International Review of Fuzzy Mathematics
Vol. 4 No. 1 (January-June, 2019)

Received: 04th October 2018 Revised: 30th October 2018 Accepted: 15th November 2018

Reviewer: Dumitru Busneag

Kyoung Ja Lee & Chul Hwan Park

A NOTE ON INTUITIONISTIC FUZZY LIE IDEALS OF LIE ALGERBAS

ABSTRACT: *The notion of intuitionistic fuzzy Lie ideal of a Lie algebra is introduced, and related properties are investigated in [6]. In this paper, further properties on intuitionistic fuzzy Lie ideal of a Lie algebra are investigated. Natural equivalence relations on the set of all intuitionistic fuzzy Lie ideals are investigated.*

2000 Mathematics Subject Classification: *17B99, 03G25, 03F55, 08A72.*

Keywords: *Lie algebra, Lie ideal, intuitionistic fuzzy Lie ideal.*

1. INTRODUCTION

Fuzzy set theory formulated by Zadeh [9] in 1965 has evoked tremendous interest among researchers from all disciplines such as weather forecasting, linguistics, economics, computer sciences, operations research, graph theory, topological spaces, algebraic structures and so on. There is hardly any field in which FST cannot be applied. The fact that algebraic structures occupy a prominent place in mathematics with manifold applications in various disciplines, for example, theoretical physics, information sciences, coding theory, to name just a few, along with the fact that FST offers greater richness in application than the ordinary set theory motivates one to study various concepts/results of algebra in the broader framework of the fuzzy setting. Yehia [7, 8] introduced fuzzy sets in the realm of Lie algebra. He/She introduced the notion of fuzzy ideal and fuzzy Lie subalgebra in Lie algebras, and discussed several useful results. He/She also constructed fuzzy quotient Lie algebra by using fuzzy ideals. Jun *et al.* [5] discussed useful properties in fuzzy setting of a Lie algebra. Using a collection of ideals with additional properties, they established a fuzzy ideal. With relation to the ascending chain of ideals, they also stated a characterization for the set of values of any fuzzy ideal to be a well-ordered subset of the closed unit interval [0, 1]. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets (IFSs) introduced by Atanassov [1] is one among them. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of non-membership, and furthermore the sum of these two degrees is not greater than 1. For more details on intuitionistic fuzzy sets, we refer the reader to [1, 2]. Since then, a great number of theoretical and practical results appeared in the area of IFSs. There are numerical applications of IFSs in various areas of computer science, for example, in artificial intelligence, as well as in medicine, chemistry, economics, astronomy, etc. In the paper [6], Jun and Park applied the concept of an intuitionistic fuzzy set to Lie ideals in Lie algebras. They introduced the notion of an intuitionistic fuzzy Lie ideal of a Lie algebra, and investigated some related properties. They also gave characterizations of an intuitionistic fuzzy Lie ideal, and made an intuitionistic fuzzy Lie ideal by using a collection of Lie ideals. In this paper we investigate further properties on intuitionistic fuzzy Lie ideals of a Lie algebra. We discuss natural equivalence relations on the set of all intuitionistic fuzzy Lie ideals

2. PRELIMINARIES

A vector space L over a field F, with an operation $\mathscr{L} \times \mathscr{L} \alpha \mathscr{L}$, denoted $(x, y) \mapsto [xy]$ and called the bracket or commutator of x and y, is called a *Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket operation is bilinear.

(L2) $[xx] = 0$ for all $x \in \mathscr{L}$.

(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for all $x, y, z \in L$.

A subspace K of a Lie algebra $\mathscr{L}$ is called a *Lie ideal* of $\mathscr{L}$ if $x \in \mathscr{L}$, $y \in K$ together imply $[xy] \in K$.

A mapping $\mu : \mathscr{L} \alpha [0, 1]$, where $\mathscr{L}$ is an arbitrary nonempty set, is called a *fuzzy set* in $\mathscr{L}$. For any fuzzy set μ in $\mathscr{L}$ and any $t \in [0, 1]$ we define two sets

$U(\mu; t) = \{x \in \mathscr{L} \mid \mu(x) \geq\geq t\}$ and $L(\mu; t) = \{x \in \mathscr{L} \mid \mu(x) \leq t\}$,

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ.

A fuzzy set μ in a Lie algebra $\mathscr{L}$ over a field F is called a *fuzzy Lie ideal* of $\mathscr{L}$ if it satisfies

(1) $(\forall x, y \in \mathscr{L})\ (\mu(x+y) \geq \min\{\mu(x), \mu(y)\})$.

(2) $(\forall x \in L)\ (\forall r \in F)\ (\mu(rx) \geq \mu(x))$.

(3) $(\forall x, y \in L)\ (\mu([xy]) \geq \mu(y))$.

As an important generalization of the notion of fuzzy sets in $\mathscr{L}$, Atanassov [1, 2] introduced the concept of an *intuitionistic fuzzy set* (IFS for short) defined on a nonempty set $\mathscr{L}$ as objects having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in \mathscr{L}\},$$

where the functions $\mu_A : \mathscr{L} \to [0, 1]$ and $\gamma_A : \mathscr{L} \to [0, 1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\gamma_A(x)$) of each element $x \in \mathscr{L}$ to the set A respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \tag{2.1}$$

for each $x \in \mathscr{L}$. For the sake of simplicity, we shall use the symbol $A = \langle \mathscr{L}, \mu_A, \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in \mathscr{L}\}$. Obviously, every fuzzy set A' corresponds to the following intuitionistic fuzzy set:

$$A' = \{\langle x, \alpha_{A'}(x), 1 - \alpha_{A'}(x)\rangle \mid x \in L\}. \tag{2.2}$$

Obviously, for an IFS $A = \langle \mathscr{L}, \mu_A, \gamma_A\rangle$ in $\mathscr{L}$, when

$$\gamma_A(x) = 1 - \mu_A(x), \text{ that is, } \mu_A(x) + \gamma_A(x) = 1 \tag{2.3}$$

for every $x \in \mathscr{L}$, the IFS A is a fuzzy set. Hence the notion of intuitionistic fuzzy set theory is a generalization of fuzzy set theory. Let $A = \langle \mathscr{L}, \mu_A, \gamma_A\rangle$ be an IFS in $\mathscr{L}$ and let $s, t \in [0, 1]$ be such that $s + t \leq 1$. Then the set

$$\mathscr{L}_A^{(s,t)} := \{x \in \mathscr{L} \mid \mu_A(x) \geq s, \gamma_A(x) \leq t\}$$

is called an *(s, t)-level subset* of $A = \langle L, \mu_A, \gamma_A\rangle$. Note that

$$\begin{aligned} \mathscr{L}_A^{(s,t)} &= \{x \in \mathscr{L} \mid \mu_A(x) \geq s, \gamma A(x) \leq t\} \\ &= \{x \in \mathscr{L} \mid \mu_A(x) \geq s\} \bigcap \{x \in X \mid \gamma_A(x) \leq t\} \\ &= U(\mu A; s) \bigcap L(\gamma A; t). \end{aligned}$$

3. INTUITIONISTIC FUZZY LIE IDEALS OF LIE ALGEBRAS

In what follows, let L denote a Lie algebra over a field F unless otherwise specified.

Definition 3.1: [6] An IFS $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ in $\mathscr{L}$ is an *intuitionistic fuzzy Lie ideal* of $\mathscr{L}$ if it satisfies the following conditions for all $x, y \in \mathscr{L}$ and $r \in F$.

(I1) $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\gamma_A(x + y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(I2) $\mu_A(rx) \geq \mu_A(x)$, $\gamma_A(rx) \leq \gamma_A(x)$,

(I3) $\mu_A([xy]) \geq \mu_A(y)$, $\gamma_A([xy]) \leq \gamma_A(y)$.

Note that if an IFS $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$, then every upper (resp. lower) t-level cut of μ_A (resp. γ_A) is a Lie ideal of $\mathscr{L}$, which is called a μ_A-upper level Lie ideal (resp. γ_A-lower level Lie ideal) of A.

Example 3.2: [6] Let $\mathscr{L}$ be the real vector space R^3. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in \mathscr{L}$. Then $\mathscr{L}$ is a Lie algebra (see [4, p. 5]). Define an IFS $A = \langle L, \mu_A, \gamma_A \rangle$ in $\mathscr{L}$ by

$$\mu_A\big((x_1, x_2, x_3)\big) = \begin{cases} t \in (0,1] & \text{if} \quad x_1 = x_2 = x_3 = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

and

$$\gamma_A\big((x_1, x_2, x_3)\big) = \begin{cases} s \in (0,1] & \text{if} \quad x_1 = x_2 = x_3 = 0, \\ 1 & \text{otherwise,} \end{cases} \tag{3.2}$$

where $s + t \in 1$. Then $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$.

Example 3.3: [6] Let K_1 and K_2 be two Lie ideals of L. Define an IFS $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ in $\mathscr{L}$ by

$$\mu_A(x) = \begin{cases} t_1 & \text{if} \;\; x = a + b \text{ for some } a \in K_1 \text{ and } b \in K_2, \\ t_2 & \text{otherwise,} \end{cases} \tag{3.3}$$

and

$$\gamma_A(x) = \begin{cases} s_1 & \text{if} \;\; x = a + b \text{ for some } a \in K_1 \text{ and } b \in K_2, \\ s_2 & \text{otherwise,} \end{cases} \tag{3.4}$$

for any $x \in \mathscr{L}$ where $t_1 > t_2$, $s_1 < s_2$ in [0, 1] and $t_i + s_i \leq 1$ for $i = 1, 2$. Then $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$.

Theorem 3.4: *Consider a chain of Lie ideals of* $\mathscr{L}$

$$P_0 \subset P_1 \subset \ldots \subset P_n = \mathscr{L}.$$

Then there exists an intuitionistic fuzzy Lie ideal of $\mathscr{L}$ *whose upper and lower level Lie ideals are exactly the Lie ideals in the above chain.*

Proof: Let $\{\alpha_k \mid k = 0, 1, \ldots, n\}$ (resp. $\{\beta_k \mid k = 0, 1, \ldots, n\}$) be a finite decreasing (resp. increasing) sequence in [0, 1] such that $\alpha_i + \beta_i \leq 1$ for $i = 0, 1, \ldots, n$. Let $A = \langle \mathscr{L}, \mu_A, \gamma A \rangle$ be an IFS in $\mathscr{L}$ defined by $\mu_A(P_0) = \alpha_0$, $\gamma_A(P_0) = \beta 0$, $\mu_A(Pk \setminus P_{k-1}) = \alpha_k$ and $\gamma_A(P_k \setminus P_{k-1}) = \beta_k$ for $0 < k \leq n$. Let $x, y \in L$. If $x, y \in P_k \setminus P_{k-1}$, then $x + y \in P_k$, $rx \in P_k$ for all $r \in F$, $\mu_A(x) = \alpha_k = \mu_A(y)$ and $\gamma_A(x) = \beta_k = \gamma_A(y)$. It follows that

$$\mu_A(x + y) \geq \alpha_k = \min\{\mu_A(x), \mu_A(y)\}, \; \mu_A(rx) \geq \alpha_k = \mu_A(x)$$

and

$$\gamma_A(x + y) \leq \beta_k = \max\{\gamma_A(x), \gamma_A(y)g, \; \gamma_A(rx) \leq \beta_k = \gamma_A(x).$$

For $i > j$, if $x \in P_i \setminus P_{i-1}$ and $y \in P_j \setminus P_{j-1}$, then $\mu_A(x) = \alpha_i < \alpha_j = \mu_A(y)$,

$\gamma_A(x) = \beta_i > \beta_j = \gamma_A(y)$, $x + y \in P_i$ and $rx \in P_i$ for all $r \in F$. Hence

$$\mu_A(x + y) \geq \alpha_i = \min\{\mu_A(x), \mu_A(y)\}, \; \mu_A(rx) \geq \alpha_i = \mu_A(x)$$

and

$$\gamma_A(x + y) \leq \beta_i = \max\{\gamma_A(x), \gamma_A(y)\}, \; \gamma_A(rx) \leq b_i = \gamma_A(x).$$

Similarly, for $i > j$, if $x \in P_j \setminus P_{j-1}$ and $y \in P_i \setminus P_{i-1}$, then we have the same results. Now, if $y \in P_k \setminus P_{k-1}$, then $[xy] \in P_k$ for all $x \in \mathscr{L}$. Hence

$$\mu_A([xy]) \geq \alpha_k = \mu_A(y) \text{ and } \gamma_A([xy]) \leq \beta_k = \gamma_A(y).$$

Therefore $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy Lie ideal of $\mathscr{L}$. Note that

$$\mathrm{Im}(\mu_A) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \text{ and } \mathrm{Im}(\gamma_A) = \{\beta_0, \beta_1, \ldots, \beta_n\}.$$

It follows that the μ_A-upper level Lie ideals and the γA-lower level Lie ideals of $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ are given by the chain of Lie ideals

$$U(\mu_A; \alpha_0) \subset U(\mu_A; \alpha_1) \subset \cdots \subset U(\mu_A; \alpha_n) = \mathscr{L}$$

and

$$L(\gamma_A; \beta_0) \subset L(\gamma_A; \beta_1) \subset \cdots \subset L(\gamma_A; \beta_n) = \mathscr{L},$$

respectively. Obviously, we have

$$U(\mu_A; \alpha_0) = \{x \in \mathscr{L} \mid \mu_A(x) \geq \alpha_0\} = P_0$$

and

$$L(\gamma_A; \beta_0) = \{x \in \mathscr{L} \mid \gamma_A(x) \leq \beta_0\} = P_0.$$

We now prove that $U(\mu_A; \alpha_k) = P_k = L(\gamma_A; \beta_k)$ for $0 < k \leq n$. Clearly $P_k \subseteq U(\mu_A; \alpha_k)$ and $P_k \,\mu\, L(\gamma_A; b_k)$. If $x \in U(\mu_A; \alpha_k)$, then $\mu_A(x) \geq \alpha_k$ and so $x = \notin P_i$ for $i > k$. Hence $\mu_A(x) \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, which implies $x \in P_j$ for some $j \leq k$. Since $P_j \subseteq P_k$, it follows that $x \in P_k$. Consequently, $U(\mu_A; \alpha_k) = P_k$ for $0 \leq k \leq n$. Now if $y \in L(\gamma_A; \beta_k)$, then $\gamma_A(y) \leq \beta_k$ and thus $y = \notin P_i$ for $i > k$. Hence $\gamma_A(y) \in \{\beta_1, \beta_2, \ldots, \beta_k\}$ and so $y \in P_j$ for some $j \leq k$. Since $P_j \subseteq P_k$, we have $y \in P_k$. Therefore $L(\gamma_A; \beta_k) = P_k$ for $0 \leq k \leq n$. This completes the proof.

Theorem 3.5: *Let $\mathscr{L}$ be a Lie algebra in which every descending chain* of *Lie ideals terminates at finite step. For an intuitionistic fuzzy Lie ideal $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ of $\mathscr{L}$, if a sequence of elements of* Im(A) *is strictly intuitionistic increasing, that is, a sequence of elements of Im(μ_A) is strictly increasing and a sequence of elements of* Im(γ_A) *is strictly decreasing, then $A = \subseteq \mathscr{L}, \mu_A, \gamma_A \rangle$ has finite number of intuitionistic values, that is, μ_A and γ_A have finite number of values.*

Proof: Suppose that Im(μ_A) is not finite. Let $\{\alpha_n\}$ be a strictly increasing sequence of elements of Im(μ_A). Then $0 \leq \alpha_1 < \alpha_2 < \cdots \leq 1$ and

$$U(\mu_A; t) := \{x \in X \mid \mu_A(x) \geq \alpha_t\}$$

is a Lie ideal of $\mathscr{L}$ for $t = 1, 2, \cdots$. Let $x \in U(\mu_A; t)$. Then $\mu_A(x) \geq \alpha_t > \alpha_{t-1}$, which implies that $x \in U(\mu_A; t-1)$. Hence $U(\mu_A; t) \subseteq U(\mu_A; t-1)$. Since $\alpha_{t-1} \in \mathrm{Im}(\mu_A)$, there exists $x_{t-1} \in \mathscr{L}$ such that $\mu_A(x_{t-1}) = \alpha_{t-1}$. It follows that $x_{t-1} \in U(\mu_A; t-1)$, but $x_{t-1} = \notin U(\mu_A; t)$. Thus $U(\mu_A; t)$ is a proper subset of $U(\mu_A; t-1)$, and so we obtain a strictly descending chain

$$U(\mu_A; 1) \supset U(\mu_A; 2) \supset U(\mu_A; 3) \supset \cdots$$

of Lie ideals of $\mathscr{L}$ which is not terminating. This is a contradiction. Now assume that Im(γA) is not finite. Let $\{\beta_n\}$ be a strictly decreasing sequence of elements of Im(γ_A). Then $1 \geq \beta_1 > \beta_2 > \beta_3 > \cdots \geq 0$. Note that

$$L(\gamma_A; k) := \{x \in \mathscr{L} \mid \gamma_A(x) \leq \beta_k\}$$

is a Lie ideal of $\mathscr{L}$ for $k = 1, 2, \cdots$. If $y \in L(\gamma_A; k)$, then $\gamma_A(y) \le \beta_k < \beta_{k-1}$ and so $y \in L(\gamma_A; k-1)$. This shows that $L(\gamma_A; k) \subseteq L(\gamma_A; k-1)$. Since $\beta_{k-1} \in \mathrm{Im}(\gamma_A)$, we have $\gamma_A(y_{k-1}) = \beta_{k-1}$ for some $y_{k-1} \in L$. Hence $y_{k-1} \in \mathscr{L}(\gamma_A; k-1)$, but $y_{k-1} \notin L(\gamma_A; k)$. Therefore $L(\gamma_A; k)$ is a proper subset of $L(\gamma_A; k-1)$, and thus we get a strictly descending chain

$$L(\gamma_A; 1) \supset L(\gamma_A; 2) \supset L(\gamma_A; 3) \supset \cdots$$

of Lie ideals of $\mathscr{L}$ which is not terminating. This is impossible, and the proof is complete.

Let $\alpha \in [0, 1]$ be fixed and let $IF(\mathscr{L})$ denote the family of all intuitionistic fuzzy Lie ideals of $\mathscr{L}$. For any $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle \mathscr{L}, \mu_B, \gamma_B\rangle$ from $IF(\mathscr{L})$ we define two binary relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on $IF(\mathscr{L})$ as follows:

$$(A, B) \in \mathfrak{U}^\alpha \Longleftrightarrow U(\mu_A; \alpha) = U(\mu_B; \alpha)$$

and

$$(A, B) \in \mathrm{L}\alpha \Longleftrightarrow L(\gamma_A; \alpha) = L(\gamma_B; \alpha).$$

These two relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ are equivalence relations. Hence $IF(\mathscr{L})$ can be divided into the equivalence classes of $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = \langle \mathscr{L}, \mu_A, \gamma_A\rangle \in IF(\mathscr{L})$, respectively. The corresponding quotient sets will be denoted by $IF(\mathscr{L})=\mathfrak{U}^\alpha$ and $IF(\mathscr{L})=\mathfrak{L}^\alpha$, respectively.

For the family $\mathfrak{I}(\mathscr{L})$ of all Lie ideals of $\mathscr{L}$ we define two maps U_α and L_α from $IF(\mathscr{L})$ to $\mathfrak{I}(\mathscr{L}) \cup \{\emptyset\}$ by putting

$$U_\alpha(A) = U(\mu_A; \alpha) \text{ and } L_\alpha(A) = L(\gamma_A; \alpha)$$

for each $A = \langle \mathscr{L}, \mu_A, \gamma_A\rangle \in IF(\mathscr{L})$.

It is not difficult to see that these maps are well-defined.

Lemma 3.6: *For any* $\alpha \in (0, 1)$ *the maps* U_α *and* L_α *are surjective.*

Proof: Note that $\mathbf{0}_\sim = \langle \mathscr{L}, \mathbf{0}, \mathbf{1}\rangle \in IF(\mathscr{L})$ and $U_\alpha(0_\sim) = L_\alpha(\mathbf{0}_\sim) = \emptyset$ for any $\alpha \in (0, 1)$. Moreover for any $K \in \mathfrak{I}(\mathscr{L})$ we have $K_\sim = \left\langle \mathscr{L}, \mathcal{X}_K, \bar{\mathcal{X}}_K \right\rangle \in IF(\mathscr{L})$, $U_\alpha(K_\sim) = U(\mathcal{X}_K; \alpha) = K$ and $L_\alpha(K_\sim) = L(\bar{\mathcal{X}}_K; \alpha) = K$ for any $\alpha \in (0, 1)$. Hence U_α and L_α are surjective.

Theorem 3.7: *For any* $\alpha \in (0, 1)$ *the sets* $IF(\mathscr{L})=\mathfrak{U}^\alpha$ *and* $IF(\mathscr{L})=\mathfrak{L}^\alpha$ *are equipotent to* $\mathfrak{I}(\mathscr{L}) \cup \{\emptyset\}$.

Proof: Let $\alpha \in (0, 1)$. Putting $U^*_\alpha([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A)$ and $L^*_\alpha([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A)$ for any $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle \in IF(\mathscr{L})$, we obtain two maps

$$U^*_\alpha : IF(\mathscr{L})/\mathfrak{U}^\alpha \to \mathfrak{I}(\mathscr{L}) \cup \{\emptyset\} \text{ and } L^*_\alpha : IF(\mathscr{L})/\mathfrak{L}^\alpha \to \mathfrak{I}(\mathscr{L}) \cup \{\emptyset\}$$

If $U(\mu_A; \alpha) = U(\mu_B; \alpha)$ and $L(\gamma_A; \alpha) = L(\gamma_B; \alpha)$ for some $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ and $B = \langle \mathscr{L}, \mu_B, \gamma_B \rangle$ from $IF(\mathscr{L})$, then $(A,B) \in \mathfrak{U}^\alpha$ and $(A,B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$, which means that U^*_α and L^*_α are injective. To show that the maps U^*_α and L^*_α are surjective, let $K \in \mathfrak{I}(\mathscr{L})$. Then for $K\sim = \left\langle \mathscr{L}, \mathcal{X}_K, \bar{\mathcal{X}}_K \right\rangle \in IF(\mathscr{L})$ we have $U^*_\alpha([K\sim]_{\mathfrak{U}^\alpha}) = U(\mathcal{X}_K; \alpha) = K$ and $L^*_\alpha([K\sim]_{\mathfrak{L}^\alpha}) = L(\bar{\mathcal{X}}_K; \alpha) = K$. Also $\mathbf{0}\sim = \langle \mathscr{L}, \mathbf{0}, \mathbf{1} \rangle \in IF(\mathscr{L})$. Moreover $U^*_\alpha([\mathbf{0}\sim]_{\mathfrak{U}^\alpha}) = U(\mathbf{0}; \alpha) = \emptyset$ and $L^*_\alpha([\mathbf{0}\sim]_{\mathfrak{L}^\alpha}) = L(\mathbf{1}; \alpha) = \emptyset$. Hence U^*_α and L^*_α are surjective. This completes the proof.

Now for any $\alpha \in [0, 1]$ we define a new relation $\mathfrak{R}^\alpha$ on $IF(\mathscr{L})$ by putting:

$$(A, B) \in \mathfrak{R}^\alpha \Longleftrightarrow U(\mu_A; \alpha) \cap L(\gamma_A; \alpha) = U(\mu_B; \alpha) \cap L(\gamma_B; \alpha),$$

where $A = \langle \mathscr{L}, \mu_A, \gamma_A \rangle$ and $B = \langle \mathscr{L}, \mu_B, \gamma_B \rangle$ are from $IF(\mathscr{L})$. Obviously $\mathfrak{R}^\alpha$ is an equivalence relation.

Lemma 3.8: *The map* $I_\alpha : IF(\mathscr{L}) \to \mathfrak{I}(\mathscr{L}) \cup \{\emptyset\}$ *defined by*

$$I_\alpha(A) = U(\mu_A; \alpha) \cap L(\gamma_A; \alpha),$$

where $A = \langle L, \mu_A, \gamma_A \rangle \in IF(\mathscr{L})$, *is surjective for any* $\alpha \in (0, 1)$.

Proof: If $\alpha \in (0, 1)$ is fixed, then for $\mathbf{0}\sim = \langle L, \mathbf{0}, \mathbf{1} \rangle \in IF(\mathscr{L})$ we have

$$I_\alpha(\mathbf{0}\sim) = U(\mathbf{0}; \alpha) \cap L(\mathbf{1}; \alpha) = \emptyset,$$

and for any $K \in \mathfrak{I}(\mathscr{L})$ there exists $K\sim = \left\langle \mathscr{L}, \mathcal{X}_K, \bar{\mathcal{X}}_K \right\rangle \in IF(\mathscr{L})$ such that $I_\alpha(K\sim) = U(\mathcal{X}_K; \alpha) \cap L(\bar{\mathcal{X}}_K; \alpha) = K$.

Theorem 3.9: *For any* $\alpha \in (0, 1)$ *the quotient set* $IF(\mathscr{L})/\mathfrak{R}^\alpha$ *is equipotent to* $\mathfrak{I}(\mathscr{L}) \cup \{\emptyset\}$.

Proof: Let $I^*_\alpha : IF(\mathscr{L})/\mathfrak{R}^\alpha \to \mathfrak{I}(\mathscr{L}) \cup \{\emptyset\}$, where $\alpha \in (0, 1)$, be defined by the formula:

$$I^*_\alpha([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A) \text{ for each } [A]_{\mathfrak{R}^\alpha} \in IF(\mathscr{L})/\mathfrak{R}^\alpha.$$

If $I^*_\alpha([A]_{\mathfrak{R}^\alpha}) = I^*_\alpha([B]_{\mathfrak{R}^\alpha})$ for some $[A]_{\mathfrak{R}^\alpha}, [B]_{\mathfrak{R}^\alpha} \in IF(\mathscr{L})/\mathfrak{R}^\alpha$, then

$$U(\mu_A; \alpha) \cap L(\gamma_A; \alpha) = U(\mu_B; \alpha) \cap L(\gamma_B; \alpha),$$

which implies $(A,B) \in \mathfrak{R}^{\alpha}$ and, in the consequence, $[A]_{\mathfrak{R}\alpha} = [B]_{\mathfrak{R}\alpha}$. Thus I_{α}^{*} is injective. It is also onto because $I_{\alpha}^{*}(0\sim) = I\alpha(0\sim) = \mathbf{0}$ for $0\sim = \langle \mathscr{L}, \mathbf{0}, \mathbf{1}\rangle \in IF(\mathscr{L})$, and $I_{\alpha}^{*}(K\sim) = I_{\alpha}(K) = K$ for $K \in \mathfrak{I}(\mathscr{L})$ and $K\sim = \left\langle \mathscr{L}, \mathcal{X}_K, \overline{\mathcal{X}}_K \right\rangle \in IF(\mathscr{L})$. This completes the proof.

REFERENCES

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986), 87–96.

[2] K. T. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems **61** (1994), 137–142.

[3] K. T. Atanassov, *Intuitionistic fuzzy sets. Theory and applications*, Studies in Fuzziness and Soft Computing, **35**. Heidelberg; Physica-Verlag 1999.

[4] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory,* Springer-Verlag, New York, 1972.

[5] Y. B. Jun, K. H. Kim and E. H. Roh, *On fuzzy ideals of Lie algebras,* J. Appl. Math. Computing **10** (2002), no. 1–2, 251–259.

[6] Y. B. Jun and C. H. Park, *Intuitionistic fuzzy Lie ideals of Lie algebras,* Honam Math. J. (to appear).

[7] S. E. Yehia, *Fuzzy ideals and fuzzy subalgebras of Lie algebras,* Fuzzy Sets and Systems **80** (1996), 237–244.

[8] S. E. Yehia, *The adjoint representation of fuzzy Lie algebras,* Fuzzy Sets and Systems **119** (2001), 409–417.

[9] L. A. Zadeh, *Fuzzy sets,* Inform. Control 8 (1965), 338–353.

K. J. Lee
School of General Education
Kookmin University
Seoul 136-702, Korea
E-mail: *lsj1109@kookmin.ac.kr*

C. H. Park
Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail: *skyrosemary@gmail.com*