mu_2) \geq \mathcal{B}(\mu_1) \odot \mathcal{B}(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$.

Remark 3.2: By the sense of Remark 2.2, a map $\beta : P(X) \to M$ is called a $(2, M)$-fuzzifying base on X if it satisfies the following conditions:

(B1) $\beta(X) = \beta(\emptyset) = \top$

(B2) $\beta(A \cap B) \geq \beta(A) \odot \beta(B)$, for all $A, B \in P(X)$.

An (L, M)-fuzzy base $\mathcal{B}$ always generates an (L, M)-topology $\mathcal{T}_{\mathcal{B}}$ on X in the following sense.

Theorem 3.3: *Let $\mathcal{B}$ be an (L, M)-fuzzy base on X. Define a map $\mathcal{T}_{\mathcal{B}} : L^X \to M$ as follows:*

$$\mathcal{T}_{\mathcal{B}}(\mu) = \bigvee \left\{ \bigwedge_{j\in\Lambda} \mathcal{B}(\mu_j) \middle| u = \bigvee_{j\in\Lambda} \mu_j \right\}.$$

Then $\mathcal{T}_{\mathcal{B}}$ is the coarsest (L, M)-topology on X such that $\mathcal{T}_{\mathcal{B}}(\lambda) \geq \mathcal{B}(\lambda)$, for all $\lambda \in L^X$.

Proof: (LO1) It is trivial from the definition of $\mathcal{T}_{\mathcal{B}}$.

(LO2) For two families $\{\lambda_j | \lambda = \bigvee_{j\in\Lambda} \lambda_j\}$ and $\{\mu_k | \mu = \bigvee_{k\in\Gamma} \mu_k\}$, since L is a completely distributive lattice, there exists a family $\{\lambda j \bigwedge \mu_k\}$ such that

$$\lambda \wedge \mu = (\bigvee_{j\in\Lambda} \lambda_j) \wedge (\bigvee_{k\in\Lambda}) = \bigvee_{j\in\Lambda, k\in\Gamma} (\lambda_j \wedge \mu_k).$$

It implies

$$\begin{aligned} \mathcal{T}_{\mathcal{B}}(\lambda \wedge \mu) &\geq \bigwedge_{j\in\Lambda, k\in\Gamma} \mathcal{B}(\lambda_j \wedge \mu_k) \\ &\geq \bigwedge_{j\in\Lambda, k\in\Gamma} (\mathcal{B}(\lambda_j \odot \mathcal{B}(\mu_k)) \text{ (by Definition 3.1 (LB2))} \\ &\geq (\bigwedge_{j\in\Lambda} \mathcal{B}(\lambda_j)) \odot (\bigwedge_{k\in\Lambda} \mathcal{B}(\mu_k)). \text{ (by Lemma 1.4(1))} \end{aligned}$$

For all families $\{\lambda_j | \lambda = \vee_{j\in\Lambda} \lambda_j\}$ and $\{\mu_k | \mu = \vee_{k\in\Gamma} \mu_k\}$, by Definition 1.2 (M3), $\mathcal{T}_{\mathcal{B}}(\lambda \wedge \mu) \geq \mathcal{T}_{\mathcal{B}}(\lambda) \odot \mathcal{T}_{\mathcal{B}}(\mu)$.

(LO3) Let $\mathcal{J}_i$ be the collection of all index sets K_i such that $\{\lambda_{i_k} \in L^X | \lambda_i = \vee_{k\in K_i} \lambda_{i_k}\}$ with $\lambda = \vee_{i\in\Gamma} \lambda_i = \vee_{i\in\Gamma} \vee_{k\in K_i} \lambda_{i_k}$. For each $i \in \Gamma$ and each $\psi \in \Pi_{i\in\Gamma} \mathcal{J}_i$ with $\psi(i) = K_i$, we have

$$\mathcal{T}_{\mathcal{B}}(\lambda) \geq \bigwedge_{i\in\Gamma}(\bigwedge_{k\in K_i} \mathcal{B}(\lambda_{i_k})). \tag{I}$$

Put $a_{i,\psi(i)} = \bigwedge k\in K_i \, \mathcal{B}(\lambda_{i_k})$. From (I),

$$\mathcal{T}_{\mathcal{B}}(\lambda) \geq \bigvee_{\psi\in\Pi_{i\in\Gamma}\mathcal{J}_i}(\bigwedge_{i\in\Gamma}\alpha_{i\psi(i)})$$

(Since L is a completely distributive lattice,)

$$= \bigwedge_{i\in\Gamma}(\bigvee_{M_i\in\mathcal{J}_i} a_{iM_i}) = \bigwedge_{i\in\Gamma}(\bigvee_{M_i\in\mathcal{J}_i}(\bigwedge_{m\in M_i}\mathcal{B}(\lambda_{i_m})))$$

$$= \bigwedge_{i\in\Gamma}\mathcal{T}_{\mathcal{B}}(\lambda_i).$$

Thus $\mathcal{T}_{\mathcal{B}}$ is an (L, M)-topology on X.

If $\mathcal{T} \geq \mathcal{B}$, for every $\lambda = \vee_{j\in\Lambda}\lambda_j$,

$$T(\lambda) \geq \bigwedge_{j\in\Lambda}\mathcal{T}(\lambda_j) \geq \bigwedge_{j\in\Lambda}\mathcal{B}(\lambda_j)$$

Thus $\mathcal{T} \geq \mathcal{T}_{\mathcal{B}}$.

From Theorem 3.3, we easily prove the following lemma.

Lemma 3.4: Let T be an (L, M)-topology on X and $\mathcal{B}$ be an (L, M)-fuzzy base on Y. Then a map $\phi : (X, \mathcal{T}) \to (Y, \mathcal{T}_{\mathcal{B}})$ is LF-continuous iff $\mathcal{T}(\phi^{\leftarrow}(\lambda)) \geq \mathcal{B}(\lambda)$, for each $\lambda \in L^Y$.

Corollary 3.5: Let β be a $(2, M)$-fuzzifying base on X. Define a map $\tau_\beta : P(X) \to M$ as follows:

$$\tau_\beta(A) = \bigvee\{\bigwedge_{j\in\Lambda}\mathcal{B}(A_j) \,\big|\, A = \bigcup_{j\in\Lambda}A_j\}.$$

Then:

(1) τ_β is the coarsest $(2, M)$-fuzzifying topology on X such that $\tau_\beta(A) \geq \beta(A)$, for all $A \in P(X)$.

(2) a map $f : (Y, \tau^*) \to (X, \tau_\beta)$ is fuzzifying continuous iff $\tau^*(f^{-1}(A)) \geq \beta(A)$, for each $A \in P(Y)$.

Theorem 3.6: *Let $\{(X_i, T_i)\}_{i\in\Gamma}$ be a family of (L, M)-topological spaces, X a set and for each $i \in \Gamma$, $\phi_i : X \to X_i$ a map. Define a map $\mathcal{B} : L^X \to M$ on X by*

$$\mathcal{B}(\mu) = \bigvee\{\odot_{j=1}^{n} \mathcal{T}_{k_j}(\nu_{k_j}) \mid \mu = \wedge_{j=1}^{n} \phi_{k_j}^{\leftarrow}(\nu_{k_j})\}$$

where $\vee$ is taken over all finite subsets $K = \{k_1, \ldots, k_n\} \subset \Gamma$.

Then:

(1) *$\mathcal{B}$ is an (L, M)-fuzzy base on X.*

(2) *The (L, M)-topology $\mathcal{T}_{\mathcal{B}}$ generated by $\mathcal{B}$ is the coarsest (L, M)-topology on X for which all $f_i, i \in \Gamma$, are LF-continuous maps.*

(3) *A map $\phi : (Y, \mathcal{T}') \to (X, \mathcal{T}_{\mathcal{B}})$ is LF-continuous iff for each $i \in \Gamma$, $\phi_i \circ \phi : (Y, \mathcal{T}') \to (X_i, T_i)$ is LF-continuous map.*

Proof: (1) Since $\lambda = \phi_i^{\leftarrow}(\lambda)$ for each $\lambda \in \{0, 1\}$, $\mathcal{B}(1) = \mathcal{B}(0) = \top$.

($B2$) For all finite subsets $K = \{k_1, \ldots, k_p\}$ and $J = \{j_1, \ldots, j_q\}$ of Γ such that

$$\lambda = \wedge_{i=1}^{p} \phi_{k_i}^{\leftarrow}(\lambda_{k_i}), \mu = \vee_{i=1}^{q} \phi_{j_i}^{\leftarrow}(\mu_{j_i}),$$

we have

$$\lambda \wedge \mu = (\wedge_{i=1}^{p} \phi_{k_i}^{\leftarrow}(\lambda_{k_i})) \wedge (\wedge_{i=1}^{q} \phi_{j_i}^{\leftarrow}(\mu_{j_i})).$$

Furthermore, we have for each $k \in K \cap J$,

$$\phi_k^{\leftarrow}(\lambda_k) \wedge \phi_k^{\leftarrow}(\mu_k) = \phi_k^{\leftarrow}(\lambda_k \wedge \mu_k).$$

Put $\lambda \wedge \mu = \wedge_{m_i \in K \cup J} \phi_{m_i}^{\leftarrow}(\rho_{m_i})$ where

$$\rho_{m_i} = \begin{cases} \lambda_{m_i} & \text{if } m_i \in K - (K \cap J) \\ \mu_{m_i} & \text{if } m_i \in J - (K \cap J) \\ \lambda_{m_i} \wedge \mu_{m_i} & \text{if } m_i \in K \cap J. \end{cases}$$

Since $\mathcal{T}_{mi}(\lambda_{mi} \wedge \mu_{mi}) \geq \mathcal{T}_{mi}(\lambda_{mi}) \odot \mathcal{T}_{mi}(\mu_{mi})$ for $m_i \in K \cap J$, we have

$$\mathcal{B}(\lambda \wedge \mu) \geq \odot_{j \in KI \cup J} \mathcal{T}_j(\rho_j)$$
$$\geq (\odot_{i=1}^{p} \mathcal{T}_{k_i}(\lambda_{k_i})) \odot_{i=1}^{q} \mathcal{T}_{j_i}(\mu_{j_i}).$$

By Definition 1.2 (M3), $\mathcal{B}(\lambda \wedge \mu) \geq \mathcal{B}(\lambda) \odot \mathcal{B}(\mu)$.

(2) For each $\lambda_i \in L^{Xi}$, one family $\{\phi_i^{\leftarrow}(\lambda_i)\}$ and $i \in \Gamma$, we have

$$\mathcal{T}_{\mathcal{B}}(\phi_i^{\leftarrow}(\lambda_i)) \geq \mathcal{B}(\phi_i^{\leftarrow}(\lambda_i)) \geq \mathcal{T}_i(\lambda_i).$$

Thus, for each $i \in \Gamma$, $\phi_i : (X, \mathcal{T}_{\mathcal{B}}) \to (X_i, \mathcal{T}_i)$ is *LF*-continuous.

Let $\phi_i : (X, \mathcal{T}^0) \to (X_i, \mathcal{T}_i)$ be *LF*-continuous, that is, for each $i \in \Gamma$ and $\lambda_i \in L^{Xi}$, $\mathcal{T}^0(\phi_i^{\leftarrow}(\lambda_i)) \geq \mathcal{T}_i(\lambda_i)$. For all finite subsets $K = \{k_1, \ldots, k_p\}$ of Γ such that $\lambda = \wedge_{i=1}^{p} \phi_{ki}^{\leftarrow}(\lambda_{ki})$, we have

$$\mathcal{T}^0(\lambda) \geq \odot_{i=1}^{p} \mathcal{T}^0(\phi_{k_i}^{\leftarrow}(\lambda_{k_i}))$$
$$\geq \odot_{i=1}^{p} \mathcal{T}_{k_i}(\lambda_{k_i}).$$

It implies $\mathcal{T}^0(\lambda) \geq \mathcal{B}(\lambda)$ for each $\lambda \in L^X$. By Theorem 3.3, $\mathcal{T}^0 \geq \mathcal{T}_{\mathcal{B}}$.

(3)($\Rightarrow$) Let $\phi : (Y, T') \to (X, \mathcal{T}_{\mathcal{B}})$ be *LF*-continuous. For each $i \in \Gamma$ and $\lambda_i \in L^{Xi}$, we have

$$\mathcal{T}'((\phi_i \circ \phi)^{\leftarrow}(\lambda_i)) = \mathcal{T}'(\phi^{\leftarrow}(\phi_i^{\leftarrow}(\lambda_i))) \geq \mathcal{T}_{\mathcal{B}}(\phi_i^{\leftarrow}(\lambda_i)) \geq \mathcal{T}_i(\lambda_i).$$

Hence $\phi_i \circ \phi : (Y, \mathcal{T}') \to (X_i, \mathcal{T}_i)$ is *LF*-continuous.

($\Leftarrow$) For all finite subsets $K = \{k_1, \ldots, k_p\}$ of Γ such that $\lambda = \wedge_{i=1}^{p} \phi_{ki}^{\leftarrow}(\lambda_{ki})$, since $\phi_{k_i} \circ \phi : (Y, \mathcal{T}') \to (X_{ki}, \mathcal{T}_{ki})$ is *LF*-continuous,

$$\mathcal{T}'(\phi^{\leftarrow}(\phi_{ki}^{\leftarrow}(\lambda_{ki}))) \geq \mathcal{T}_{ki}(\lambda_{ki}). \qquad \text{(II)}$$

Hence we have

$$\mathcal{T}'(\phi^{\leftarrow}(\lambda)) = \mathcal{T}'(\phi^{\leftarrow}(\wedge_{i=1}^{p} \phi_{ki}^{\leftarrow}(\lambda_{ki}))) = \mathcal{T}'(\wedge_{i=1}^{p} \phi^{\leftarrow}(\phi_{k_i}^{\leftarrow}(\lambda_{ki})))$$
$$\geq \odot_{i=1}^{p} \mathcal{T}'(\phi^{\leftarrow}(\phi_{ki}^{\leftarrow}(\lambda_{ki}))) \geq \odot_{i=1}^{p} \mathcal{T}_{k_i}(\lambda_{ki}). \text{ (by (II))}$$

It implies $\mathcal{T}'(\phi^{\leftarrow}(\lambda)) \geq \mathcal{B}(\lambda)$ for all $\lambda \in L^X$. By Lemma 3.4, $\phi : (Y, \mathcal{T}') \to (X, \mathcal{T}_{\mathcal{B}})$ is *LF*-continuous.

Theorem 3.7: *The forgetful functor* $V : (L, M)$-**TOP** $\to$ Set *defined by* $V(X, \mathcal{T}) = X$ *and* $V(\phi) = \phi$ *is topological.*

Proof: By Theorem 3.6, every V-structured source $(\phi_i : X \to (X_i, \mathcal{T}_i))_{i\in\Gamma}$ has a unique V-initial lift $(\phi_i : (X, \mathcal{T}_{\mathcal{B}}) \to (X_i, \mathcal{T}_i))_{i\in\Gamma}$ such that $V(X, \mathcal{T}_{\mathcal{B}}) = X$ and $V(\phi_i) = \phi_i$.

From Theorem 3.6, we can define a product (L, M)-topology and a subspace of (L, M)-topology.

Definition 3.8: Let $\{(X_i, \mathcal{T}_i)\}_{i\in\Gamma}$ be a family of (L, M)-topological spaces, $X = \Pi_{i\in\Gamma} X_i$ a product set and for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection map. The *product* (L, M)*-topology* is the coarsest (L, M)-topology on X for which all $\pi_i, i \in \Gamma$, are LF-continuous maps. Let $(X, \mathcal{T})$ be an (L, M)-topological space, A a subset and $i : A \to X$ an inclusion map. Define a map $\mathcal{T}_A : L^A \to M$ on A by

$$\mathcal{T}_A(\mu) = \bigvee \{\mathcal{T}(\nu) \,|\, \mu = i^{\leftarrow}(\nu)\}.$$

Then $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$.

Theorem 3.9: *Let* $\phi : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ *and* $\psi : (X, \mathcal{T}) \to (Z, \mathcal{T}_2)$ *be LF-continuous. Define a function* $h : X \to Y \times Z$ *by*

$$h(x) = (\phi(x), \psi(x)).$$

Then $h : (X, \mathcal{T}) \to (Y \times Z, \mathcal{T}_1 \otimes \mathcal{T}_2)$ *is LF-continuous where* $\mathcal{T}_1 \otimes \mathcal{T}_2$ *is a product* (L, M)*-topology of* $(Y, \mathcal{T}_1)$ *and* $(Z, \mathcal{T}_2)$.

Proof: Suppose there exists $\rho \in L^{Y\times Z}$ such that

$$\mathcal{T}(h^{\leftarrow}(\rho)) \not\geq \mathcal{T}_1 \otimes \mathcal{T}_2(\rho).$$

Let $\mathcal{B}$ be an (L, M)-fuzzy base for $\mathcal{T}_1 \otimes \mathcal{T}_2$. By the definition of $\mathcal{T}_1 \otimes \mathcal{T}_2$, there exists a family $\{\rho_i \mid \rho = \bigvee_{i\in\Gamma} \rho_i\}$ such that

$$\mathcal{T}(h^{\leftarrow}(\rho)) \not\geq \bigvee_{i\in\Gamma} \mathcal{B}(\rho_i)$$

By the definition of $\mathcal{B}$, for each $i \in \Gamma$, there exist $\lambda_i \in L^Y$ and $\mu_i \in L^Z$ with $\rho_i = \pi_1^{-1}(\lambda_i) \wedge \pi_2^{-1}(\mu_i)$ such that

$$\mathcal{T}(h^{-1}(\rho)) \not\geq \bigwedge_{i\in\Gamma} (\mathcal{T}_1)(\lambda_i) \odot \mathcal{T}_2(\mu_i)) \tag{III}$$

On the other hand, $(\pi_1 \circ h)^{\leftarrow}(\lambda_i)(x) = \lambda_i(\pi_1(h(x))) = \lambda_i(\phi(x)) = \phi^{\leftarrow}(\lambda_i)(x)$ for all $x \in X$, similarly, $(\pi_2 \circ h)^{\leftarrow}(\mu_i) = \psi^{\leftarrow}(\mu_i)$. Thus, we have

$$h^{\leftarrow}(\rho_i) = h^{\leftarrow}(\pi_1^{\leftarrow}(\lambda_i) \wedge \pi_2^{\leftarrow}(\mu_i)) = h^{\leftarrow}(\pi_1^{\leftarrow}(\lambda_i)) \wedge h^{\leftarrow}(\pi_2^{\leftarrow}(\mu_i))$$
$$= (\pi_1 \circ h)^{\leftarrow}(\lambda_i) \wedge (\pi_2 \circ h)^{\leftarrow}(\mu_i) = \phi^{\leftarrow}(\lambda_i) \wedge \psi^{\leftarrow}(\mu_i).$$

It follows

$$\mathcal{T}(h^{\leftarrow}(\rho)) = \mathcal{T}(h^{\leftarrow}(\bigvee_{i\in\Gamma}\rho_i)) \geq \bigwedge_{i\in\Gamma}\mathcal{T}(h^{\leftarrow}(\rho_i))$$

$$= \bigwedge_{i\in\Gamma}\mathcal{T}(\phi^{\leftarrow}(\lambda_i)\wedge\psi^{\leftarrow}(\mu_i))) \geq \bigwedge_{i\in\Gamma}(\mathcal{T}(\phi^{\leftarrow}(\lambda_i)) \odot \mathcal{T}(\psi^{\leftarrow}(\mu_i)))$$

(Since ϕ and ψ are *LF*- continuous,)

$$\geq \bigwedge_{i\in\Gamma}(\mathcal{T}_1(\lambda_1) \odot \mathcal{T}_2(\mu_i)).$$

It is a contradiction for the equation (III).

From Theorems 3.6 and 3.7, we can obtain the following corollaries.

Corollary 3.10: Let $\{(X_i, \tau_i)\}_{i\in\Gamma}$ be a family of (2, *M*)-fuzzifying topological spaces, *X* a set and for each $i \in \Gamma$, $f_i : X \to X_i$ a map. Define a map $\beta : P(X) \to M$ on *X* by

$$\beta(A) = \bigvee_{i\in\Gamma}\{\odot_{j=1}^{n}\tau_{k_j}(B_{k_j}) \Big| A = \bigcap_{j=1}^{n} f_{k_j}^{-1}(B_{k_j})\}.$$

where *W* is taken over all finite subsets $K = \{k_1, \ldots, k_n\} \subset \Gamma$.

Then:

(1) β is a (2, *M*)-fuzzifying base on *X*.

(2) (2, *M*)-fuzzifying topology τ_β generated by β is the coarsest (2,*M*)-fuzzifying topology on *X* for which all $f_i, i \in \Gamma$, are fuzzifying continuous.

(3) A map $f : (Y, \tau') \to (X, \tau_\beta)$ is fuzzifying continuous iff for each $i \in \Gamma$, $f_i \circ f$: $(Y, \tau') \to (X_i, \tau_i)$ *is fuzzifying continuous.*

Corollary 3.11: The forgetful functor *W* : (2, *M*)-**TOP** → **Set** defined by $W(X, \tau) = X$ and $W(f) = f$ is topological.

Theorem 3.12: *Let* $(X, \mathcal{T})$ *be an* (L, M)*-topological space. We define a function* $\beta\tau : P(X) \to M$ *as follows:*

$$\beta_{\mathcal{T}}(A) = \bigvee\{\odot_{i=1}^{m}(\bigvee_{r\in\Gamma}\bigvee\{\mathcal{T}(\lambda) \,|\, \lambda \in L^X, \lambda_r = \beta_i\}) \,|\, A = \bigcap_{i=1}^{m} B_i\}.$$

Then:

(1) $b_{\mathcal{T}}$ *is a* (2, *M*)*-fuzzifying base on X.*

(2) $\mathcal{T}_{\tau_{\beta_{\mathcal{T}}}} \geq \mathcal{T}$.

Proof: (1) (B1) It is trivial.

(B2) Suppose there exist $A,B \in P(X)$ such that

$$\beta_{\mathcal{T}}(A \cap B) \not\geq \beta_{\mathcal{T}}(A) \odot \beta_{\mathcal{T}}(B).$$

By definition of $\beta_{\mathcal{T}}$ and (M3), there exist two finite families $\{A_i | A = \bigcap_{i=1}^{m} A_i\}$ and $\{B_j \mid B = \bigcap_{j=1}^{n} B_j\}$ such that

$$\beta_{\mathcal{T}}(A \cap B) \not\geq \left(\odot_{i=1}^{m}(\bigvee_{r\in\Gamma}\bigvee\{\mathcal{T}(\lambda_i) \,|\, \lambda \in L^X, (\lambda_i)_r = A_i\})\right)$$

$$\odot\left(\odot_{j=1}^{n}(\bigvee_{s\in L}\bigvee\{\mathcal{T}(\mu_j) \,|\, \lambda \in L^X, (\mu_j)_s = B_j\})\right)$$

Also, there exist $r, s \in L$ such that

$$\beta_{\mathcal{T}}(A \cap B) \not\geq \left(\odot_{i=1}^{m}(\bigvee_{r\in\Gamma}\{\mathcal{T}(\lambda_i) \,|\, \lambda \in L^X, (\lambda_i)_r = A_i\})\right)$$

$$\odot\left(\odot_{j=1}^{n}(\bigvee\{\mathcal{T}(\mu_j) \,|\, \lambda \in L^X, (\mu_j)_s = B_j\})\right)$$

On the other hand, since $A \cap B = (\bigcap_{i=1}^{m} A_i) \cap (\bigcap_{j=1}^{n} B_j) = \bigcap_{j=1}^{m} \bigcap_{j=1}^{n} (A_i \cap B_j)$, we have

$$\beta_{\mathcal{T}}(A \cap B) = \bigvee\{\odot_{i,j}(\bigvee_{r\in\Gamma}\bigvee\{\mathcal{T}(\lambda_i \wedge \mu_j) \,|\, (\lambda_i \wedge \mu_j)_r = A_i \cap B_j\})\}$$

$$\geq \{\odot_{i,j}(\bigvee_{r\in L}\bigvee\{\mathcal{T}(\lambda_i \wedge \mu_j) \big| (\lambda_i \wedge \mu_j)_r = A_i \cap B_j\})\}$$

$$\geq \{\odot_{i,j}(\bigvee_{r\in L}\bigvee\{\mathcal{T}(\lambda_i)\odot\mathcal{T}(\mu_j) \big| (\lambda_i \wedge \mu_j)_r = A_i \cap B_j\})\}$$

$$\geq \{\odot_{i=1}^{m}(\bigvee_{r\in L}\bigvee\{\mathcal{T}(\lambda_i) \big| \lambda_i \in L^X, (\lambda_i)_r = A_i\})\}$$

$$\odot\left(\odot_{j=1}^{n}(\bigvee_{r\in L}\bigvee\{\mathcal{T}(\mu_j) \big| \lambda \in L^X, (\mu_j)_r = B_j\})\right)$$

$$\geq \left(\odot_{i=1}^{m}(\bigvee\{\mathcal{T}(\lambda_i) \big| \lambda \in L^X, (\lambda_i)_r = A_i\})\right)$$

$$\odot\left(\odot_{j=1}^{n}(\bigvee\{\mathcal{T}(\mu_j) \big| \lambda \in L^X, (\mu_j)_s = B_j\})\right)$$

It is a contradiction. Thus, the condition (B2) holds.

(2) Since $\mathcal{T}_{\tau_{\beta_{\mathcal{T}}}}(\lambda) = \wedge_{r\in L}\tau_{\beta_{\mathcal{T}}}((\lambda)_r)$ for all $\lambda \in L^X$, by the definition of $\beta_{\mathcal{T}}((\lambda)_r)$, there exists a family $\{\lambda_r \mid \lambda_r \in P(X)\}$ such that $\beta_{\mathcal{T}}((\lambda)_r) \geq \mathcal{T}(\lambda)$. Hence $\mathcal{T}_{\tau_{\beta_{\mathcal{T}}}} \geq \mathcal{T}$.

Theorem 3.13: *Let τ be a (2, M)-fuzzifying topological space. Then* $\mathcal{T}_{\tau_{\beta_{\mathcal{T}}}} = \tau$.

Proof: For each $A \in P(X)$, there exists $1_A \in L^X$ such that $\wedge_{r\in L}\tau((1_A)_r) = \tau(A)$. It implies $\tau_{\beta_{\mathcal{T}_\tau}}(A) \geq \tau(A)$.

Conversely, suppose there exist $B \in P(X)$ such that

$$\tau_{\beta_{\mathcal{T}_\tau}}(B) \not\leq \tau(B).$$

By definition of $\tau_{\beta_{\mathcal{T}_\tau}}$ and (M3), there exists a family $\{B_i \mid B = \bigcup_{i\in\Gamma} Bi\}$ such that

$$\tau(B) \not\geq \bigwedge_{i\in\Gamma}\beta_{\mathcal{T}_\tau}(B_i).$$

Since M is a completely distributive lattice, for each $i \in \Gamma$, by definition of $\beta_{\mathcal{T}_\tau}(Bi)$ and (M3), there exists a finite family $\{B_{i_j} \mid B_i = \bigcap_{j=1}^{m} B_{i_j}\}$ such that

$$\tau(B) \not\geq \bigwedge_{i\in\Gamma}\left(\odot_{i=1}^{m}(\bigvee_{r\in\Gamma} \bigvee\{\mathcal{T}_\tau(\lambda_{i_j}) \Big| \lambda_{i_j} \in L^X, (\lambda_{i_j})_r = B_{i_j}\})\right)$$

Also, there exists $r \in L$ such that

$$\tau(B) \not\geq \bigwedge_{i\in\Gamma}\left(\odot_{i=1}^{m}\{\mathcal{T}_\tau(\lambda_{i_j}) \Big| \lambda_{i_j} \in L^X, (\lambda_{i_j})_r = B_{i_j}\})\right)$$

$$= \bigwedge_{i\in\Gamma}\left(\odot_{i=1}^{m}\{\bigwedge_{s\in L}\tau((\lambda_{i_j})_s \Big| \lambda_{i_j} \in L^X, (\lambda_{i_j})_r = B_{i_j}\})\right)$$

Since $\bigwedge_{s\in L}\tau((\lambda_{i_j})_s \leq \tau((\lambda_{i_j})_r = \tau(B_{i_j}),$ we have

$$\tau(B) \not\geq \bigwedge_{i\in\Gamma}\left(\odot_{i=1}^{m} \tau(B_{i_j})\right).$$

On the other hand, since $B = \bigcup_{i\in\Gamma}(\bigcap_{j=1}^{m} B_{i_j})$, we have

$$\tau(B) \geq \bigwedge_{i\in\Gamma}\left(\odot_{i=1}^{m} \tau(B_{i_j})\right).$$

It is a contradiction. Hence $\tau_{\beta_{\mathcal{T}_\tau}} \leq \tau.$

Theorem 3.14: *Let* $(X, \mathcal{T}_1)$ *and* $(Y, \mathcal{T}_2)$ *be* (L, M)*-topological spaces.*

If $\phi : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ *is LF-continuous, then* $\phi : (X, \tau_{\beta_{\mathcal{T}_1}}) \to (Y, \tau_{\beta_{\mathcal{T}_2}})$ *is fuzzifying* continuous.

Proof: For each $A \in P(X)$, we have

$$\beta_{\mathcal{T}_2}(A) = \bigvee\{\odot_{i=1}^{m}(\bigvee_{r\in L}\bigvee\{\mathcal{T}_2(\lambda_i) \Big| \lambda_i \in L^X, (\lambda_i)_r = B_i\}) \Big| A = \bigcap_{i=1}^{m} B_i\}$$

$$\leq \bigvee\{\odot_{i=1}^{m}(\bigvee_{r\in L}\bigvee\{\mathcal{T}_1(\phi^{\leftarrow}(\lambda_i)) \Big| (\phi^{\leftarrow}(\lambda_i))_r = \phi^{\leftarrow}(B_i)\}) \Big| \phi^{\leftarrow}(A) = \bigcap_{i=1}^{m} \phi^{\leftarrow}(B_i)\}$$

$$\leq \beta_{\mathcal{T}_1}(\phi^{\leftarrow}(A)).$$

It implies $\tau_{\beta_{\mathcal{T}_2}}(A) \leq \tau_{\beta_{\mathcal{T}_1}}(\phi^{\leftarrow}(A))$.

From Theorems 2.4 and 2.6, a functor $G : (2, M)$-**TOP** $\to (L, M)$-**TOP** is defined by $G(X, \tau) = (X, \mathcal{T}_\tau)$ and $G(f) = f$. From Theorems 3.12 and 3.14, a functor $H : (L, M)$-**TOP** $\to (2, M)$-**TOP** is defined by $H(X, \mathcal{T}) = (X, \tau_{\beta_{\mathcal{T}}})$ and $H(\phi) = \phi$.

Theorem 3.15: *A functor* $H : (L, M)$-**TOP** $\to (2, M)$-**TOP** *is a left adjoint of the functor G.*

Proof: For each $(X, \mathcal{T}) \in (L, M)$-**TOP**, since $G \circ H(\mathcal{T}) = \mathcal{T}_{\tau_{\beta_{\mathcal{T}}}} \geq \mathcal{T}$ from Theorem 3.12(2), then $1_X : (X, \mathcal{T}) \to (X, G \circ H(\mathcal{T}))$ is *LF*-continuous. In fact, 1_X is the universal map for $(X, \mathcal{T})$. Let $\phi : (X, \mathcal{T}) \to G(Y, \tau)$ be a morphism in (L, M)-**TOP**. Then $\phi = H(\phi) : (X, \mathcal{T}\beta_{\mathcal{T}}) \to (Y, \tau) = H \circ G(Y, \tau)$ is fuzzifying continuous. Hence the result follows.

We may consider $(2, M)$-TOP as a bireflective subcategory of (L, M)-**TOP**.

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