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FUZZY BANACH ALGEBRA

ABSTRACT: *The notion of Fuzzy Banach algebra has been introduced and some of its properties are studied.*

Key words: *Algebra, Fuzzy normed algebra, Banach algebra and Fuzzy Banach algebra.*

1. INTRODUCTION

Fuzzy normed linear space have been introduced and studied by Felbin ([3]). Some further studies in this direction are in ([2], [7]). Fuzzy algebra ([1]) and Fuzzy algebras over Fuzzy fields ([4],[6]) have been defined and studied. We define Fuzzy Banach algebra below, in the line of [3] and study some of its properties. First we define a few concepts that are to be used in this context.

Definition 1.1: A *Fuzzy real number* η is a Fuzzy set on R (the set of real numbers), that is a mapping $\eta : R \to I(=[0,1])$, associating each real number t with its grade of membership $\eta(t)$.

Definition 1.2: A Fuzzy real number η is called *convex,* if

$\eta(t) \geq \eta(s) \wedge \eta(r) = \min\{\eta(s), \eta(r)\}$, where $s < t < r$.

Definition 1.3: If there exists a $t_0 \in R$ such that $\eta(t_0) = 1$, then a Fuzzy real number η is called *normal.*

Definition 1.4: The α–*level* set of a Fuzzy real number η, $0 < \alpha \leq 1$, denoted by $[\eta]_\alpha$ is defined as $[\eta]_\alpha = \{t \in R : \eta(t) \geq \alpha\}$.

Definition 1.5: A Fuzzy real number η is said to be *upper semi-continuous,* if for each $\in > 0, \eta^{-1}([0, a+ \in])$, for all $a \in I$ is open in the usual topology of *R*.

The set of all upper semi-continuous, normal and convex Fuzzy real numbers is denoted by *R(I)*.

The set *R* can be embedded in *R(I)* if we define $\overline{r} \in R(I)$ by

$$\overline{r}(t) = \begin{cases} 1, & if\ t = r; \\ 0, & if\ t \neq r. \end{cases}$$

Definition 1.6: A Fuzzy real number η is called *nonnegative*, if $\eta(t) = 0$ for all $t < 0$.

The set of all non negative Fuzzy real numbers of *R(I)* is denoted by $R^*(I)$.

The arithmetic operations $\oplus, \Theta, \odot$ and $\oslash$ on R(I) x R(I) can be defined in terms of $\alpha-$cuts as follows:

Proposition 1.1: ([5], Lemma 2.1) Let $\eta,\ \delta \in R(I)$, $[\eta]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[\delta]_\alpha = [a_2^\alpha, b_2^\alpha],\ \alpha \in (0,1]$. Then

$$[\eta \oplus \delta]_\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha];$$

$$[\eta \Theta \delta]_\alpha = [a_1^\alpha - b_2^\alpha, b_1^\alpha - a_2^\alpha];$$

$$[\eta \odot \delta]_\alpha = \left[\min_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha, \max_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha \right];$$

$$[\eta \oslash \delta]_\alpha = \left[\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha} \right],\ a_2^\alpha > 0.$$

2. FUZZY BANACH ALGEBRA

Definition 2.1: Let X be algebra over R. We define $\|\bullet\| : X \to R^*(I)$ such that

1. $\|x\| = \overline{0}$ if and only if x = 0;

2. $\|rx\| = |r|\|x\|, x \in X, r \in R$ and

3. For all $x, y \in X, \|x+y\| \leq \|x\| \oplus \|y\|$.

If further $\|x.y\| \leq \|x\| \odot \|y\|$, we say that $(X, \|\bullet\|)$ is a *Fuzzy normed algebra* over *R*.

Definition 2.2: Let $(X, \|.\|)$ be a Fuzzy normed algebra. An element $1 \in X$ is said to be the multiplicative identity of X, if $1 \cdot x = x \cdot 1 = x$ for all $x \in X$.

A Fuzzy normed algebra with identity is having a multiplicative identity '1' such that $\|1\| = \bar{1}$.

Definition 2.3: Let $(X, \|\bullet\|)$ be a Fuzzy normed algebra. A sequence $\{x_n\} \in X$ is said to converge to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \|x_n - x\| = \bar{0}$, that is $\lim_{n\to\infty} \|\|x_n - x\|\|_1^\alpha = \lim_{n\to\infty} \|\|x_n - x\|\|_2^\alpha = 0,\ for\ \alpha \in (0,1],$ where, for any $y \in X$, we write $\|y\|^\alpha = \left[\|\|y\|\|_1^\alpha, \|\|y\|\|_2^\alpha \right]$.

Following Fellbin ([3], note on page 243), $\lim_{n\to\infty} \|x_n - x\| = \bar{0}$ is equivalent to $\|\|x_n - x\|\|_2^\alpha \to 0\ as\ n \to \infty$, for each $\alpha \in (0,1]$.

Definition 2.4: A sequence $\{x_n\}$ in a Fuzzy normed algebra $(X, \|.\|)$ is said to be Cauchy if $\lim_{\substack{n\to\infty \\ m\to\infty}} \|x_m - x_n\| = \bar{0}$ that is $\lim_{\substack{n\to\infty \\ m\to\infty}} \|\|x_m - x_n\|\|_2^\alpha = \bar{0}$ for $\alpha \in (0,1]$.

Definition 2.5: A Fuzzy normed algebra $(X, \|\bullet\|)$ is said to be *complete* if every Cauchy sequence in X converges in X.

A complete Fuzzy normed algebra is called a *Fuzzy Banach algebra.*

Definition 2.6: Let $(X, \|\bullet\|)$ be a Fuzzy normed algebra. Y is called a Fuzzy *subalgebra* (F- subalgebra) of X if Y is a subalgebra of X considered as an algebra, with the Fuzzy norm obtained by restricting the Fuzzy norm on X to the subset Y.

Proposition 2.1: ([3], Theorem 4.1) every finite dimension F-subspace Y of a Fuzzy normed linear space X is complete. In particular every finite dimension Fuzzy normed linear space is complete.

Corollary 2.1: Every finite dimensional F-subalgebra of a Fuzzy normed algebra is complete. In particular every finite dimensional Fuzzy normed algebra is a Fuzzy Banach algebra.

Theorem 2.1: Let $(X, \|\bullet\|)$ be a Fuzzy normed algebra with identity 1 and $x \in X$ be such that $\|1-x\| < \bar{1}$. Then x^{-1} exists and $\left\|\|x^{-1}\|\right\|_2^{\alpha} \leq \left(\frac{1}{1-\left\|\|1-x\|\right\|_2^{\alpha}}\right)$ for each $\alpha \in (0,1]$.

Proof: For $N \geq M$,

$$\left\|\sum_{n=0}^{N}(1-x)^n - \sum_{n=0}^{M}(1-x)^n\right\| = \left\|\sum_{n=M+1}^{N}(1-x)^n\right\| \leq \sum_{n=M+1}^{N}\left\|(1-x)^n\right\| \leq \sum_{n=M+1}^{N}\|1-x\|^n .$$

We have $\|1-x\| < \bar{1} \Rightarrow \left\|\|1-x\|\right\|_2^{\alpha} < 1$ for each $\alpha \in (0,1]$.

So, $\sum_{n=M+1}^{N}\left\|\|1-x\|\right\|_2^{n,\alpha} \to 0 \; as \; N, M \to \infty$ for each $\alpha \in (0,1]$.

This implies that $\sum_{n=M+1}^{N}\|1-x\|^n \to 0 \; as \; N, M \to \infty.$

So, $\left\|\sum_{n=0}^{N}(1-x)^n - \sum_{n=0}^{M}(1-x)^n\right\| \to 0 \; as \; N, M \to \infty.$

Hence $\left\{\sum_{n=0}^{N}(1-x)^n\right\}_{N=0}^{\infty}$ is a Cauchy sequence is X. But X is complete, so,

$$\sum_{n=0}^{\infty}(1-x)^n = y \text{ (say)}, \ y \in X.$$

Again, $xy = [1-(1-x)]\left[\sum_{n=0}^{\infty}(1-x)^n\right]$

$$= \lim_{N\to\infty}[1-(1-x)]\left[\sum_{n=0}^{\infty}(1-x)^n\right]$$

$$= \lim_{N\to\infty}(1-(1-x)^{N+1}) = 1,$$

Since $\lim_{N\to\infty}\left\|\left\|(1-x)^{N+1}\right\|\right\|_2^{\alpha} \to 0$ for each $\alpha \in (0,1] \Rightarrow \lim_{N\to\infty}\left\|(1-x)^{N+1}\right\| = 0.$

Similarly, it can be shown that $yx = 1$.

Finally, $\|\|y\|\|_2^{\alpha} = \lim_{N\to\infty}\left\|\left\|\sum_{n=0}^{N}(1-x)^n\right\|\right\|_2^{\alpha} \le \lim_{N\to\infty}\sum_{n=0}^{N}\|\|1-x\|\|_2^{n,\alpha} = \dfrac{1}{1-\|\|x\|\|_2^{\alpha}}, \alpha \in (0,1].$

3. FUZZY NORMED ALGEBRA VALUED SEQUENTIAL ALGEBRA

Let X be a Fuzzy normed algebra. We denote by $\Omega(X)$, the set of all sequences in X, which is an algebra with respect to point-wise addition, multiplication and scalar multiplication. Any subalgebra $\lambda(X)$ of $\Omega(X)$ is called a *Fuzzy normed valued sequential algebra*.

Definition 3.1: Let $(X, \|\bullet\|)$ be a Fuzzy normed algebra. We denote by $c_F(X)$, the set of all convergent sequences in $(X, \|\bullet\|)$. $c_F^0(X)$ is the set of all sequences in $(X, \|\bullet\|)$ which are convergent to $\overline{0}$.

Proposition 3.1: ([5], Theorem 4.1) If X is a Fuzzy normed linear space then $c_F(X)$ is a Fuzzy normed linear space valued sequence space.

Definition 3.2: Let $(X, \|\bullet\|)$ be a Fuzzy normed algebra. We denote by $\ell_F^\infty(X)$, the set of all bounded sequences in $(X, \|\bullet\|)$. By a bounded sequence in X, we mean $x = \{x_n\}$ such that $\sup\|x_n\| < \infty$. Here we note that each $\|x_n\|$ is a Fuzzy real number of the special type, that is a member of $R(I)$. On $R(I)$ we define an ordering as, follows:

For $\lambda, \mu \in R(I), \lambda \le \mu$ if and only if $\lambda_\alpha \le \mu_\alpha$, for all $\alpha \in (0,1]$; equivalently, $[a_\alpha, b_\alpha] \subseteq [c_\alpha, d_\alpha]$ for all $\alpha \in (0,1]$, where $\lambda_\alpha = [a_\alpha, b_\alpha]$ and $\mu_\alpha = [c_\alpha, d_\alpha]$. This is in turn is true if and only if $a_\alpha \le c_\alpha$ and $b_\alpha \le d_\alpha$, for all $\alpha \in (0,1]$. This ordering is a partially ordered relation on $R(I)$. We mean that the set $\{\|x_n\|, \quad n \in N\}$ is bounded if and only if there exist two Fuzzy real numbers $\beta, \delta \in R(I)$ such that $\beta \le \|x_n\| \le \delta$, for all $n \in N$.

By $\sup\limits_n \|x_n\|$, we mean the least upper bound of the set $\{\|x_n\|, \ n \in N\}$ in the usual sense.

The compatibility of the definition of the boundness of members of $R(I)$ in terms of $\alpha-$level sets can be seen from the fact that the $\alpha-$level sets of upper semi-continuous, convex, normal Fuzzy real numbers for each α, $0 < \alpha \le 1$ is a closed interval $[a^\alpha, b^\alpha]$, where $a^\alpha = -\infty$ and $b^\alpha = +\infty$ are admissible ([3]). In this connection, the following result is worth noting:

Proposition 3.2: ([3], Lemma 2.2) Let $[a^\alpha, b^\alpha], 0 < \alpha \le 1,$ be a given family of nonempty intervals. Suppose

(a) for all $0 < \alpha_1 \leq \alpha_2$

$$\left[a^{\alpha_1}, b^{\alpha_1}\right] \supset \left[a^{\alpha_2}, b^{\alpha_2}\right];$$

(b) for any increasing sequence $\{\alpha_k\}$ in $(0,1]$ converging to α,

$$\left[\lim_{k\to\infty} a^{\alpha_k}, \lim_{k\to\infty} b^{\alpha_k}\right] = \left[a^{\alpha}, b^{\alpha}\right].$$

Then the family $\left[a^{\alpha}, b^{\alpha}\right]$ represents the $\alpha-$level sets of a Fuzzy real number η in $R(I)$.

Conversely, if $\left[a^{\alpha}, b^{\alpha}\right]$, $o < \alpha \leq 1$, are $\alpha-$level sets of a Fuzzy real number $\eta \in R(I)$, then the conditions (a) and (b) are satisfied.

Definition 3.3: For any $\overline{x} = \{x_n\} \in c_F\ (X)\ or\ c_F^0(X)\ or\ l_F^{\infty}(X)$ we define $\|\overline{x}\| = \sup_n \|x_n\|$.

The following theorems on Fuzzy normed linear spaces have been established in ([7]).

Proposition 3.3: ([5], theorem 4.2) If $(X, \|\bullet\|)$ is a Fuzzy normed linear space, then $c_F(X) \subseteq \ell_F^{\infty}(X)$.

Proposition 3.4: ([5], Theorem 4.3) For a Fuzzy normed linear space X, $c_F(X)$ is complete if X is complete.

Theorem 3.2: $c_F(X)$ is commutative Fuzzy Banach algebra with identity.

Proof: By Proposition 3.4 $c_F(X)$ is a complete Fuzzy normed linear space valued sequence space. Sine X is a Fuzzy Banach algebra $\|xy\| \leq \|x\| \odot \|y\| \in\ for\ x, y \in X$. So, for $\overline{x}, \overline{y} \in c_F(X)$,

$$\|\bar{x}.\bar{y}\| = \sup_n \|x_n y_n\| \le \sup_n \left(\|x_n\| \odot \|y_n\|\right)$$

$$\le \sup_n \|x_n\| \odot \sup_n \|y_n\| = \|\bar{x}\| \odot \|\bar{y}\|.$$

Again, suppose $\bar{x} = \{x_n\}, \bar{y} = \{y_n\}$ be any two elements of $c_F(X)$. Then there exists $x, y \in X$ such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ *as* $n \to \infty$. Hence

$$\left\|\|x_n - x\|\right\|_2^\alpha \to 0, \left\|\|y_n - y\|\right\|_2^\alpha \to 0 \text{ as } n \to \infty, \text{ for each } \alpha \in (0,1].$$

So, $\left\|\|x_n y_n - xy\|\right\|_2^\alpha \le \left\|\|x_n y_n - xy_n\|\right\|_2^\alpha + \left\|\|xy_n - xy\|\right\|_2^\alpha \to 0$ as $n \to \infty$, $\alpha \in (0,1]$,

as $\sup_n \left\|\|y_n\|\right\|_2^\alpha < \infty$.

It may be noted that $\left\|\|y_n - y\|\right\|_2^\alpha \to 0 \Rightarrow \{y_n\}$ is convergent in $\left\|\|\bullet\|\right\|_2^\alpha$ for each α. So, $\{y_n\}$ is bounded in $\left\|\|\bullet\|\right\|_2^\alpha$ for each α. Hence $\|x_n y_n - xy\| \to \bar{0}$ *as* $n \to \infty$. So, $\{x_n y_n\} \in c_F(X)$.

This proves that $c_F(X)$ is closed under multiplication. The associative property under multiplication holds trivially.

Also, if $1 \in X$ then $\bar{1} = \{1, 1, ...\} \in c_F(X)$, which is the multiplicative identity.

Corollary 3.1: $c_F^0(X)$ is a commutative sequential Fuzzy Banach algebra if X is a commutative Fuzzy Banach algebra.

Definition 3.5: A Fuzzy normed linear space valued sequence space $A(X)$ is called *normal* if $\bar{x} = \{x_i\} \in A(X)$ and $\|y_i\| \le \|x_i\|$ for all i, implies that $\bar{y} = \{y_i\} \in A(X)$.

Definition 3.6: For a subsequence *J* of *N* and a Fuzzy normed linear space valued sequence space *A(X)*, we define $A_j(X) = \{\mu_i\}$: there exists $\{\eta_i\} \in A(X)$ with $\mu_i = \eta_{n_i}$ for all $n_i \in J$ and call $A_J(X)$, the *J-step space* of *A(X)*.

If $\{\mu_i\} \in A_j(X)$ then the *canonical pre-image* of $\{\mu_i\}$ is the sequence $\{\bar{\mu}_j\}$ which agrees with $\{\mu_i\}$ on the indices of *J* and is $\bar{0}$ elsewhere.

The canonical pre-image of $A_J(X)$ is the space $\bar{A}_J(X)$ containing the canonical pre-images of the elements of $A_J(X)$.

Definition 3.7: A Fuzzy normed linear space valued sequence space *A(X)* is said to be monotone if *A(X)* contains the canonical pre-image of all its subspaces.

We can define a Fuzzy normed linear space valued sequential algebra to be *normal* or *monotone* depending upon whether the underlying Fuzzy normed linear space valued sequence space is normal or monotone.

The following properties are immediate from [7]:

Theorem 3.3 (i) The Fuzzy normed linear algebra valued sequential algebra $l_F^\infty(X)$ is both monotone and normal.

(ii) The Fuzzy normed linear algebra valued sequential algebras $c_F(X)$ and $c_F^0(X)$ are monotone.

Definition 3.8: Let $(X, \|\cdot\|)$ be a Fuzzy normed linear algebra. We define

$m_F^0(X) = \{\{x_i\}$, where $x_i = \sum_{j=0}^{n} y_i a^i{}_j$ and $a_j^i = \bar{0}$ *or* $\bar{1}$, $y_i \in X\}$.

It can be shown as in [7] that $m_F^0(X)$ is monotone.

Also, we can show (as in [7] Theorem 4.2) that $c_F(X) \subseteq l_F^\infty(X)$, for any Fuzzy normed linear algebra $(X, \|.\|)$.

CONCLUSION

In this paper we introduced the notion of Fuzzy Banach algebra and Fuzzy normed algebra valued sequential algebra. Also, we studied their properties. More results on properties of Fuzzy normed algebra valued sequential algebra can be obtained by taking special cases.

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