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FUZZY ESSENCES OF SUBTRACTION ALGEBRAS

ABSTRACT: *The notion of fuzzy essences in subtraction algebras is introduced, and several properties are investigated. Relations between fuzzy subalgebra, fuzzy ideals, and fuzzy essences are considered.*

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1. INTRODUCTION

B. M. Schein [7] considered systems of the form (Φ; o, \), where Φ is a set of functions closed under the composition "o" of functions (and hence (Φ; o) is a function semigroup) and the set theoretic subtraction "\" (and hence (Φ; \) is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [8] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [5] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In this paper, we introduce the notion of fuzzy essences in subtraction al-gebras, and investigate several properties. We consider relations between fuzzy subalgebra, fuzzy ideals, and fuzzy essences.

2. PRELIMINARIES

By a *subtraction algebra* we mean an algebra $(X;-)$ with a single binary operation "–" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \le b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X;\cdot)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$b \vee c = (b' \wedge c')' = a - ((a - b) \wedge (a - c))$$

$$= a - ((a - b) - ((a - b) - (a - c))).$$

In a subtraction algebra, the following are true (see [5]):

(a1) $(x - y) - y = x - y$.

(a2) $x - 0 = x$ and $0 - x = 0$.

(a3) $(x - y) - x = 0$.

(a4) $x - (x - y) \le y$.

(a5) $(x - y) - (y - x) = x - y$.

(a6) $x - (x - (x - y)) = x - y$.

(a7) $(x - y) - (z - y) \le x - z$.

(a8) $x \le y$ if and only if $x = y - w$ for some $w \in X$.

(a9) $x \le y$ implies $x - z \le y - z$ and $z - y \le z - x$ for all $z \in X$.

(a10) $x, y \le z$ implies $x - y = x \wedge (z - y)$.

(a11) $(x \wedge y) - (x \wedge z) \le x \wedge (y - z)$.

Definition 2.1: [5] A nonempty subset A of a subtraction algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:

(b1) $a - x \in A$ for all $a \in A$ and $x \in X$.

(b2) for all $a, b \in A$, whenever $a \lhd b$ exists in X then $a \vee b \in A$.

Proposition 2.2: [5] A nonempty subset A of a subtraction algebra X *is* an ideal of X if and only if it satisfies:

(b3) $0 \in A$,

(b4) $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Lemma 2.3: An ideal A of a subtraction algebra X has the following property:

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

Lemma 2.4: [4] Let X be a subtraction algebra. If $0 \in H \subseteq X$, then

$$(\forall G \subseteq X)(G \subseteq G - H).$$

Definition 2.5: [4] If a nonempty subset K of a subtraction algebra X satisfies $K - X = K$, then we say that K is an essence of X

3. FUZZY ESSENCES

In what follows let X be a subtraction algebra unless otherwise specified. For any subsets G and H of X, we define

$$G - H := \{x - y \mid x \in G, y \in H\}.$$

Lemma 3.1: [4] For any subsets A, B and E of X, we have

(i) $A \subseteq B \Rightarrow A - E \subseteq B - E, E - A \subseteq E - B$.

(ii) $(A \cap B) - E \subseteq (A - E) \cap (B - E)$.

(iii) $E - (A \cap B) \subseteq (E - A) \cap (E - B)$.

(iv) $(A \cup B) - E = (A - E) \cup (B - E)$.

(v) $E - (A \cup B) = (E - A) \cup (E - B)$.

For any $\alpha \in [0, 1]$, we know $U(\mathscr{A}; \alpha) = \{x \in X\} \mathscr{A}(x) \geq \alpha\}$([6]).

Definition 3.2: A fuzzy set $\mathscr{A}$ in X is called a *fuzzy essence* of X if it satisfies:

$$(\forall\ \alpha \in [0, 1])\ (U(\mathscr{A}; a) \neq \emptyset \Rightarrow U(\mathscr{A}; \alpha) - X = U(\mathscr{A}; \alpha)). \qquad (3.1)$$

Example 3.3: Let $\mathscr{A}$ be a fuzzy set in X given by

$$\mathscr{A}(x) = \begin{cases} m & \text{if } x = 0, \\ n & \text{otherwise} \end{cases}$$

for all $x \in X$, where $m, n \in [0, 1]$ with $m > n$. Then $\mathscr{A}$ is a fuzzy essence of X

Example 3.4: Let $X = \{0, a, b, c\}$ be a set with the following Cayley table.

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then X is a subtraction algebra.

(i) Let $\mathscr{A}$ be a fuzzy set in X given by

$$\mathscr{A}(x) = \begin{cases} m & \text{if } x \in \{0, a\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in X$, where $m, n \in [0, 1]$ with $m > n$. Then $\mathscr{A}$ is a fuzzy essence of X.

(ii) Let $\mathscr{B}$ be a fuzzy set in X given by

$$\mathscr{B}(x) = \begin{cases} m & \text{if } x \in \{0, a, b\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in X$, where $m, n \in [0, 1]$ with $m > n$. Then $\mathscr{B}$ is a fuzzy essence of X.

(iii) Let $\mathscr{C}$ be a fuzzy set in X given by

$$\mathscr{C}(x) = \begin{cases} m & \text{if } x \in \{0, c\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in X$, where $m, n \in [0, 1]$ with $m > n$. Then $\mathscr{C}$ is not a fuzzy essence of X since $U(\mathscr{C}; m) - X = \{0, c\} - X = X \neq U(\mathscr{C}; m)$.

Lemma 3.5: [4] Let G be an essence of X. Then

$$(\forall x \in X)\ (\forall a \in G)\ (x \leq a \Rightarrow x \in G).$$

Theorem 3.6: *Let G be an essence of X. For every $a \in X \setminus G$, define a fuzzy set* $\mathscr{A}_a$ on X by

$$\mathscr{A}_a(x) = \begin{cases} \alpha & \text{if } x \in \{y \in X \mid y - a \in G\}, \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$, *where* $\alpha > \beta$ *in* $[0, 1]$. *Then* $\mathscr{A}_a$ *is a fuzzy essence of* X.

Proof: Let $\gamma \in [0, 1]$. If $\gamma > \alpha$, then $U(\mathscr{A}_a; \gamma) = \emptyset$. If $\beta < \gamma \leq \alpha$, then $U(\mathscr{A}_a; \gamma) = \{y \in X \mid y - a \in G\}$. Let $y \in U(\mathscr{A}_a; \gamma)$ and $x \in X$. Then $(y - x) - a = (y - a) - x \leq y - a$. Since $y - a \in G$ and G is an essence, it follows from Lemma 3.5 that $(y - x) - a \in G$ so that $y - x \in U(\mathscr{A}_a; \gamma)$. This shows that $U(\mathscr{A}_a; \gamma) - X \subseteq U(\mathscr{A}_a; \gamma)$. The reverse inclusion follows from Lemma 2.4. Hence $U(\mathscr{A}_a; \gamma) - X = U(\mathscr{A}_a; \gamma)$. If $\gamma \leq \beta$, then $U(\mathscr{A}_a; \gamma) = X$ and thus $U(\mathscr{A}_a; \gamma) - X = U(\mathscr{A}_a; \gamma)$. Therefore $\mathscr{A}_a$ is a fuzzy essence of X.

Theorem 3.7: *For any* $a \in X$, *let* $\mathscr{B}$ *be a fuzzy set in* X *given by*

$$\mathscr{B}(x) = \begin{cases} \alpha & \text{if } x \leq a, \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$, *where* $\alpha > \beta$ *in* $[0, 1]$. *Then* $\mathscr{B}$ *is a fuzzy essence of* X.

Proof: Let $\gamma \in [0, 1]$. If $\gamma > \alpha$, then $U(\mathscr{B}; \gamma) = \emptyset$. If $\beta < \gamma \leq \alpha$, then $U(\mathscr{B}; \gamma) = \{x \in X \mid x \leq a\}$. Let $x \in U(\mathscr{B}; \gamma)$ and $y \in X$. Then $x \leq a$, and so $x - y \leq a - y \leq a$. Hence $x - y \in U(\mathscr{B}; \gamma)$, which shows that $U(\mathscr{B}; \gamma) - X \subseteq U(\mathscr{B}; \gamma)$. The reverse inclusion follows from Lemma 2.4. Hence $U(\mathscr{B}; \gamma) - X = U(\mathscr{B}, \gamma)$. If $\gamma \leq \beta$, then clearly $U(\mathscr{B}; \gamma) - X = U(\mathscr{B}, \gamma)$. Hence $\mathscr{B}$ is a fuzzy essence of X.

Proposition 3.8: Every fuzzy essence $\mathscr{A}$ of X satisfies the following inequality.

$$(\forall x \in X)\ (\mathscr{A}(0) \geq \mathscr{A}(x)).$$

Proof*:* If $\mathscr{A}$ is a fuzzy essence of X, then $U(\mathscr{A}; \alpha) - X = U(\mathscr{A}; \alpha)$ for all $\alpha \in \text{Im}(\mathscr{A})$. Since $U(\mathscr{A}; \alpha) \neq \emptyset$, there exists $x \in U(\mathscr{A}; \alpha)$. Hence

$$0 = x - x \in U(\mathscr{A}; \alpha) - X = U(\mathscr{A}; \alpha),$$

and so $\mathscr{A}(0) \geq \mathscr{A}(x)$ for all $x \in X$.

Lemma 3.9: A fuzzy set $\mathscr{A}$ in X is a fuzzy subalgebra (resp. fuzzy ideal) of X if and only if for every $\alpha \in [0, 1]$, the nonempty level set $U(\mathscr{A}; \alpha)$ is a subalgebra (resp. ideal) of X.

Proof: It follows from the Transfer Principle (see [6, Theorem 2.1]).

Theorem 3.10: *Every fuzzy essence of X is a fuzzy subalgebra of X.*

Proof: Let $\mathscr{A}$ be a fuzzy essence of X. Assume that $U(\mathscr{A}; \alpha) \neq \emptyset$ for all $\alpha \in [0, 1]$. For any $x, y \in U(\mathscr{A}; \alpha)$, we have

$$x - y \in U(\mathscr{A}; \alpha) - U(\mathscr{A}; \alpha) \subseteq U(\mathscr{A}; \alpha) - X = U(\mathscr{A}; \alpha).$$

Thus $U(\mathscr{A}; \alpha)$ is a subalgebra of X, and so $\mathscr{A}$ is a fuzzy subalgebra of X.

The following example shows that the converse of Theorem 3.10 is not true in general.

Example 3.11: The fuzzy set C in Example 3.4(iii) is a fuzzy subalgebra of X which is not a fuzzy essence of X.

Theorem 3.12: *Every fuzzy ideal of X is a fuzzy essence of X.*

Proof: Let $\mathscr{A}$ be a fuzzy ideal of X. Assume that $U(\mathscr{A}; \alpha) \neq \emptyset$ for all $\alpha \in [0, 1]$. Let $x \in X$ and $y \in U(\mathscr{A}; \alpha)$. Since $y - x \leq y$ and $U(\mathscr{A}; \alpha)$ is an ideal, it follows from Lemma 2.3 that $y - x \in U(\mathscr{A}; \alpha)$. This shows that

$$U(\mathscr{A}; \alpha) - X \subseteq U(\mathscr{A}; \alpha). \tag{3.2}$$

Combining (3.2) and Lemma 2.4, we have $U(\mathscr{A}; \alpha) - X = U(\mathscr{A}; \alpha)$. Hence $\mathscr{A}$ is a fuzzy essence of X.

The converse of Theorem 3.12 may not be true as seen in the following example.

Example 3.13: The fuzzy set B in Example 3.4(ii) is a fuzzy essence of X. Note that

$$\mathscr{B}(c) = n < m = \min\{\mathscr{B}(c - a), \mathscr{B}(a)\}.$$

Hence $\mathscr{B}$ is not a fuzzy ideal of X.

Theorem 3.14: *Let $\mathscr{A}$ and $\mathscr{B}$ be fuzzy essences of X. Then $\mathscr{A} \wedge \mathscr{B}$ and $\mathscr{A} \vee \mathscr{B}$ are fuzzy essences of X.*

Proof: Let $\alpha \in [0, 1]$ be such that $U(\mathscr{A} \wedge \mathscr{B}; \alpha) \neq \emptyset$. Then there exists $x \in U(\mathscr{A} \wedge \mathscr{A}; \alpha)$, and so

$$\alpha \leq (\mathscr{A} \wedge \mathscr{B})(x) = \min\{\mathscr{A}(x), \mathscr{B}(x)\}.$$

It follows that $x \in U(\mathscr{A}; \alpha)$ and $x \in U(\mathscr{B}; \alpha)$ so that $U(\mathscr{A}; \alpha) - X = U(\mathscr{A}; \alpha)$ and $U(\mathscr{B}; \alpha) - X = U(\mathscr{B}; \alpha)$. Using Lemma 3.1(ii), we have

$$\begin{aligned} U(\mathscr{A} \wedge \mathscr{B}; \alpha) - X &= (U(\mathscr{A}; \alpha) \bigcap U(\mathscr{B}; \alpha)) - X \\ &\subseteq (U(\mathscr{A}; \alpha) - X) \bigcap (U(\mathscr{B}; \alpha) - X) \\ &= U(\mathscr{A}; \alpha) \bigcap U(\mathscr{B}; \alpha) = U(\mathscr{A} \wedge \mathscr{B}; \alpha). \end{aligned}$$

The reverse inclusion follows from Lemma 2.4. Hence

$$U(\mathscr{A} \wedge \mathscr{B}; \alpha) - X = U(\mathscr{A} \wedge \mathscr{B}; \alpha),$$

i.e., $\mathscr{A} \wedge \mathscr{B}$ is a fuzzy essence of X. Now assume that $U(\mathscr{A} \vee \mathscr{B}; \beta) \neq \emptyset$ for all $\beta \in [0, 1]$. Then there exists $y \in U(\mathscr{A} \vee \mathscr{B}; \beta)$, and so

$$\beta \leq (\mathscr{A} \vee \mathscr{B})(y) = \max\{\mathscr{A}(y), \mathscr{B}(y)\}.$$

Hence $\mathscr{A}(y)$ ¸ β or $\mathscr{B}(y) \geq \beta$. We may assume that $\mathscr{A}(y) \geq \beta$ without loss of generality. Then $y \in U(\mathscr{A}; \beta)$, and thus $U(\mathscr{A}; \beta) - X = U(\mathscr{A}; \beta)$. If $U(\mathscr{B}; \beta) = \emptyset$, then

$$\begin{aligned} U(\mathscr{A} \vee \mathscr{B}; \beta) - X &= (U(\mathscr{A}; \beta) \bigcup U(\mathscr{B}; \beta)) - X \\ &= U(\mathscr{A}; \beta) - X = U(\mathscr{A}; \beta) \\ &= U(\mathscr{A}; \beta) \bigcup U(\mathscr{B}; \beta) \\ &= U(\mathscr{A} \vee \mathscr{B}; \beta). \end{aligned}$$

If $U(\mathscr{B}; \beta) \neq \emptyset$, then $U(\mathscr{B}; \beta) - X = U(\mathscr{B}; \beta)$. Hence

$$\begin{aligned} U(\mathscr{A} \vee \mathscr{B}; \beta) - X &= (U(\mathscr{A}; \beta) \bigcup U(\mathscr{B}; \beta)) - X \\ &= ((U(\mathscr{A}; \beta) - X) \bigcup ((U(\mathscr{B}; \beta) - X) \\ &= U(\mathscr{A}; \beta) \bigcup U(\mathscr{B}; \beta) = U(\mathscr{A} \vee \mathscr{B}; \beta). \end{aligned}$$

Therefore $\mathscr{A} \vee \mathscr{B}$ is a fuzzy essence of X.

In general, the union of two fuzzy ideals of X may not be a fuzzy ideal of X. For example, in Example 3.4, the fuzzy set $\mathscr{A}$ in X is a fuzzy ideal of X. Now let $\mathscr{D}$ be a fuzzy set in X given by

$$\mathscr{D}(x) = \begin{cases} m & \text{if } x \in \{0, b\}, \\ n & \text{otherwise} \end{cases}$$

for all $x \in X$, where $m, n \in [0, 1]$ with $m > n$. Then $\mathscr{D}$ is a fuzzy ideal of X. Note that $\mathscr{A} \vee \mathscr{D} = \mathscr{B}$ which is not a fuzzy ideal of X. But we know that Theorem 3.12 and Theorem 3.14 induce the following corollary.

Corollary 3.15: The union and intersection of two fuzzy ideals of X are fuzzy essences of X.

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