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FUZZY IDEALS OF PSEUDO MV-ALGEBRAS

ABSTRACT. The notion of fuzzy (implicative) ideals of a pseudo MV-algebra is introduced, and its characterizations are established. Conditions for a fuzzy set to be a fuzzy ideal are given. Given a fuzzy set $\mu$, the least fuzzy ideal containing $\mu$ is constructed.

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1. INTRODUCTION

The ideal theory of pseudo MV-algebras is studied in [1] and [2]. In particular, the second author gave characterizations of ideals, and introduced the notion of implicative ideals in pseudo MV-algebras (see [2]). In this paper, we introduce the notion of fuzzy (implicative) ideals in a pseudo MV-algebra. We give characterizations of fuzzy (implicative) ideals, and provide conditions for a fuzzy set to be a fuzzy ideal. Given a fuzzy set $\mu$, we make the least fuzzy ideal containing $\mu$.

2. PRELIMINARIES

A pseudo MV-algebra is an algebra $(M; \oplus, -, \cdot, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^- :$$

(a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(a2) $x \oplus 0 = 0 \oplus x = x$,

(a3) $x \oplus 1 = 1 \oplus x = 1$,

(a4) $1^- = 0$, $1^- = 0$,
(a5) \((x^- \oplus y^-)^- = (x^- \oplus y^-)^-\),

(a6) \(x \oplus x^- \odot y = y \oplus y^- \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x\),

(a7) \(x \odot (x^- \oplus y) = (x \oplus y^-) \odot y\),

(a8) \((x^-)^- = x\).

If we define \(x \leq y\) if and only if \(x^- \oplus y = 1\), then \(\leq\) is a partial order such that \(M\) is a bounded distributive lattice with the join \(x \vee y\) and the meet \(x \wedge y\) given by

\[
x \vee y = x \oplus x^- \odot y = x \odot y^- \oplus y,
\]

\[
x \wedge y = x \odot (x^- \oplus y) = (x \oplus y^-) \odot y.
\]

Let \(M\) be a pseudo MV-algebra \(M\) and \(x, y, z \in M\). Then the following properties are valid (see [1]).

(b1) \(x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y\).

(b2) \((x \vee y)^- = x^- \wedge y^-\).

(b3) \(x \leq y \Rightarrow z \odot x \leq z \odot y, x \odot z \leq y \odot z\).

(b4) \(z \oplus (x \wedge y) = (z \oplus x) \wedge (z \oplus y)\).

(b5) \(z \odot (x \oplus y) \leq z \odot x \oplus y\).

(b6) \((x^-)^- = x\).

(b7) \(x \odot 1 = 1 \odot x = x\).

(b8) \(x \oplus x^- = 1, x^- \oplus x = 1\).

(b9) \(x \odot x^- = 0, x^- \odot x = 0\).

(b10) \(x \odot (y \odot z) = (x \odot y) \odot z\).

A subset \(I\) of a pseudo MV-algebra \(M\) is called an ideal of \(M\) (see [2]) if it satisfies:

(c1) \(0 \in I\),

(c2) If \(x, y \in I\), then \(x \oplus y \in I\),

(c3) If \(x \in I, y \in M\) and \(y \leq x\), then \(y \in I\).

For every subset \(W \subseteq M\), we denote by \(\langle W \rangle\) the ideal of \(M\) generated by \(W\), that is, \(\langle W \rangle\) is the smallest ideal containing \(W\). By [1, Lemma 2.5],

\[
\langle W \rangle = \{ x \in M \mid x \leq y_1 \oplus \ldots \oplus y_k \text{ for some } y_1, \ldots, y_k \in W \}.
\]
3. FUZZY IDEALS

We give the definition of a fuzzy ideal in a pseudo $MV$-algebra.

**Definition 3.1.** A fuzzy set $\mu$ in a pseudo $MV$-algebra $M$ is called a fuzzy ideal of $M$ if it satisfies:

(d1) $(\forall x, y \in M) (\mu(x \oplus y) \geq \min \{\mu(x), \mu(y)\})$,

(d2) $(\forall x, y \in M) (y \leq x \implies \mu(y) \geq \mu(x))$.

It is easily seen that (d2) forces

(d3) $(\forall x \in M) (\mu(0) \geq \mu(x))$.

**Example 3.2.** Let $I$ be an ideal of a pseudo $MV$-algebra $M$ and let $\mu_I$ be a fuzzy set in $M$ defined by

$$\mu_I(x) = \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Let $x, y \in M$. If $x, y \in I$, then $x \oplus y \in I$ and so

$$\mu_I(x \oplus y) = \alpha = \min \{\mu_I(x), \mu_I(y)\}.$$ 

If $x \not\in I$ or $y \not\in I$, then $\mu_I(x) = \beta$ or $\mu_I(y) = \beta$. Thus

$$\mu_I(x \oplus y) \geq \beta = \min \{\mu_I(x), \mu_I(y)\}.$$ 

Let $x, y \in M$ be such that $y \leq x$. If $y \in I$, then $\mu_I(y) = \alpha \geq \mu_I(x)$. Assume that $y \not\in I$. Then $x \not\in I$, and thus $\mu_I(y) = \beta = \mu_I(x)$. Therefore $\mu_I$ is a fuzzy ideal of $M$.

**Proposition 3.3.** Let $\mu$ be a fuzzy ideal of a pseudo $MV$-algebra $M$. Then

(i) $(\forall x, y \in M) (\mu(x \circ y) \geq \min \{\mu(x), \mu(y)\})$.

(ii) $(\forall x, y \in M) (\mu(x \land y) \geq \min \{\mu(x), \mu(y)\})$.

(iii) $(\forall x, y \in M) (\mu(x \land y) = \min \{\mu(x), \mu(y)\})$.

(iv) $(\forall x, y \in M) (\mu(x \circ y) = \min \{\mu(x), \mu(y)\})$.

**Proof.** Since $x \circ y \leq x \land y \leq x \lor y \leq x \oplus y$ for all $x, y \in M$, it follows from (d1) and (d2) that

$$\mu(x \circ y) \geq \mu(x \land y) \geq \mu(x \lor y) \geq \mu(x \oplus y) \geq \min \{\mu(x), \mu(y)\}.$$
Since \( x \oplus y \geq x \lor y \geq x, y \) for all \( x, y \in M \), we have \( \mu(x \oplus y) \leq \mu(x), \mu(y) \) and \( \mu(x \lor y) \leq \mu(x), \mu(y) \) by (d2). This completes the proof.

**Theorem 3.4.** Let \( \mu \) be a fuzzy set in a pseudo MV-algebra \( M \). Then \( \mu \) is a fuzzy ideal of \( M \) if and only if it satisfies (d1) and

\[
\text{(d4) } (\forall x, y \in M) \ (\mu(x \lor y) \geq \mu(x)).
\]

**Proof.** Let \( \mu \) be a fuzzy ideal of \( M \) and let \( x, y \in M \). Since \( x \land y \leq x \), it follows from (d2) that \( \mu(x \land y) \geq \mu(x) \). Suppose that \( \mu \) satisfies (d1) and (d4). Let \( x, y \in M \) be such that \( y \leq x \). Then \( x \land y = y \) and so \( \mu(y) = \mu(x \land y) \geq \mu(x) \) by (d4). Hence \( \mu \) is a fuzzy ideal of \( M \).

**Proposition 3.5.** Every fuzzy ideal \( \mu \) of a pseudo MV-algebra \( M \) satisfies the following inequality

\[
(\forall x, y \in M) \ (\mu(y) \geq \min \{\mu(x), \mu(x^\lor y)\}).
\]

**Proof.** Let \( \mu \) be a fuzzy ideal of a pseudo MV-algebra \( M \). Since \( y \leq x \lor y = x \oplus x^\lor \circ y \) for all \( x, y \in M \), it follows from (d1) and (d2) that

\[
\mu(y) \geq \mu(x \oplus x^\lor \circ y) \geq \min \{\mu(x), \mu(x^\lor \circ y)\}.
\]

This completes the proof.

**Proposition 3.6.** Let \( \mu \) be a fuzzy set in a pseudo MV-algebra \( M \) that satisfies (d3) and (1). Then \( \mu \) satisfies the condition (d2) and

\[
(\forall x, y \in M) \ (\mu(y) \geq \min \{\mu(x), \mu(y \circ x^\lor)\}).
\]

**Proof.** Assume that \( \mu \) satisfies (d3) and (1). Let \( x, y \in M \) be such that \( y \leq x \). Using (b3) and (b9), we have \( x^\lor \circ y \leq x^\lor \circ x = 0 \) and so \( x^\lor \circ y = 0 \). It follows from (d3) and (1) that

\[
\mu(y) \geq \min \{\mu(x), \mu(x^\lor \circ y)\} = \min \{\mu(x), \mu(0)\} = \mu(x)
\]

so that (d2) is valid. Note that

\[
(y \circ x^\lor)^\lor \circ (y \circ x^\lor \oplus x) \leq (y \circ x^\lor)^\lor \circ (y \circ x^\lor) \oplus x = 0 \oplus x = x
\]

so from (d2) that \( \mu(x) \leq \mu((y \circ x^\lor)^\lor \circ (y \circ x^\lor \oplus x)) \). Now since

\[
x^\lor \circ y \leq x \oplus x^\lor \circ y = y \circ x^\lor \oplus x,
\]

it follows from (d2) that \( \mu(x^\lor \circ y) \geq \mu(y \circ x^\lor \oplus x) \) so that

\[
\mu(y) \geq \min \{\mu(x), \mu(x^\lor \circ y)\} \geq \min \{\mu(x), \mu(y \circ x^\lor \oplus x)\}
\]
\[ \mu \geq \min \{ \mu(x), \min \{ \mu(y \odot x^\sim), \mu((y \odot x^\sim) \odot (y \odot x^\sim \odot x)) \} \} \]
\[ \geq \min \{ \mu(x), \min \{ \mu(y \odot x^\sim), \mu(x) \} \} \]
\[ = \min \{ \mu(x), \mu(y \odot x^\sim) \}. \]

This completes the proof.

**Proposition 3.7.** If a fuzzy set \( \mu \) in a pseudo MV-algebra \( M \) satisfies conditions (d3) and (2), then \( \mu \) is a fuzzy ideal of \( M \).

**Proof.** Let \( x, y \in M \) be such that \( y \leq x \). Then \( y \odot x^\sim \leq x \odot x^\sim = 0 \) by (b3) and (b9), and thus \( y \odot x^\sim = 0 \). Using (d3) and (2), we have
\[ \mu(y) \geq \min \{ \mu(x), \mu(y \odot x^\sim) \} = \min \{ \mu(x), \mu(0) \} = \mu(x). \]

Thus (d2) is valid. Note that
\[ (x \oplus y) \odot y^\sim = (x \oplus (y^\sim)^\sim) \odot y^\sim = x \wedge y^\sim \leq x \]
for all \( x, y \in M \) so from (2) and (d2) that
\[ \mu(x \oplus y) \geq \min \{ \mu(y), \mu((x \oplus y) \odot y^\sim) \} \geq \min \{ \mu(y), \mu(x) \}. \]
Hence (d1) is valid, and \( \mu \) is a fuzzy ideal of \( M \).

Combining Propositions 3.5, 3.6 and 3.7, we have the following characterization of a fuzzy ideal in a pseudo MV-algebra.

**Theorem 3.8.** For a fuzzy set \( \mu \) in a pseudo MV-algebra \( M \), the following are equivalent:

(i) \( \mu \) is a fuzzy ideal of \( M \).

(ii) \( \mu \) satisfies the conditions (d3) and (1).

(iii) \( \mu \) satisfies the conditions (d3) and (2).

**Proposition 3.9.** Let \( \mu \) be a fuzzy set in a pseudo MV-algebra \( M \). If \( \mu \) satisfies conditions (d3) and
\[ (\forall x, y, z \in M) (\mu(x \odot y) \geq \min \{ \mu(x \odot y \odot z), \mu(z^\sim \odot y) \}), \tag{3} \]
then \( \mu \) is a fuzzy ideal of \( M \). Moreover, \( \mu \) satisfies:

(i) \( (\forall x, y \in M) (\mu(x \odot y) = \mu(x \odot y \odot y)) \),

(ii) \( (\forall x \in M) (\forall n \in \mathbb{N}) (\mu(x) = \mu(x^n)), \) where \( x^n = x^{n-1} \odot x = x \odot x^{n-1} \) and \( x^0 = 1 \).
Proof. Taking $x = y$, $y = 1$ and $z = x^\sim$ in (3) and using (a8) and (b7), we have
\[
\mu(y) = \mu(y \odot 1) \geq \min \{ \mu(y \odot 1 \odot x^\sim), \mu((x^\sim) \odot y) \} = \min \{ \mu(y \odot x^\sim), \mu(x) \}.
\]

It follows from Theorem 3.8 that $\mu$ is a fuzzy ideal of $M$. Now taking $z = y$ in (3) and using (b9) and (d3), we get
\[
\mu(x \odot y) \geq \min \{ \mu(x \odot y \odot y), \mu(y^\sim \odot y) \}
\]
\[
= \min \{ \mu(x \odot y \odot y), \mu(0) \}
\]
\[
= \mu(x \odot y \odot y).
\]

On the other hand, since $x \odot y \odot y \leq x \odot y$, we see that $\mu(x \odot y \odot y) \geq \mu(x \odot y)$. Then (i) holds.

The proof of (ii) is by induction on $n$. If $n = 1$, then (ii) is obviously true. If we put $x = 1$ and $y = x$ in (i), then
\[
\mu(x) = \mu(1 \odot x) = \mu(1 \odot x \odot x) = \mu(x^2).
\]
Now assume that (ii) is valid for every positive integer $k > 2$. Then
\[
\mu(x^{k+1}) = \mu(x^{k-1} \odot x \odot x) = \mu(x^{k-1} \odot x) = \mu(x^k) = \mu(x).
\]
Therefore (ii) is true.

Lemma 3.10. For any fuzzy set $\mu$ in a pseudo $MV$-algebra $M$, the condition (3) is equivalent to the following condition:
\[
(\forall x, y, z \in M) (\mu(x \odot y) \geq \min \{ \mu(x \odot y \odot z^\sim), \mu(z \odot y) \}). \quad (4)
\]
Proof. (3) $\Rightarrow$ (4): Let $x, y, z \in M$. By (3),
\[
\mu(x \odot y) \geq \min \{ \mu(x \odot y \odot z^\sim), \mu((z^\sim) \odot y) \}.
\]
Since $(z^\sim)^\sim = z$, we have (4).

(4) $\Rightarrow$ (3): Applying (4) we see that
\[
\mu(x \odot y) \geq \min \{ \mu(x \odot y \odot (z^\sim)^\sim), \mu(z^\sim \odot y) \}.
\]
From this we obtain (3), because $(z^\sim)^\sim = z$ by (b6).

In [2] we introduced the notion of implicative ideals in pseudo $MV$-algebras. An ideal $I$ of a pseudo $MV$-algebra $M$ is said to be implicative if it satisfies the following implication:
\[
(\forall x, y, z \in M) (x \odot y \odot z \in I, z^\sim \odot y \in I \Rightarrow x \odot y \in I).
\]
Definition 3.11. Let μ be a fuzzy ideal of a pseudo MV-algebra M. We say that μ is fuzzy implicative if it satisfies the condition (3) (or (4)).

Proposition 3.12. Let I be an ideal of a pseudo MV-algebra M. Then I is implicative if and only if the fuzzy set μ, which is described in Example 3.2 is a fuzzy implicative ideal of M.

Proof. Straightforward.

Lemma 3.13. Let μ be a fuzzy ideal of a pseudo MV-algebra M. Then

\[(\forall x, y \in M) (\mu(x \circ y) \geq \min \{\mu(x \circ y), \mu(y \wedge y^\gamma)\}).\]

Proof. Applying (b7) and (b8) we have

\[x \circ y = (x \circ y) \circ 1 = (x \circ y) \circ (y \oplus y^\gamma),\]

and so \(x \circ y \leq (x \circ y) \circ y \oplus y^\gamma\) by (b5). Using (b4) we obtain

\[x \circ y \leq y \wedge (x \circ y \circ y \oplus y^\gamma)\]
\[\leq (x \circ y \circ y \oplus y) \wedge (x \circ y \circ y \oplus y^\gamma)\]
\[= x \circ y \circ y \oplus (y \wedge y^\gamma).\]

It follows from (d2) and (d1) that

\[\mu(x \circ y) \geq \mu(x \circ y \circ y \oplus (y \wedge y^\gamma) \geq \min \{\mu(x \circ y), \mu(y \wedge y^\gamma)\}.\]

This completes the proof.

Theorem 3.14. Let μ be a fuzzy ideal of a pseudo MV-algebra M. Then the following statements are equivalent:

(i) μ is fuzzy implicative.

(ii) \((\forall x, y \in M) (\mu(x \circ y) = \mu(x \circ y \circ y)).\)

(iii) \((\forall x \in M) (x^2 = 0 \Rightarrow \mu(x) = \mu(0)).\)

(iv) \((\forall x \in M) (\mu(x \wedge x^\gamma) = \mu(0)).\)

(v) \((\forall x \in M) (\mu(x \wedge x^\gamma) = \mu(0)).\)

Proof. (i) \(\Rightarrow\) (ii): This is by Proposition 3.9.

(ii) \(\Rightarrow\) (iii): Taking \(x = 1\) and \(y = x\) in (ii), we get

\[\mu(x) = \mu(1 \circ x) = \mu(1 \circ x \circ x) = \mu(x^2).\]
Then (iii) is obviously true.

(iii) \( \Rightarrow \) (iv): Using (b3) and (b9) we have

\[
(x \wedge x^\sim)^2 = (x \wedge x^\sim) \circ (x \wedge x^\sim) \leq x \circ x^\sim = 0.
\]

Consequently, \( (x \wedge x^\sim)^2 = 0 \). Hence \( \mu(x \wedge x^\sim) = \mu(0) \).

(iv) \( \Rightarrow \) (v): Since \( x \circ x^\sim = x^\sim \circ x = (x^\sim)^\sim \), it follows from (iv) that \( \mu(x \wedge x^\sim) = \mu(0) \).

(v) \( \Rightarrow \) (i): By Lemma 3.13, \( \mu(x \circ y) \geq \min \{\mu(x \circ y \circ y), \mu(y \circ y^\sim)\} \). Therefore \( \mu(x \circ y) \geq \min \{\mu(x \circ y \circ y), \mu(0)\} = \mu(x \circ y \circ y) \). Applying (b3) and (b5) we get

\[
x \circ y \circ y \leq x \circ y \circ (z \vee y) = x \circ y \circ (z \oplus z^\sim \circ y) \leq x \circ y \circ z \oplus z^\sim \circ y.
\]

Since \( \mu \) is a fuzzy ideal, we have

\[
\mu(x \circ y \circ y) \geq \mu(x \circ y \circ z \oplus z^\sim \circ y) \geq \min \{\mu(x \circ y \circ z), \mu(z^\sim \circ y)\}.
\]

Thus \( \mu \) satisfies the condition (3), i.e., \( \mu \) is fuzzy implicative.

Using the level subset of a fuzzy set, we give a characterization of a fuzzy ideal.

**Theorem 3.15.** Let \( \mu \) be a fuzzy set in a pseudo MV-algebra \( M \). Then \( \mu \) is a fuzzy ideal of \( M \) if and only if its nonempty level subset

\[
U(\mu; \alpha) := \{x \in M \mid \mu(x) \geq \alpha\}
\]

is an ideal of \( M \) for all \( \alpha \in [0, 1] \).

**Proof.** Assume that \( \mu \) is a fuzzy ideal of \( M \) and let \( \alpha \in [0, 1] \) be such that \( U(\mu; \alpha) \neq 0 \). Obviously \( 0 \in U(\mu; \alpha) \). Let \( x, y \in M \) be such that \( x, y \in U(\mu; \alpha) \). Then \( \mu(x) \geq \alpha \) and \( \mu(y) \geq \alpha \). It follows from (d1) that

\[
\mu(x \oplus y) \geq \min \{\mu(x), \mu(y)\} \geq \alpha
\]

so that \( x \oplus y \in U(\mu; \alpha) \). Let \( x, y \in M \) be such that \( x \in U(\mu; \alpha) \) and \( y \leq x \). Then \( \mu(y) \geq \mu(0) \geq \alpha \) by (d2), and so \( y \in U(\mu; \alpha) \). Therefore \( U(\mu; \alpha) \) is an ideal of \( M \).

Conversely suppose that \( U(\mu; \alpha) \) is a nonempty ideal of \( M \) for all \( \alpha \in [0, 1] \). If (d1) is not valid, then there exist \( a, b \in M \) such that \( \mu(a \oplus b) < \min \{\mu(a), \mu(b)\} \). Taking

\[
\beta = \frac{1}{2} (\mu(a \oplus b) + \min \{\mu(a), \mu(b)\}),
\]

but
we get $\mu(a \odot b) < \beta < \min \{\mu(a), \mu(b)\}$. Therefore $a, b \in U(\mu; \beta)$ but $a \oplus b \notin U(\mu; \beta)$. This is a contradiction, and (d1) is valid. Finally let $x, y \in M$ be such that $y \leq x$.

Assume that $\mu(y) < \mu(x)$ and let $\gamma = \frac{1}{2} (\mu(y) + \mu(x))$. Then $\mu(y) < \gamma < \mu(x)$ and thus $x \in U(\mu; \gamma)$ and $y \notin U(\mu; \gamma)$. This is impossible, and $\mu$ is a fuzzy ideal of $M$.

**Corollary 3.16.** If $\mu$ is a fuzzy ideal of a pseudo MV-algebra $M$, then the set

$$M_a := \{x \in M \mid \mu(x) \geq \mu(a)\}$$

is an ideal of $M$ for every $a \in M$.

*Proof.* Straightforward.

**Theorem 3.17.** Let $\mu$ be a fuzzy set in a pseudo MV-algebra $M$. Then $\mu$ satisfies the condition (3) if and only if for all $x, y, z \in M$ and $\alpha \in [0, 1]$, whenever $x \odot y \odot z \in U(\mu; \alpha)$ and $z^{-} \odot y \in U(\mu; \alpha)$ then $x \odot y \in U(\mu; \alpha)$.

*Proof.* Let $x, y, z \in M$ and $\alpha \in [0, 1]$ be such that $x \odot y \odot z \in U(\mu; \alpha)$ and $z^{-} \odot y \in U(\mu; \alpha)$. Then $\mu(x \odot y \odot z) \geq \alpha$ and $\mu(z^{-} \odot y) \geq \alpha$. It follows from (3) that

$$\mu(x \odot y) \geq \min \{\mu(x \odot y \odot z), \mu(z^{-} \odot y)\} \geq \alpha$$

so that $x \odot y \in U(\mu; \alpha)$. Conversely, if the condition (3) is not valid, then

$$\mu(a \odot b) < \min \{\mu(a \odot b \odot c), \mu(c^{-} \odot b)\}$$

for some $a, b, c \in M$. Let

$$\beta := \frac{1}{2} (\mu(a \odot b) + \min \{\mu(a \odot b \odot c), \mu(c^{-} \odot b)\}).$$

Then $\mu(a \odot b) < \beta < \min \{\mu(a \odot b \odot c), \mu(c^{-} \odot b)\}$, and so $a \odot b \odot c \in U(\mu; \beta)$ and $c^{-} \odot b \in U(\mu; \beta)$ but $a \odot b \notin U(\mu; \beta)$. This is a contradiction.

Theorems 3.15 and 3.17 together yield the following corollary.

**Corollary 3.18.** Let $\mu$ be a fuzzy set in a pseudo MV-algebra $M$. Then $\mu$ is a fuzzy implicative ideal of $M$ if and only if for each $\alpha \in [0, 1]$, $U(\mu; \alpha) = \emptyset$ or $U(\mu; \alpha)$ is an implicative ideal of $M$.

**Theorem 3.19.** For a fuzzy set $\mu$ in a pseudo MV-algebra $M$, let $\mu^*$ be a fuzzy set in $M$ defined by
\[ \mu^*(x) := \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \]

for all \( x \in M \). Then \( \mu^* \) is the least fuzzy ideal of \( M \) that contains \( \mu \).

**Proof.** For any \( \beta \in \text{Im}(\mu^*) \), let \( \beta_n = \beta - \frac{1}{n} \) for \( n \in \mathbb{N} \). Let \( x \in U(\mu^*; \beta) \). Then

\[ \mu^*(x) \geq \beta, \]

which implies that

\[ \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \geq \beta - \frac{1}{n} = \beta_n, \ \forall n \in \mathbb{N}. \]

Hence for any \( n \in \mathbb{N} \) there exists \( \gamma_n \in \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \) such that \( \gamma_n > \beta_n \). Thus \( x \in \langle U(\mu; \gamma_n) \rangle \) for all \( n \in \mathbb{N} \). Consequently, \( x \in \bigcap_{n \in \mathbb{N}} \langle U(\mu; \gamma_n) \rangle \). On the other hand, if \( x \in \bigcap_{n \in \mathbb{N}} \langle U(\mu; \gamma_n) \rangle \), then \( \gamma_n \in \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \) for any \( n \in \mathbb{N} \). Therefore

\[ \beta - \frac{1}{n} = \beta_n < \gamma_n \leq \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} = \mu^*(x), \ \forall n \in \mathbb{N}. \]

Since \( n \) is arbitrary, it follows that \( \beta \leq \mu^*(x) \) so that \( x \in U(\mu^*; \beta) \). Hence \( U(\mu^*; \beta) = \bigcap_{n \in \mathbb{N}} \langle U(\mu; \gamma_n) \rangle \), which is an ideal of \( M \). Therefore we conclude that \( \mu^* \) is a fuzzy ideal of \( M \) by Theorem 3.15. We now prove that \( \mu^* \) contains \( \mu \). For any \( x \in M \), let \( \beta \in \{ \alpha \in [0, 1] \mid x \in U(\mu; \alpha) \} \}. \) Then \( x \in U(\mu; \beta) \) and thus \( x \in \langle U(\mu; \beta) \rangle \). Therefore \( \beta \in \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \}, \) which implies that

\[ \{ \alpha \in [0, 1] \mid x \in U(\mu; \alpha) \} \subseteq \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \}. \]

It follows that

\[ \mu(x) \leq \sup \{ \alpha \in [0, 1] \mid x \in U(\mu; \alpha) \} \]
\[ \leq \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \]
\[ = \mu^*(x) \]

which shows that \( \mu^* \) contains \( \mu \). Finally, let \( v \) be a fuzzy ideal of \( M \) containing \( \mu \). Let
then let $x \in M$ and $\mu^*(x) = \beta$. Then $x \in \bigcap_{n \in \mathbb{N}} \langle U(\mu; \gamma_n) \rangle$, and so $x \in \langle U(\mu; \gamma_n) \rangle$ for all $n \in \mathbb{N}$. Consequently,

$$x \leq y_1 \oplus \ldots \oplus y_k$$

for some $y_1, \ldots, y_k \in U(\mu, \gamma_n)$ (see the last paragraph of Section 2). It is easy to check that

$$\mu(x) \geq \min \{ \mu(y_1), \ldots, \mu(y_k) \} \geq \gamma_n.$$

Then $v(x) \geq \mu(x) \geq \gamma_n \geq \beta_n = \beta - \frac{1}{n}$ for every $n \in \mathbb{N}$, so that $v(x) \geq \beta = \mu^*(x)$ since $n$ is arbitrary. This shows that $\mu^* \subseteq v$, and the proof is complete.

**REFERENCES**


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