International Review of Fuzzy Mathematics
Vol. 3 No. 1 (June, 2018)

Received: 03rd June 2017 Revised: 14th August 2017 Accepted: 01st November 2017

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RΓ-SUBMODULES IN FUZZY SETTING

ABSTRACT: *In this paper we introduce the notion of a fuzzy RΓ-submodule and investigate some of its properties. Also, we give the idea of a faithful and regular fuzzy RΓ-submodule and establish a relation between the two properties therein. Further, an existence and uniqueness theorem of a left (right) quasi-faithful fuzzy ideal of the corresponding Γ-ring with the help of an isomorphism defined on its RΓ-module is obtained. We derive RΓ-submodules, which are characterized by some functional equation . The notion of a R-normal fuzzy RΓ-submodule is given in the end and we investigate some related properties.*

Key words: Fuzzy *RΓ*-submodule, faithfull fuzzy *RΓ*-submodule, Fuzzy regular *RΓ*-submodule, *R*-normal fuzzy *RΓ*- submodule.

2000 MCS: *13C99.*

1. INTRODUCTION

In 1964, Nobusawa [4] in his premier work on generalization of ring theory developed the notion of Γ-ring which was later termed as Γ-ring of Nobusawa. The notion of Γ-rings and RΓ-modules was discussed by Ravishankar and Shukhla [7] in 1979. They investigated the properties of a faithful and regular RΓ-module and its annihilator ideal pertaining to the development of Jacobson-radical for a Γ-ring via modules. Ravisankar et al. [7] also have shown that the Jacobson radical of a Γ-ring behaves in a similar fashion as its corresponding object in rings. Jun and Hong [6] discussed fuzzy Γ-ideals in 1995. They showed that their results in fuzzy sense adhered to the most generalized version of their counterparts in ordinary ring theory. Banerjee and Borkotokey [1] introduced the concept of a fuzzy Γ-semigroup and discussed its various properties. In this paper we introduce the notion of a fuzzy RΓ-submodule. We have defined a regular and faithful fuzzy RΓ-submodule and discussed its various characteristics. We have also introduced the notion of a finitely generated fuzzy RΓ-submodule and obtained few existence theorems of faithful fuzzy Γ-ideal and faithful fuzzy RΓ-submodule. In the end, the concept of R-normal fuzzy RΓ-submodule is introduced and some related properties are discussed. The process of fuzzyfication we apply here is more general than that taken up by Jun et al. [6].

2. PRELIMINARIES

In this section we give the needed definitions and results from [1-8], which are used in this paper.

Definition 2.1 [7]: Let $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be an additive abelian group. A Γ-ring is an additive group $R=\{x,y,z,\ldots\}$ together with a composition $x\alpha y$ in R defined for x, y in R and α in Γ satisfying the following conditions:

(i) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z)=x\alpha y + x\alpha z$

(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Remark 2.2: Through out the paper we assume R to be a Γ-ring.

Definition 2.3 [6]: An additive subgroup I of R is called a left (right) ideal of R if $I\Gamma R \subseteq I$ ($R\Gamma I \subseteq I$).

Definition 2.4 [7] A Γ-ring R is simple if $R\Gamma R \neq 0$ and the only ideals of R are 0 and R.

Definition 2.5 [7]: Let R be a Γ-ring. An additive abelian group M will be called an RΓ-module , if there exists a map $\phi : M\times G\times R \rightarrow M$ satisfying ($\phi(m, \alpha, x)$ is denoted by $m\alpha x$ in short)

(1) $(m+n)\alpha x = m\alpha x + n\alpha x$.

(2) $m\alpha(x+y) = m\alpha x + m\alpha y$.

(3) $m\alpha(x\beta y) = (m\alpha x)\beta y$. for all x, y in R, α,β in Γ and m,n in M.

Definition 2.6 [7]: A subset N of M is said to be an RΓ- submodule of M if N is itself an RΓ-module under the operations of M.

Definition 2.7 [7]: An RΓ-module M is said to be simple if $M\Gamma R\neq 0$ and M has no proper sub-modules.

Remark 2.8: Through out the paper we denote M as an RΓ-module.

Definition 2.9 [8]: A Γ-near ring is a triple (R,+, Γ), where

(i) (R,+) is a (not necessarily abelian) group;

(ii) Γ is a non empty set of binary operations on R such that for each $\gamma \in \Gamma$, (R, +, γ) is a near ring, i.e (R, γ) is a semi group with the right (left) distributive property.

(iii) $x \gamma (y \mu z) = (x \gamma y)\mu z$, for all $x,y,z \in R$ and $\gamma, \mu \in \Gamma$.

Definition 2.10 [8]: Let (G,+) be a group. If, for all $x, y \in R$, $\gamma, \mu \in \Gamma$ and $g \in G$, it holds

(i) $x \gamma g \in G$ ($g \gamma x \in G$)

(ii) $(x+y) \gamma g = x \gamma g + y \gamma g$,

(iii) $x \gamma (y \mu g) = (x \gamma y) \mu g$.

Then G is called an RΓ-group.

Definition 2.11 [8]: Let G be an RΓ-group, $H \subseteq G$ is an RΓ-ideal if

(i) (H, +) is a normal subgroup of (G,+).

(ii) $\forall x \in M, \gamma \in \Gamma, h \in H, g \in G, x \gamma (g + h) - x \gamma g \in H$.

Definition 2.12 [8]: An RΓ-group homomorphism is a map $f : G \to H$, where G and H are RΓ-groups such that f satisfies the following conditions:

(1) $f(m + n) = f(m) + f(n)$.

(2) $f(m\alpha r) = f(m)\alpha r$, for all m, n in G, α in Γ and r in R.

Remark 2.13: We assume through out this paper, that G satisfies both left and right closure properties, i.e $x \gamma g \in G$ and also $g \gamma x \in G$, $\forall x \in R, \gamma \in \Gamma, g \in G$.

3. FUZZY RΓ-SUBMODULES

We assume that every fuzzy set maps into the lattice ([0,1], ∧,∨), where [0,1] is the set of reals between 0 and 1 and $x \wedge y = \inf(x,y)$, $x \vee y = \sup(x,y)$.

Definition 3.1: Let μ be a fuzzy subset of M, ν, a fuzzy subset of Γ and ξ, a fuzzy subset of R. μ is said to be a fuzzy RΓ-submodule of M with respect to the pair (ν, ξ) if the following hold:

(i) $\mu(m\text{-}n) \geq \mu(m) \wedge \mu(n)$

(ii) $\mu(m\alpha x) \geq \mu(m) \vee \nu(\alpha) \vee \xi(x)$, for all m,n in M, α in Γ and x in R.

We denote the fuzzy RΓ-submodule μ by the triplet (μ, ν, ξ), to distinguish it from ordinary fuzzy submodules.

Remark 3.2: If μ_1 and μ_2 are two fuzzy RΓ- submodules then $(\mu, \nu, \xi)_1, \cap (\mu,\nu,\mu)_2$ and $(\mu, \nu, \xi)_1 + (\mu, \nu, \xi)_2$ are also fuzzy RΓ-submodules, where

$$(\mu,\nu,\xi)_1 + (\mu,\nu,\xi)_2 (z) = \mu_1(z) \wedge \mu_2 (z),\ (\mu,\nu,\xi)_1 \cup (\mu,\nu,\xi)_2(z) = \mu_1(z) \vee \mu_2 (z) \text{ and}$$

$$(\mu,\nu,\xi)_1 + (\mu,\nu,\xi)_2(z) = \vee\{\mu_1(u) \wedge \mu_2 (v) \mid u, v\in M, u + v = z\}, \forall x, z \in M.$$

Definition 3.3: M is said to be simple in the fuzzy sense if every fuzzy RΓ-submodule of M is constant.

Proposition 3.4: If (μ, ν, ξ) is a fuzzy RΓ-submodule of M then the level subset $(\mu, \nu, \xi)_t = \mu_t = \{x\in M \mid \mu(x) \geq t\}$ is always an RΓ-submodule of M for t in [0,1] called the level submodule of M, but the converse is not true as it is entirely dependent on the choice of ν and ξ. Moreover every RΓ-submodule of M can be expressed as a level submodule of M.

Remark 3.5: Proposition 3.4 provides a deviation to the usual correspondence found trivially between an ordinary fuzzy ideal and its level ideal of the crisp ring. This correspondence can be established between a fuzzy RΓ-submodule μ and its level subset only with a particular choice of (ν, ξ).

Lemma 3.6: Every RΓ-submodule of an RΓ-module can be expressed as a level submodule.

Proof: Let N be an RΓ-submodule of an RΓ-module. For $t \in [0,1]$, Define the fuzzy set μ on M as:

$$\mu(m) = \begin{cases} t, & m \in N \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mu_t = N$.

Theorem 3.7: An RΓ-module M is simple if and only if it is simple in the fuzzy sense.

Proof: Let M be simple in the fuzzy sense, and (μ, ν, ξ) a fuzzy RΓ-submodule of M. Then by definition (μ, ν, ξ) is constant. Thus μ(x) = a, a constant for all x in M. Clearly MΓR ≠ 0. From 3.4 we can infer that $0 \neq \mu_a = \{x \in M \mid \mu(x) = a\}$ is an RΓ-submodule of M. But then $M = \mu_a$. Since every RΓ-submodule of M can be expressed as a level submodule of M, M has no proper RΓ-submodule other than 0. Conversely, let M be simple. Then M has no proper RΓ-submodule. We show that every (μ, ν, ξ) is constant. Suppose (μ, ν, ξ) takes at least two distinct values t and s, say t > s. We have correspondingly two non zero RΓ-sub modules $(\mu, \nu, \xi)_t$ and $(\mu, \nu, \xi)_s$ such that $(\mu, \nu, \xi)_s \subset (\mu, \nu, \xi)_t$, which is a contradiction to our hypothesis that M is simple. So we have t = s.

Definition 3.8: A fuzzy RΓ-submodule (μ, ν, ξ) is said to be regular if, for all m in M, there exists e in R and α in Γ, we have μ(m-mαe) = μ(m).

Definition 3.9: A fuzzy RΓ-submodule (μ, ν, ξ) is said to be faithful if it satisfies the following conditions:

(1) For all m in M, there exists e in R and α in Γ, we have μ(mαe) = μ(m).

(2) μ(m-n) = μ(m) ∧ μ(n), for all m, n in M.

Example 3.10: Let $M = Z_6 = \{0,1,2,3,4,5\}$, the ring of integer modulo 6.

$R = Z_3 = \{0,1,2\}$, the ring of integer modulo 3.

And $\Gamma = Z_2 = \{0,1\}$, the ring of integer modulo 2.

We define the fuzzy subsets μ, ν and ξ of M, Γ and R respectively, as follows:

μ (0) = μ (2) = μ (4) = 0.6 and μ (1) = μ (3) = μ (5) = 0.5.

ν(0) = ν(2) = 0.4 and ν(1)= 0.3. ξ(0) = 0.2 and ξ(1) = 0.3.

Then (μ, ν, ξ) is a faithful fuzzy RG-submodule of M.

Theorem 3.11: A faithful fuzzy RΓ-submodule (μ, ν, ξ) is regular. However not every regular fuzzy RΓ-submodule of M is faithful.

Proof: For m in M, there exist e in R and α in Γ, such that μ(mαe) = μ(m).

We have $\mu(m\text{-}m\alpha e) = \mu(m) \wedge \mu(m\alpha e) = \mu(m)) \wedge \mu(m) = \mu(m)$. For the converse part we present the following counter example.

Example 3.12: Let M = Z, the set of integers, R = Γ = E, the set of even integers and the composition be the usual product of the integers.

Consider the fuzzy subsets :

$\mu(m) = \{a_0$, if x is even.
$\{a_1$, otherwise for all m in M.

$\nu(\alpha) = \{b_0$, if $\alpha \geq 20$ $\quad\xi(x) = \{b_0$, if $x \geq 20$

$\{b_1$, otherwise $\quad\{b_1$, otherwise, for all α in Γ and x in R.

Wherever $a_0 > a_1 > b_0 > b_1$. It is easily seen that (μ, ν, ξ) is a fuzzy RΓ-submodule of M.

Since : for any m in M, α in Γ and e in R,

Case I: If m is even ,then so is (m-mαe), so that $\mu(m\text{-}m\alpha e) = \mu(m)= a_0$.

Case II: If m is odd, then so is (m-mαe), so that $\mu(m\text{-}m\alpha e) = \mu(m)= a_1$.

Thus in both the cases (μ, ν, ξ) is regular. However for an odd m in M, for each α in Γ and e in R, mαe is always even so that $\mu(m\alpha e) \neq \mu(m)$.

Corollary 3.13: If M is simple, then every fuzzy RΓ-submodule of M, being constant is faithful.

Proposition 3.14: If $(\mu, \nu, \xi)_1$ and $(\mu, \nu, \xi)_2$ are regular fuzzy RΓ-submodules of M and also $(\mu, \nu, \xi)_2$ is faithful, then $(\mu, \nu, \xi)_1 \cap (\mu, \nu, \xi)_2$, $(\mu, \nu, \xi)_1 \cup (\mu, \nu, \xi)_2$ and $(\mu, \nu, \xi)_1 + (\mu, \nu, \xi)_2$ are regular, where ∩ and ∪ are the usual t-norms and co-norms denoting minimum and maximum respectively.

4. FUZZY ANNIHILATOR, QUASI FAITHFUL IDEAL AND FINITELY GENERATED RΓ-SUBMODULES

Definition 4.1: A fuzzy Γ-ideal ξ of R is a fuzzy subset of R with respect to the fuzzy subset γ of Γ such that

(1) $\xi(r\alpha s) \geq \xi(r) \vee \gamma(\alpha) \vee \xi(s)$.

(2) $\xi(r\text{-}s) \geq \xi(r) \wedge \xi(s)$., for all r, s in R and a in Γ.

Definition 4.2: A fuzzy Γ-ideal ξ_μ of R is said to be the fuzzy annihilator ideal of R with respect to the fuzzy RΓ-submodule μ of M, if $\mu(a\Gamma r) = \xi_\mu(r)$, for all a in M and r in R.

Remark 4.3: Let ξ_μ be the fuzzy annihilator ideal of R, with respect to (μ,γ,ξ). If (μ,γ,ξ) is faithful, we have for all a in M, there exists an α in Γ and r in R such that $\mu(a\alpha r) = \mu(a)$. Moreover, for each a in M, r in R and α in Γ, $\mu(a\alpha r) = \xi_\mu(r)$ by definition 4.2. Thus we can infer that for each a in M, there is a and r in R, such that $\mu(a\alpha r) = \mu(a) = \xi_\mu(r)$. Thus there exists a one to one map $h : M \to R$ such that $\mu = \xi_\mu o\, h$, where the composition ξ_μ o h is defined on M as $(\xi_\mu \text{ o } h)(m) = \xi_\mu\,(h(m))$, for every m in M. Moreover if h is a surjective homomorphism, then for each s in R, there is an m in M such that $h(m) = s$. Also for this m, there exists $\alpha \in \Gamma$, $r \in R$: $\mu(m\alpha r) = \mu(m)$ and $\xi_\mu(s\alpha r) = \xi_\mu(h(m)\alpha r) = \xi_\mu(h(m\alpha r) = \xi_\mu o h(m\alpha r) = \mu(m\alpha r) = \mu(m) = \xi_\mu(h(m)) = \xi_\mu(s)$. Also $\xi_\mu(r\text{-}s) = \mu(m\alpha(r\text{-}s)) = \mu(m\alpha r\text{-}m\alpha s) = \mu(m\alpha r) \wedge \mu(m\alpha s) = \xi_\mu(r) \wedge \xi_\mu(s)$, for all $m \in M$ and $\alpha \in \Gamma$. Thus we can define a quasi faithful fuzzy ideal of R as follows:

Definition 4.4: A fuzzy Γ-ideal ξ of R is said to be fuzzy Γ-left(right) quasi faithful ideal, if

(1) for each s in R, there exist α in Γ & r in R, such that $\xi(s\alpha r) = \xi(s)$. $(\xi(r))$.

(2) for r and s in R, $\xi(r\text{-}s) = \xi(r) \wedge \xi(r)$.

Definition 4.5 [7]: A Γ-ring R is commutative if $s\alpha r = r\alpha s$, for all r, s in R, α in Γ.

Remark 4.6: If R is commutative then every fuzzy right quasi faithful ideal is also left quasi faithful. Thus we have the following existence theorem:

Proposition 4.7: Every isomorphism h: M→R, R taken as an RΓ-module determines uniquely a left (right) quasi faithful fuzzy ideal of R, with respect to a faithful fuzzy RΓ-submodule of M.

Proof: Let μ be a faithful fuzzy RΓ-submodule of M. As h is a surjection for each r in R there is an m in M, such that h (m) = r. We define a fuzzy subset ξ of R as follows:

$$\xi(r) = \mu(m), \text{ such that } h(m) = r.\ m \in M,\ r \in R.$$

Since μ is faithful, for each such m, there exist α in Γ and s in R so that $\mu(m\alpha s) = \mu(m)$. Then for $r \in R$, $\exists\, m \in M$: $h(m) = r$ and $\xi(r) = \mu(m)$. Moreover $r\alpha s \in R$ and we have

$$\xi(r\alpha s) = \xi(h(m)\alpha s) = \xi(h(m\alpha s)) = \mu(m\alpha s) = \mu(m) = \xi(r).$$

Also for r, $s \in R$, $\exists\, m,n \in M$: $h(m) = r$, $h(n) = s$ and $\xi(r) = \mu(m)$, $\xi(s) = \mu(n)$. We have $\xi(r-s) = \xi(h(m-n)) = \mu(m-n) = \mu(m) \wedge \mu(n) = \xi(r) \wedge \xi(s)$. Hence ξ is left quasi faithful.

Definition 4.8: A fuzzy RΓ-submodule (μ, γ, ξ) of the RΓ-module M is said to be finitely generated if there is a finite set $\{x_i \mid i \in N, 0 \le 1 \le n\}$, n being a fixed element of the set **N** of natural number, of elements of M, such that for every element $m \in M$, $\exists$ unique $\alpha_i \in \Gamma$ and $r_i \in R$, $1 \le i \le n$,

$$m = \sum_{i=1}^{n} x_i \alpha_i r_i, \quad \sum_{i=1}^{n} \mu(x_i) \le 1, \quad \text{and} \quad \mu(m) = \sum_{i=1}^{n} \mu(x_i)\gamma(\alpha_i)\xi(r_i).$$

Remark 4.9: As $\gamma(\alpha_i) \le 1$ and $\xi(r_i) \le 1$, for all $\alpha_i \le \Gamma$ and $r_i \in R$, $\mu(m) \le \sum_{i=1}^{n} \mu(x_i)$.

Therefore, we require the extra condition, namely $\sum_{i=1}^{n} \mu(x_i) \le 1,$ in definition 4.8.

Definition 4.10: Let (μ, γ, ξ) be a fuzzy RΓ-submodule of M, and ψ be an RΓ-module endomorphism on M. The composition $\mu o \psi$ is defined on M as $(\mu o \psi)(m) = \mu(\psi(m))$ for every m in M.

Proposition 4.11: Let (μ, γ, ξ) be a finitely generated fuzzy RΓ-submodule of M, and ψ be an RΓ-module endomorphism on M. Then $(\mu o \psi, \gamma, \xi)$ is again a fuzzy RΓ-submodule of M, and μ and $\mu o \psi$ are connected by a functional equation of the form:

$$\mu o \psi + A\{\mu \times \mu o \psi\} + B\mu = 0, \text{ where A and B are some real numbers.}$$

Where $\mu \times \mu o \psi$ is another fuzzy RΓ-submodule of M, defined by

$$(\mu \times \mu o\psi)(m) = \mu(m) \wedge \mu o\psi(m), \text{ for all m in M.}$$

Proof: Let for every m in M, there is a finite set $\{x_i \mid 0 \le i \le n\}$, n being a finite natural number, of elements of M, such that ∃ unique $\alpha_i \in \Gamma$ and $r_i \in R$, $1 \le i \le n$,

$m = \sum_{i=1}^{n} x_i \alpha_i r_i, \sum_{i=1}^{n} \mu(x_i) \le 1,$ we have $\mu(m) = \sum_{i=1}^{n} \mu(x_i)\gamma(\alpha_i)\xi(r_i).$ Given that ψ is an RΓ-module endomorphism on M . So by the hypothesis ∃ unique $\alpha_i \in \Gamma$, $a_{ij} \in R$: i, j = 1,2,3,…n.

$$\psi(x_j) = \sum_{i=1}^{n} a_{ij}\, \alpha_i\, x_j,\ 1 \le j \le n.$$

$$\mu(\psi(x_j)) = \sum_{i=1}^{n} \mu(x_i)\gamma(\alpha_i)\xi(a_{ij}),\ 1 \le j \le n.$$

$$\Rightarrow \sum_{i=1}^{n} \{(\mu o\psi)\delta_{ij} - \gamma(\alpha_i)\xi(a_{ij})\mu\}(x_i) = 0 \qquad \text{...(a)}$$

As $M \neq 0$, at least one $x_i \neq 0$, hence we get a non zero solution for x_i from (a) only when Determinant $\{\{(\mu \text{ o } \psi)\delta_{ij} - \gamma(\alpha_i)\, \xi(a_{ij})\, \mu\} = 0$.

Expanding the determinant, using the facts that

$$(\mu \times \mu o\, \psi)(m) = \mu(m) \wedge \mu o\, \psi(m), \text{ for all m in M.}$$

$$(\mu o\, \psi) \times (\mu o\, \psi) = (\mu \text{ o } \psi) \text{ and } \mu \times \mu = \mu.$$

We obtain the required result as $\mu o\, \psi + A\, \{\mu \times \mu o\, \psi\} + B\, \mu = 0$, where A and B are real numbers.

Lemma 4.12: Let M be a simple RΓ-module. Then there exists a unique x in M such that every m in M can be expressed as $m = x\alpha r$, for $\alpha \in \Gamma$ and $r \in R$.

Proof: We have $0 \neq x\Gamma R$, is an RΓ-submodule of M. M being simple, $x\Gamma R = M$, and hence proved the result.

Remark 4.13: Since, M is fuzzy simple i.e every fuzzy RΓ-submodule of M is constant implies and implied by M is simple, hence by lemma 4.11, there exists a unique x in M such that every m in M can be expressed as $m = x\alpha r$, for $\alpha\in\Gamma$ and $r\in R$. In other words M is spanned by a singleton.

Proposition 4.14: Let M be spanned by a singleton $x \in M$ and (μ, γ, ξ), a finitely generated fuzzy RΓ-submodule of M. Let also $\alpha\in\Gamma$ be fixed such that every m in M is expressed as $m = x\alpha r$, $r\in R$. Then (μ, γ, ξ) is faithful only when ξ is a right(left) quasi faithful ideal of R. If the generator set of M consists of more than one element, (μ, γ, ξ) will no more be faithful.

Proof: Let $a \in M$, so that $a = x\alpha r$, for $r\in R$ so that by definition 4.8, $\mu(a)= \mu(x)\gamma(\alpha)\xi(r)$. Then as ξ is left quasi faithful, there exist β and s such that $\xi(r\beta s) = \xi(r)$ and we have $\mu(a\beta s) = \mu(x\alpha r\beta s) = \mu(x)\gamma(\alpha)\xi(r\beta s) = \mu(x)\gamma(\alpha)\xi(r) = \mu(a)$. Let $m, n \in M$, so that $m = x\alpha r$, $n = x\alpha s$ for $r, s \in R$, then $\mu(m\text{-}n) = \mu(x\, \alpha\, (r\text{-}s)) = \mu(x)\gamma(\alpha)\xi(r\text{-}s) = \mu(x)\gamma(\alpha)\xi(r) \wedge \mu(x)\gamma(\alpha)\xi(s) = \mu(m) \wedge \mu(n)$ {ξ is left quasi faithful}.

Example 4.15: We show that keeping other conditions unchanged, if M is spanned by at least two elements $\{x_1 , x_2\}$, then also the faithfulness condition of (μ, γ, ξ) is not satisfied.

For $m , n \in M$, we have $m = x_1\, \alpha\, r_1 + x_2\, \alpha\, r_2$ and $n = x_1\, \alpha\, s_1 + x_2\, \alpha\, s_2$. Therefore $m\text{-}n = x_1\, \alpha\, (r_1\text{-}s_1) + x_2\, \alpha\, (r_2\text{-}s_2)$. Thus $\mu(m\text{-}n) = \mu(x_1)\, \gamma(\alpha)\xi\, (r_1\text{-}s_1))+ \mu(x_2)\, \gamma\, (\alpha)\xi(r_2\text{-}s_2)$.

Let $\xi\, (r_1\text{-}s_1) = \xi\, (r_1)$ and $\xi\, (r_2\text{-}s_2) = \xi\, (s_2)$. In general, we have

$$\mu(m\text{-}n) \;=\; \mu(x_1)\, \gamma(\alpha)\, \xi\, (r_1) + \mu(x_2)\, \gamma\, (\alpha)\, \xi\, (s_2)$$

$$\neq \mu(x_1)\, \gamma(\alpha)\xi\, (r_1)+ \mu(x_2)\, \gamma\, (\alpha)\xi(r_2) \wedge \mu(x_1)\, \gamma(\alpha)\xi\, (s_1) + \mu(x_2)\, \gamma\, (\alpha)\xi(s_2).$$

5. R-NORMAL FUZZY SUBGROUPS AND SUBMODULES

Definition 5.1: A fuzzy subset μ of an RΓ-group G is called a fuzzy subgroup of G if the condition: $\mu(x\text{-}y) \geq \mu(x) \wedge m(y)$, holds.

Definition 5.2: Let G be an RΓ-group where R is a Γ-near ring with unity, γ, a fuzzy subset of Γ, ξ a fuzzy subset of R. A fuzzy subgroup μ of G is called R-normal if

$$\mu(a^{-1}\, \alpha\, g\, \alpha\, a) \geq \mu(g) \vee \gamma(\alpha) \vee \xi(a),\ \forall\, g \in G,\ \alpha \in \Gamma \text{ and for every unit } a \in R.$$

Remark 5.3: We denote the fuzzy R-normal subgroup μ with respect to the fuzzy subsets γ of Γ, ξ of R by the triplet (μ, γ, ξ).

Example 5.4: Let R ={a, b, c, d}, Γ = {α}, set of a single operation, composition in R with respect to the binary operations "+" and "α" being given in the tables 5.4.1 and 5.4.2 , so that (R,+,α) is a Γ-near ring.

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Table 5.4.1

α	a	b	c	d
b	c	d	b	a
c	a	b	c	d
d	b	a	d	c

Table 5.4.2

+	x	y
x	x	x
y	y	x

Table 5.4.3

α	a	b	c	d
x	x	x	x	x
y	y	y	x	y

Table 5.4.4

α	x	y
b	x	x
c	x	y
d	x	y

Table 5.4.5

Let G = {x, y} be a set and the composition among the elements of G are defined in table 5.4.3 and among the elements of G and R are given in table 5.4.4 and 5.4.5.

We define the fuzzy sets μ, γ and ξ as : μ (x) = 0.6, μ (y) = 0.5. ξ (a) = 0.6, ξ (b) = 0.5, ξ (c) = 0.4, ξ (d) = 0.5. and γ (a) = 0.4. Here a^{-1} = b and b^{-1} = a while c^{-1} = c and d^{-1} = d. Then (μ, γ, ξ) is an R- normal fuzzy subgroup of G.

Remark 5.5: If *f*: G →H is an RΓ-homomorphism and (μ, γ, ξ) is an R-normal fuzzy subgroup of G then (f^{-1}(μ), γ, ξ) and (*f*(μ), γ, ξ) are also R-normal fuzzy subgroups of H. If G is an RΓ-group, H its RΓ-ideal, then we can define the quotient RΓ-group G/H in the same way as we define a quotient group in ordinary group structure, with the more general condition that: a α (g + H) α b = a α g α b + H, ∀ a, b ∈ R, α ∈ Γ and g ∈ G.

Proposition 5.6: Let G be an RΓ-group and (μ, γ, ξ), an R-normal fuzzy subgroup of G. Then for H an RΓ-ideal of G, μ induces an R-normal fuzzy subgroup μ′ of G/H given by

$$\mu' (g + H) = \mu(g) \wedge \mu (H), \text{ where } \mu (H) = \vee \{\mu(h) \mid h \in H\}$$

Remark 5.7: The RΓ-module M can be treated as an RΓ-group also, where R is a Γ-ring instead of a Γ-near ring. Also if (μ, γ, ξ) is a fuzzy RΓ-submodule of M then we can treat it as a fuzzy RΓ-subgroup of M. Moreover, if the fuzzy subset ξ of R is a fuzzy RΓ-ideal of R, then (μ, γ, x) is an R-normal fuzzy subgroup of M.

Lemma 5.8: Let F(M) be the set of all R-normal fuzzy submodules of M. Then F(M) is closed under addition of fuzzy submodules.

Proof: Let x be a unit in R, $\alpha \in \Gamma$, $g \in M$ and $\mu_1, \mu_2 \in F(M)$.

We have $(\mu_1 + \mu_2)(x^{-1} \alpha g \alpha x) = \vee\{\mu_1 (z) \wedge \mu_2 (y) | z + y = x^{-1} \alpha g \alpha x, y,z \in M\}$

$= \vee\{\mu_1 (z) \wedge \mu_2 (y) | x\alpha(y+z)\alpha x^{-1} = g\} = \vee \{\mu_1 (z) \wedge \mu_2 (y) | x\alpha y\alpha x^{-1} + x\alpha z\alpha x^{-1} = g\}$

$= \vee \{\mu_1 (x^{-1} \alpha u\alpha x) \wedge \mu_2 (x^{-1} \alpha v\alpha x) | u + v = g\}$

$\geq \vee \{\{\mu_1 (u) \vee \xi (x) \vee \gamma (\alpha)\} \wedge \{\mu_2 (v) \vee \xi(x) \vee \gamma(\alpha)\} | u + v = g\}$

$\geq \{\vee \{\mu_1 (u) \wedge \mu_2 (v) | u + v = g\}\} \vee \xi(x) \vee \gamma(\alpha) = (\mu_1 + \mu_2)(g) \vee \xi(x) \vee \gamma(\alpha)$.

Remark 5.9: Under the binary operation "+", F(M) possesses a semigroup structure. So we can define the notion of an additive function on F(M).

Definition 5.10: A map f: A→B, between two semigroups is said to be an additive function, if $f(a + b) = f(a) + f(b)$, $\forall$ a, b $\in$ A.

Proposition 5.11: There exists an additive function between F(M) and F(M/H), the semigroups consisting of all R-normal fuzzy RΓ-submodules of M and M/H respectively with respect to the fuzzy subset γ of Γ and fuzzy RΓ-ideal ξ of R, the elements of M/H being R-normal fuzzy RΓ-submodules induced by those of M as defined in 5.7.

Proof: For μ_1 and $\mu_2 \in F(M)$, we have by lemma 5.9, $\mu_1 + \mu_2 \in F(M)$. We define the function f : F(M) → F(M/H) as: $f(\mu) = \mu'$, where $\mu' (g + H) = \mu(g) \wedge \mu (H)$, and $\mu (H) = \vee \{\mu(h) | h \in H\}$ for every $g \in G$.

Now we discuss an interesting uniqueness property under isomorphism of a fuzzy map as defined in [2].

Definition 5.12 [2]: A fuzzy map f from a set X to a set Y is an ordinary map from X to the set of all fuzzy subsets of Y satisfying the following conditions:

(i) for all $x \in X$, there exists $y_x \in Y$ such that $(f(x))(y_x)=1$

(ii) for all $x \in X$, $(f(x))(y_1)=(f(x))(y_2)$ implies $y_1 = y_2$.

Remark 5.13 [2]: A fuzzy map f from X to Y gives rise to a unique ordinary map $\mu_f : X \times Y \to I$ given by $\mu_f(x, y) = (f(x))(y)$ where I is the interval [0,1]

Definition 5.14: Let M and N be two RΓ-modules and f: M → N be a fuzzy mapping. Then f is said to be a fuzzy homomorphism if the following hold:

(1) $\mu_f(m_1 + m_2, n) = \bigvee_{n=n_1+n_2} \{\mu_f(m_1, n_1) \wedge \mu_f(m_2, n_2)\}$

(2) $\mu_f(m, r\alpha\, n) \geq \mu_f(m, n)$, for all $m, m_1, m_2 \in M$, $\alpha \in \Gamma$ and $n, n_1, n_2 \in N$, $r \in R$.

Proposition 5.15: Let f : F(M) → F(M/H) be the additive function, $\phi : M \to M$ a fuzzy homomorphism, such that $\phi(m)$ is an R-normal fuzzy RΓ-submodule of M for every $m \in M$. Then the composite map $f o\, \phi : M \to F(M/H)$ defined by $f o\, \phi\, (m) = f(\phi(m))$, $\forall\, m \in M$ is also a fuzzy homomorphism.

We generalize the proposition 5.15 as follows:

Proposition 5.16: Let $\lambda : M \to M'$ be a surjective RΓ-homomorphism between the RΓ- modules M and M′, f : F (M) → F (M′), an additive function and $\phi : M \to M'$, a fuzzy homomorphism such that $\phi(m)$ is an R-normal fuzzy RΓ-submodule of M for every $m \in M$. Then the composite map $f o\, \phi$ is again a fuzzy homomorphism.

Proof: We have the composite map $f o\, \phi : M \to F(M')$ defined by $f o\, \phi\, (m) = f(\phi(m))$, $\forall m \in M$. Moreover for $\mu \in F(M)$ there is a $\mu' \in F(M')$ such that $f(\mu) = \mu'$, whenever for $g \in M$, $\exists\, g' \in M'$, $\lambda(g) = g'$, we have $\mu'(g') = \mu(g)$.

(a) μ' thus defined is R-normal RΓ-fuzzy submodule of M′ with respect to the fuzzy subsets γ of G , and the fuzzy RΓ-ideal ξ of R .

(b) $f o\, \phi$ is well defined i.e it is an R-normal fuzzy RΓ-submodule of M′.

(c) Since ϕ is a fuzzy map, for $g \in M$, $\exists\, g^* \in M : \phi(g)(g^*) = 1$. λ being surjective $\exists\, g' \in M' : \lambda(g^*) = g'$. Hence $f o\, \phi\, (g)(g') = f(\phi(g))(g') = \phi(g)(g^*) = 1$.

(d) $f o\, \phi\, (g)(g_1') = f o\, \phi\, (g)(g_2') \Rightarrow f(\phi(g))(g_1') = f(\phi(g))(g_2')$

$\Rightarrow$ There exist g_1 and g_2 : $\lambda(g_1) = g_1'$ and $\lambda(g_2) = g_2'$ respectively and we have $\phi(g)(g_1) = \phi(g)(g_2) \Rightarrow g_1 = g_2$. Thus $f o\, \phi$ is a fuzzy homomorphism.

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