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G-FUZZY ROUGH IDEALS IN A RING

ABSTRACT: *In this paper, the notion of an* G*-fuzzy rough subring (resp.ideal) is defined, which is an extended notion of a fuzzy rough subring (resp.ideal) in [1], and some basic properties are discussed. Meanwhile, the concept of threshold of a fuzzy rough set is introduced, and a algorithm to calculate threshold is provided.*

Key Words: *Rough set; Fuzzy rough point;* G*-fuzzy rough subring;* G*-fuzzy rough ideal; Threshold.*

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1. INTRODUCTION

Theory of fuzzy sets initiated by Zadeh[9] and the theory of rough sets initiated by Pawlak[6] provided useful means of describing and modeling of vagueness in ill defined environment. Dubois and Prade[2] made an investigation around these two notions and reported that they are not rival theories but two different tools, and aim to two different purposes. As a natural need, Dubois and Prade[2] combined the two theories and so rough fuzzy sets and fuzzy rough sets are defined. Recently, Davvaz[1] gives notions of fuzzy rough subrings and ideals, and obtains some fundamental results. It is now natural to investigate generalizations of existing fuzzy rough subsystems. For this goal, using the notion of "fuzzy rough point", the concepts of *G*-fuzzy rough subrings and ideals are introduced in the present paper, and some basic properties are obtained. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is an *G*-fuzzy rough subring (resp.ideal) iff $(\mathscr{A}_t, \overline{\mathscr{A}}_t)$ is a rough subring (resp.ideal) of $Apr(X)$ for every $0 \leq t \leq 0.5$. This shows that *G*-fuzzy rough subrings (resp.ideals) are extended concepts of fuzzy rough subrings (resp.ideals) in [1]. Meanwhile, the concept of

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threshold of a fuzzy rough set is introduced, a algorithm to calculate the threshold of a fuzzy rough set is provided.

2. ROUGH SETS AND ROUGH IDEALS

Let U be a nonempty set, by $\mathscr{P}(U)$ we mean the power-set on U. Let θ be an equivalence relation on U, then θ induces a partition of U, we use $[x]_\theta$ to denote the equivalence class of θ determined by x, i.e. $[x]_\theta = \{y \in U | x\theta y\}$.

Definition 2.1. A pair(U, θ) where $U \neq \phi$ and θ is an equivalence relation on U, is called an approximation space.

Definition 2.2. For an approximation space (U, θ), by a rough approximation in (U, θ) we mean a mapping $Apr : \mathscr{P}(U) \to \mathscr{P}(U) \times \mathscr{P}(U)$ defined by for every $X \in \mathscr{P}(U), Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where $\underline{Apr}(X) = \{x \in X | [x]_\theta \subseteq X\}$, $\overline{Apr}(X) = \{x \in X | [x]_\theta \cap X \neq \phi\}$. $\underline{Apr}(X)$ and $\overline{Apr}(X)$ are called, respectively, lower rough approximation and upper rough approximation of X in (U, θ).

Definition 2.3. Given an approximation space (U, θ), a pair $(A, B) \in \mathscr{P}(U) \times \mathscr{P}(U)$ is called a rough set in (U, θ) iff $(A, B) = Apr(X)$ for some $X \in \mathscr{P}(U)$.

The properties of rough sets will be found in [6,7].

Let R be a ring and X a nonempty subset of R. Let I be an ideal of R. It is easy to see that $a \equiv b(\text{mod } I)$ is an equivalence relation on R. Therefore, when $U = R$ and θ is the above equivalence relation, then the pair (R, I) is an approximation space. $\underline{Apr}_I(X) = \{x \in R \mid x + I \subseteq X\}$, $\overline{Apr}_I(X) = \{x \in R \mid (x + I) \cap X \neq \phi\}$ are called, respectively, lower and upper approximations of the set X with respect to I.

Definition 2.4. Let I be an ideal of R and $Apr_I(X) = (\underline{Apr}_I(X), \overline{Apr}_I(X))$ a rough set in the approximation space (R, I). If $\underline{Apr}_I(X)$ and $\overline{Apr}_I(X)$ are subsubrings (resp. ideals) of R, then we call $Apr_I(X)$ a rough subring (resp. ideal). A rough subring also called a rough ring.

In present paper, for the sake of convenience, we will denote $Apr_I(X)$, $\underline{Apr}_I(X)$, $\overline{Apr}_I(X)$ by $Apr(X)$, $(\underline{Apr}(X), \overline{Apr}(X))$, respectively, in (R, I).

Theorem 2.1. (i)Let I, J be two ideals of R, then $Apr_I(J)$ and $Apr_J(I)$ are rough ideals. (ii) Let I be an ideal and J is a subring of R, then $Apr_I(J)$ is rough ring.

The more details of rough subring (resp. ideal) will be seen in [1].

3. FUZZY ROUGH SETS AND FUZZY ROUGH IDEALS

Definition 3.1. Let (U, θ) be an approximation space and $Apr(X)$ a rough set in (U, θ). A fuzzy rough set $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ in $Apr(X)$ is characterized by a pair of maps

$$\mu_{\underline{Apr}(A)} : \underline{Apr}(X) \to [0,1] \quad \text{and} \quad \mu_{\overline{Apr}(A)} : \overline{Apr}(X) \to [0,1]$$

with the property that

$$\mu_{\underline{Apr}(A)}(x) \leq \mu_{\overline{Apr}(A)}(x) \quad \text{for all } x \in \underline{Apr}(X).$$

$(\underline{\mathscr{A}}_t, \overline{\mathscr{A}}_t)$ is called a level rough set of $Apr(A)$, where

$$\underline{\mathscr{A}}_t = \{x \in \underline{Apr}(X) \mid \mu_{\underline{Apr}(A)}(x) \geq t\}, \quad \overline{\mathscr{A}}_t = \{x \in \overline{Apr}(X) \mid \mu_{\overline{Apr}(A)}(x) \geq t\}.$$

The relations and the operations of two fuzzy rough sets and the complement of a fuzzy rough set are studied by Nanda and Majumdar in [8].

Definition 3.2. Let $Apr(A)$ be a fuzzy rough set in $Apr(X)$. Then we define

$$\overline{\mu}_{\underline{Apr}(A)}(x) = \begin{cases} \mu_{\underline{Apr}(A)}(x) & \text{if } x \in \underline{Apr}(X), \\ 0 & \text{if } x \in \overline{Apr}(X) \setminus \underline{Apr}(X). \end{cases}$$

so we have an interval-valued map $\overline{\mu}_{\mathscr{A}}$ in $\overline{Apr}(X)$ given by: for $\forall\, x \in \overline{Apr}(X)$

$$\overline{Apr}(X) \to C([0,1]), \quad \text{i.e. } \overline{\mu}_{\mathscr{A}}(x) = [\overline{\mu}_{\underline{Apr}(A)}(x), \mu_{\overline{Apr}(A)}(x)]$$

where $C([0, 1])$ denotes the family of all closed subintervals of [0, 1]. if $\overline{\mu}_{\overline{Apr}(A)}(x) = \mu_{\overline{Apr}(A)}(x) = c,\ 0 \leq c \leq 1,$ then we have $\overline{\mu}_{\mathscr{A}}(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $C([0, 1])$. Thus $\overline{\mu}_{\mathscr{A}}(x) \in C([0,1])$ for all $x \in \overline{Apr}(X)$.

For any $y, t \in [0, 1]$, define a map $y^t = \begin{cases} y & \text{if } y \leq t, \\ t & \text{if } y > t. \end{cases}$

Definition 3.3. Let $C_1 = [a_1, b_1]$, $C_2 = [a_2, b_2]$ and $C = [a, b]$ be elements of $C([0, 1])$, then we define

rmax$(C_1, C_2) = [a_1 \vee a_2, b_1 \vee b_2]$, rmin$(C_1, C2) = [a_1 \wedge a_2, b_1 \wedge b_2]$,

$C^t = [a, b]^t = [a^t, b^t]$, $C_1 + C_2 = [a_1 + a_2, b_1 + b_2]^1$,

If $a_2 \leq a_1$ and $b_2 \leq b_1$, we call $C_2 \leq C_1$ or $C_1 \geq C_2$.

Davvaz[1] introduced the following definitions and results about fuzzy rough subrings and fuzzy rough ideals:

Definition 3.4. Let $Apr(X)$ be a rough ring. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is called a fuzzy rough subring (resp.ideal) if for all $x, y \in \overline{Apr}(X)$, the following hold:

(i) $\bar{\mu}_{\mathscr{A}}(x-y) \geq$ rmin $(\bar{\mu}_{\mathscr{A}}(x), \bar{\mu}_{\mathscr{A}}(y))$,

(ii) $\bar{\mu}_{\mathscr{A}}(xy) \geq$rmin $\bar{\mu}_{\mathscr{A}}(x), \bar{\mu}_{\mathscr{A}}(y))$. (resp. (ii) $\bar{\mu}_{\mathscr{A}}(xy) \geq$rmax$(\bar{\mu}_{\mathscr{A}}(x), \bar{\mu}_{\mathscr{A}}(y)))$

Theorem 3.1. Let $Apr(X)$ be a rough ring. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is a fuzzy rough subring (resp.ideal) iff for every $0 \leq t \leq 1$, $(\underline{\mathscr{A}}_t, \overline{\mathscr{A}}_t)$ is a rough subring (resp.ideal) of $Apr(X)$.

4. FUZZY ROUGH POINTS AND *G*-FUZZY ROUGH IDEALS

Definition 4.1. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is said to be a fuzzy rough point with support $x \in \overline{Apr}(X)$ and interval-value $[r, t]$ $(r \leq t)$ and denoted by $x_{[r,t]}$ if

$$\bar{\mu}_{\mathscr{A}}(y) = \begin{cases} [r,t] & \text{if } y = x. \\ [0,0] & \text{if } y \neq x. \end{cases}$$

Definition 4.2. A fuzzy rough point $x_{[r,t]}$ is said to belong to (resp.be quasi-coincident with) a fuzzy set $Apr(A)$, written as $x_{[r,t]} \in Apr(A)$ (resp.$x_{[r,t]}\ q\ Apr(A)$) if $\bar{\mu}_{\mathscr{A}}(x) \geq [r,t]$ (resp. $\bar{\mu}_{\mathscr{A}}(x)+[r,t] > [\delta^1, 1]$, where $\delta = \bar{\mu}_{\underline{Apr}(A)}(x)+r$). If $x_{[r,t]} \in Apr(A)$ or $x_{[r,t]}\ q\ Apr(A)$, then we write $x_{[r,t]} \in \vee q\ Apr(A)$.

Definition 4.3. $Apr(A)$ is said to be an G-fuzzy rough subring of $Apr(X)$ if for all $x, y \in \overline{Apr}(X)$ and $C_1 = [a_1, b_1]$, $C_2 = [a_2, b_2]$, $C = [a, b] \in C([0, 1])$, the following conditions are satisfied:

(i) $x_{C_1}, y_{C_2} \in Apr(A) \Rightarrow (x+y)_{r\min(C_1,C_2)} \in\vee qApr(A)$;

(ii) $x_C \in Apr(A) \Rightarrow (-x)_C \in \vee qApr(A)$;

(iii) $x_{C_1}, y_{C_2} \in Apr(A) \Rightarrow (xy)_{r\min(C_1,C_2)} \in\vee qApr(A)$.

Theorem 4.1. In the definition 4.3, condition (i) is equivalent to $(I)\overline{\mu}_{\mathscr{A}}(x+y) \geq$ rmin $(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0.5})$, condition (ii) is equivalent to (II) $\overline{\mu}_{\mathscr{A}}(-x) \geq \overline{\mu}_{\mathscr{A}}(x)^{0.5}$, and condition (iii) is equivalent to (III) $(\overline{\mu}_{\mathscr{A}}(xy) \geq$ rmin $(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0..5})$.

Proof. (i) $\Rightarrow$ (I): Let $x, y \in \overline{Apr}(X)$, rmin $(\overline{\mu}_{\mathscr{A}}(x), \overline{\mu}_{\mathscr{A}}(y)) = [l,r]$, then we have rmin $(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0.5}) = [l,r]$ if $l \leq 0.5$, $r \leq 0.5$ or rmin $(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0.5}) = [l, 0.5]$ if $l \leq 0.5$, $r > 0.5$ or rmin $(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0.5}) = [0.5, 0.5]$ if $l > 0.5$, $r > 0.5$. Here we only prove the first case, the other cases can be proved in the similar way.

Let $\overline{\mu}_{\mathscr{A}}(x+y) = [u,v]$. Assume that $\overline{\mu}_{\mathscr{A}}(x+y) \ngeq [l,r]$, then $u < l$ or $v < r$. If $u < l$, $v \geq r$, choose s such that $u < s < l$, then $x_{[s,r]}$, $y_{[s,r]} \in Apr(A)$, but $(x+y)_{[s,r]} \overline{\in\vee q} Apr(A)$, a contradiction. If $u \geq l$, $v < r$, choose s such that $v < s < r$, then $x_{[l,s]}$, $y_{[l,s]} \in Apr(A)$, but $(x+y)_{[l,s]} \overline{\in\vee q} Apr(A)$, a contradiction. If $u < l$, $v < r$, choose s_1, s_2 such that $u < s_1 < l$, $v < s_2 < r$, then $x_{[s_1,s_2]}, y_{[s_1,s_2]} \in Apr(A)$, but $(x+y)_{[s_1,s_2]} \overline{\in\vee q} Apr(A)$, a contradiction. Hence (I) holds.

(I) $\Rightarrow$ (i):Let $x_{C_1}, y_{C_2} \in Apr(A)$. Then $\overline{\mu}_{\mathscr{A}}(x+y) \geq$ rmin $(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0.5} \geq$ rmin $(C_1^{0.5}, C_2^{0.5})$. Thus, $\overline{\mu}_{\mathscr{A}}(x+y) \geq$ rmin(C_1, C_2), if $C_1^{0.5} = C_1$, $C_2^{0.5} = C_2$ and $\overline{\mu}_{\mathscr{A}}(x+y) \geq [(a_1 \wedge a_2)^{0.5}, 0.5]$, if $b_1 > 0.5$, $b_2 > 0.5$. Hence $(x+y)_{\text{rmin}(C_1,C_2)} \in \vee qApr(A)$.

Similarly, it can be shown that (iii) is equivalent to (III).

(ii) $\Rightarrow$ (II): Let $x \in \overline{Apr}(X)$, $\overline{\mu}_{\mathscr{A}}(x) = [l_x, r_x]$, then $\overline{\mu}_{\mathscr{A}}(x)^{0.5} = [l_x, r_x]$ if $l_x \leq 0.5$, $r_x \leq 0.5$ or $\overline{\mu}_{\mathscr{A}}(x)^{0.5} = [l_x, 0.5]$ if $l_x \leq 0.5$, $r_x > 0.5$ or $\overline{\mu}_{\mathscr{A}}(x)^{0.5} = [0.5, 0.5]$ if $l_x > 0.5$, $r_x > 0.5$. Here we only prove the first case, the other cases can be proved in the similar way.

Assume that $[l_{-x}, r_{-x}] = \overline{\mu}_{\mathscr{A}}(-x) \ngeq \overline{\mu}_{\mathscr{A}}(x)^{0.5} = \overline{\mu}_{\mathscr{A}}(x) = [l_x, r_x]$, then $l_{-x} < l_x$ or $r_{-x} < r_x$. If $l_{-x} < l_x$, $r_{-x} \geq r_x$, choose s such that $l_{-x} < s < l_x$, then $x_{[s,r_x]} \in Apr(A)$, but $(-x)_{[s,r_x]} \overline{\in \vee q} Apr(A)$, a contradiction. If $l_{-x} \geq l_x$, $r_{-x} < r_x$, choose s such that $r_{-x} < s < r_x$, then $x_{[l_x,s]} \in Apr(A)$, but $(-x)_{[l_x,s]} \overline{\in \vee q} Apr(A)$, a contradiction. If $l_{-x} < l_x$, $r_{-x} < r_x$, choose s_1, s_2 such that $l_{-x} < s_1 < l_x$, $r_{-x} < s_2 < r_x$, then $x_{[s_1,s_2]} \in Apr(A)$, but $(-x)_{[s_1,s_2]} \overline{\in \vee q} Apr(A)$, a contradiction. Hence (II) holds.

(II) $\Rightarrow$ (ii):Let $x_C \in Apr(A)$. Then $\overline{\mu}_{\mathscr{A}}(-x) \geq \overline{\mu}_{\mathscr{A}}(x)^{0.5} \geq C^{0.5}$, i.e. $\overline{\mu}_{\mathscr{A}}(-x) \geq C^{0.5} = C$ or $\overline{\mu}_{\mathscr{A}}(-x) \geq [l_x^{0.5}, 0.5]$ according as $r_x \leq 0.5$ or $r_x > 0.5$. So, $(-x)_C \in \vee q Apr(A)$.

Theorem 4.2. Let $Apr(X)$ be a rough ring. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is an G-fuzzy rough subring iff $\overline{\mu}_{\mathscr{A}}(x-y), \overline{\mu}_{\mathscr{A}}(xy) \geq \text{rmin}(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0.5})$ for all $x, y \in \overline{Apr}(X)$.

Proof: Follows from Theorem 4.1.

Definition 4.4. Let $Apr(X)$ be a rough ring. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is called an G-fuzzy rough ideal if (i) $Apr(A)$ is an G-fuzzy rough subring; (ii) for any $x, y \in \overline{Apr}(X)$, $C \in C([0, 1])$, $x_C \in Apr(A) \Rightarrow (xy)_C, (yx)_C \in \vee q Apr(A)$.

Theorem 4.3. Let $Apr(X)$ be a rough ring. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is an G-fuzzy rough ideal iff for all $x, y \in \overline{Apr}(X)$, the following conditions hold: (i) $\overline{\mu}_{\mathscr{A}}(x-y) \geq r\min(\overline{\mu}_{\mathscr{A}}(x)^{0.5}, \overline{\mu}_{\mathscr{A}}(y)^{0.5})$, $(ii)\, \overline{\mu}_{\mathscr{A}}(xy), \overline{\mu}_{\mathscr{A}}(yx) \geq \overline{\mu}_{\mathscr{A}}(x)^{0.5}$.

Proof: Straightforward.

Theorem 4.4. Let $Apr(X)$ be a rough ring. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is an G-fuzzy rough subring (resp.ideal) iff for every $0 \leq t \leq 0.5$, $(\mathscr{A}_t, \overline{\mathscr{A}}_t)$ is a rough subring (resp.ideal) of $Apr(X)$.

Proof: We only prove the result in the case of fuzzy rough subring. Let $Apr(A)$ be an G-fuzzy rough subring and $0 \leq t \leq 0.5$. We show that $\mathscr{A}_t$ and $\overline{\mathscr{A}}_t$ are subrings. For any $x, y \in \underline{\mathscr{A}}_t$, we have $\mu_{\underline{Apr}(A)}(x) \geq t$ and $\mu_{\underline{Apr}(A)}(y) \geq t$, so $\overline{\mu}_{\mathscr{A}}(x-y) \geq$

rmin $(\bar{\mu}_{\mathscr{A}}(x)^{0.5}, \bar{\mu}_{\mathscr{A}}(y)^{0.5}) \geq [t, r^{0.5}], \bar{\mu}_{\mathscr{A}}(xy) \geq [t, r^{0.5}]$. Since $x, y \in \underline{Apr}(X)$ we have $x-y, xy \in \underline{Apr}(X)$, so $\mu_{\underline{Apr}(A)}(x-y), \mu_{\underline{Apr}(A)}(xy) \geq t, x-y, xy \in \mathscr{A}_t$. Therefore $\underline{\mathscr{A}}_t$ is a subring.

Let $x, y \in \overline{\mathscr{A}}_t$, then $\mu_{\overline{Apr}(A)}(x) \geq t, \mu_{\overline{Apr}(A)}(y) \geq t, \bar{\mu}_{\mathscr{A}}(x-y), \bar{\mu}_{\mathscr{A}}(xy) \geq$ rmin $(\bar{\mu}_{\mathscr{A}}(x)^{0.5}, \bar{\mu}_{\mathscr{A}}(y)^{0.5}) \geq [0, t]$, so $\mu_{\overline{Apr}(A)}(x-y), \mu_{\overline{Apr}(A)}(xy) \geq t$. Therefore $x-y, xy \in \overline{\mathscr{A}}_t, \overline{\mathscr{A}}_t$ is a subring.

Conversely, let $0 \leq t \leq 0.5$ and $(\underline{\mathscr{A}}_t, \overline{\mathscr{A}}_t)$ be a rough subring of $Apr(X)$. Assume $\exists\ x, y \in \overline{Apr}(X)$ such that $[u, v] = \bar{\mu}_{\mathscr{A}}(x-y) \ngeq$ rmin $\bar{\mu}_{\mathscr{A}}(x)^{0.5}, \bar{\mu}_{\mathscr{A}}(y)^{0.5})$, w.l.g. suppose $v < r^{0.5}$, then $x, y \in \overline{\mathscr{A}}_v$ and hence $x-y \in \overline{\mathscr{A}}_v$ (Since $\overline{\mathscr{A}}_t$ is a subring). Thus $\mu_{\overline{Apr}(A)}(x-y) \geq v$, this is a contradiction. So $\bar{\mu}_{\mathscr{A}}(x-y) \geq$ rmin $(\bar{\mu}_{\mathscr{A}}(x)^{0.5}, \bar{\mu}_{\mathscr{A}}(y)^{0.5})$. Similarly, it can be shown that $\bar{\mu}_{\mathscr{A}}(xy) \geq$ rmin $(\bar{\mu}_{\mathscr{A}}(x)^{0.5}, \bar{\mu}_{\mathscr{A}}(y)^{0.5})$.

Remark According to Theorem 4.4 and Theorem3.1, the notion of an *G*-fuzzy rough subring (resp.ideal) is an extended notion of a fuzzy rough subring (resp.ideal) in [1].

5. AN EXTENSION OF *G*-FUZZY ROUGH IDEALS

In fact, in Theorem 4.2 and Theorem 4.3, if we replace 0.5 with $\alpha \in [0, 1]$, then we have:

Definition 5.1. Let *Apr*(*X*) be a rough ring. A fuzzy rough set *Apr*(*A*) in *Apr*(*X*) is called an G_{α}- fuzzy rough subring if for all $x, y \in \overline{Apr}(X)$, (i) $\bar{\mu}_{\mathscr{A}}(x-y) \geq$ rmin $(\bar{\mu}_{\mathscr{A}}(x)^{\alpha}, \bar{\mu}_{\mathscr{A}}(y)^{\alpha})$, (ii) $\bar{\mu}_{\mathscr{A}}(xy) \geq r\min(\bar{\mu}_{\mathscr{A}}(x)^{\alpha}, \bar{\mu}_{\mathscr{A}}(y)^{\alpha})$.

Definition 5.2. Let *Apr*(*X*) be a rough ring. A fuzzy rough set *Apr*(*A*) in *Apr*(*X*) is called an G_{α} – fuzzy rough ideal if for all $x, y \in \overline{Apr}(X)$, (i) $\bar{\mu}_{\mathscr{A}}(x-y) \geq$ rmin $(\bar{\mu}_{\mathscr{A}}(x)^{\alpha}, \bar{\mu}_{\mathscr{A}}(y)^{\alpha})$, (ii) $\bar{\mu}_{\mathscr{A}}(xy)\ \bar{\mu}_{\mathscr{A}}(yx) \geq \bar{\mu}_{\mathscr{A}}(x)^{\alpha}$.

Theorem 5.1. Let $Apr(X)$ be a rough ring. A fuzzy rough set $Apr(A)$ in $Apr(X)$ is an G_α – fuzzy rough subring (resp.ideal) iff for every $0 \le t \le \alpha$, $(\underline{\mathscr{A}}_t, \overline{\mathscr{A}}_t)$ is a rough subring (resp.ideal) of $Apr(X)$.

Proof: Similar to that of Theorem 4.4.

In view of Theorem 5.1, we have the following definition:

Definition 5.3. Let $Apr(X)$ be a rough ring and $Apr(A)$ a fuzzy subset in $Apr(X)$. If there exists the greatest τ, such that $Apr(A)$ is an G_τ-fuzzy rough ideal, then we call τ is the threshold of $Apr(A)$, denoted by $TH(Apr(A)) = \tau$.

Theorem 5.2. Let (R, I) be a finite approximation space and $Apr(X)$ a rough set in (R, I). Then a fuzzy rough set $Apr(A)$ in $Apr(X)$ has only one threshold.

Proof. Straightforward.

Let (R, I) be a finite approximation space and $Apr(X)$ a rough set in (R, I). The threshold of a fuzzy rough set $Apr(A)$ in $Apr(X)$ will be given in the following algorithm:

(1) without loss of generality, we line the membership functions of elements from the smallest to the biggest in sequence, suppose $|\overline{Apr}(X)| = n$ and $t_n < t_{n-1} < \ldots < t_1$, $t_k \in Im(\underline{Apr}(A))$; $s_n < s_{n-1} < \ldots < s_1$, $s_k \in Im(\overline{Apr}(A))$ $(k = 1, 2, \ldots, n)$;

(2) $i := 0$;

(3) $i := i + 1$; $j := t_{n-i}$;

(4) if $\underline{\mathscr{A}}_j$ and $\overline{\mathscr{A}}_j$ are, respectively, ideals of $\underline{Apr}(X)$ and $\overline{Apr}(X)$, then (3), otherwise (5);

(5) output $TH(Apr(A)) = t_{n-i-1}$.

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