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ON THE FUZZY IMPULSIVE FUNCTION

ABSTRACT: *In this paper we study Laplace transform of the fuzzy Dirac delta function and example of fuzzy impulsive differential equation.*

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Key words and Phrases: *Laplace transform, fuzzy Dirac delta function, fuzzy impulsive differential equations, fuzzy number.*

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1. INTRODUCTION

Many applications in engineering and physics are often acted upon by an external force of large magnitude that acts only for a very short period of time.

For example, a vibrating airplane wing could be struck by lighting, a mass on a spring could be given a sharp blow by a ball peen hammer, a ball could be sent soaring when struck violently by some kind of club.

To solve such a fuzzy logical problem mathematically, we can define the fuzzy function

$$\tilde{\delta}_a(t-t_0)=\begin{cases}0 & 0\le t\le t_0-a,\\ \dfrac{1}{\tilde{2}a} & t_0-a\le t\le t_0+a,\\ 0 & t_0+a\le t,\end{cases} \tag{1.1}$$

where $\tilde{\cdot}$ is about $\cdot$. The function $\tilde{\delta}_a(t-t_0)$ is called a unit fuzzy impulse since it possesses the integration property

$$\int_0^\infty \tilde{\delta}(t-t_0)dt := \tilde{1} \tag{1.2}$$

where $\tilde{1}$ is about 1.

In practice it is convenient to work with another type of unit fuzzy impulse that is defined by the limit

$$\tilde{\delta}(t-t_0)=\lim_{a\to 0}\tilde{\delta}(t-t_0) \tag{1.3}$$

The expression $\tilde{\delta}(t-t_0)$ said to be the fuzzy Dirac delta function which is useful in representing an instantaneous impulse at time $t = t_0$.

It is possible to obtain the Laplace transform of the fuzzy Dirac delta function by the formal assumption that

$$\mathcal{L}\{\tilde{\delta}(t-t_0)\}=\lim_{a\to 0}\ \mathcal{L}\{\tilde{\delta}_a(t-t_0)\}. \tag{1.4}$$

In this paper we investigate Laplace transform of the fuzzy Dirac delta function and an example of fuzzy impulsive differential equation.

2. LAPLACE TRANSFORM OF THE FUZZY DIRAC DELTA FUNCTION

A fuzzy number A is express $A=\int_{x\in R}\mu_A(x)/x$ with the understanding that $\mu_A(x)\in$ [0, 1] represents the grade of membership of A and $\int$ denotes the union of $\mu_A(x)/x$'s. If a fuzzy number $\tilde{2}$ which denotes "about 2" will be given as

$$\tilde{2}=\int_1^2 x-1/x+\int_2^3 3-x/x \tag{2.1}$$

where + stands for the union, then the interval of confidence at the level α is given by

$$[\tilde{2}]^\alpha=[\alpha+1,3-\alpha]. \tag{2.2}$$

Therefore

$$[\frac{2}{\tilde{2}}]^\alpha=[\frac{2}{\frac{2}{3-\alpha}},\frac{2}{\alpha+1}] \tag{2.3}$$

From this we obtain that

$$\frac{2}{\tilde{2}}=\int_{\frac{2}{3}}^1 3-\frac{2}{x}/x+\int_1^2\frac{2}{x}-1/x:=\tilde{1} \tag{2.4}$$

Theorem 2.1. For $t_0 > 0$ and $a > 0$

$$\int_0^\infty \tilde{\delta}_\alpha(t-t_0)dt := \tilde{1} \tag{1}$$

$$\mathcal{L}\{\tilde{\delta}(t-t_0)\} := e^{-st_0}.\tilde{1} \tag{2}$$

Proof. (1) From the definition of the unit fuzzy impulse we get

$$[\int_0^\infty \tilde{\delta}_a(t-t_0)dt]^\alpha = \int_{t_0-a}^{t_0+a} [\frac{1}{\tilde{2}a}]^\alpha dt$$

$$= \int_{t_0-a}^{t_0+a} [\frac{1}{a(3-\alpha)}, \frac{1}{a(\alpha+1)}]dt$$

$$= [\frac{2}{3-\alpha}, \frac{2}{\alpha+1}] := [\tilde{1}]^\alpha.$$

By using resolution identity,

$$\int_0^\infty \tilde{\delta}_a(t-t_0)dt := \tilde{1}.$$

(2) To begin, we can write $\tilde{\delta}_a(t-t_0)$ in terms of the unit function by

$$[\tilde{\delta}_a(t-t_0)]^\alpha$$

$$= [\frac{1}{\tilde{2}a}(u(t-(t_0-a)) - u(t-(t_0+a))]^\alpha$$

$$= [\frac{1}{a(3-\alpha)}(u(t-(t_0-a)) - u(t-(t_0+a))),$$

$$\frac{1}{a(\alpha+1)}[u(t-(t_0-a)) - u(t-(t_0+a)))].$$

where

$$u(t-b) = \begin{cases} 0 & t < b, \\ 1 & t \geq b. \end{cases}$$

By properties of the Laplace transform we have

$$
\begin{aligned}
&[\mathcal{L}\{\tilde{\delta}(t-t_0)\}]^{\alpha}\\
=\ & [\mathcal{L}(\frac{1}{a(3-\alpha)}(u(t-(t_0-a))-u(t-(t_0+a)))),\\
&\qquad \mathcal{L}(\frac{1}{a(\alpha+1)}(u(t-(t_0-a))-u(t-(t_0+a))))]\\
=\ & [\frac{1}{a(3-\alpha)}\mathcal{L}((u(t-(t_0-a))-u(t-(t_0+a)))),\\
&\qquad \frac{1}{a(\alpha+1)}\mathcal{L}((u(t-(t_0-a))-u(t-(t_0+a))))]\\
=\ & [\frac{1}{a(3-\alpha)}e^{-st_0}(\frac{e^{s\alpha}-e^{-sa}}{s}),\frac{1}{a(\alpha+1)}e^{-st_0}(\frac{e^{sa}-e^{-sa}}{s})]\\
=\ & e^{-st_0}(\frac{e^{sa}-e^{-sa}}{sa})[\frac{1}{3-\alpha},\frac{1}{\alpha+1}].
\end{aligned}
$$

Apply L'Hopital's rule we obtain

$$
\begin{aligned}
&[\mathcal{L}\{\tilde{\delta}(t-t_0)]^{\alpha}\\
=\ & \lim_{a\to 0}[[\mathcal{L}\{\tilde{\delta}_a(t-t_0)]^{\alpha}\\
=\ & \lim_{a\to 0} e^{-st_0}(\frac{e^{sa}-e^{-sa}}{sa})[\frac{1}{3-\alpha},\frac{1}{\alpha+1}]\\
=\ & e^{-st_0}2[\frac{1}{3-\alpha},\frac{1}{\alpha+1}]\\
=\ & e^{-st_0}[\frac{2}{3-\alpha},\frac{2}{\alpha+1}]\\
:=\ & e^{-st_0}[\tilde{1}]^{\alpha}.
\end{aligned}
$$

By using resolution identity,

$$\mathcal{L}\{\tilde{\delta}(t-t_0)\}:=e^{-st_0}.\tilde{1}.$$

Example. Solve the initial value problem

$$y'' + y = c\tilde{\delta}(t - 2\pi) \tag{2.6}$$

subject to $y(0) = 1$ and $y'(0) = 0$, where $c \in R$ is positive constant. It could serve as models for describing the motion of a fuzzy mass on a spring moving in a medium in which damping negligible. The fuzzy mass is release from rest 1 unit below the equilibrium position and at $t = 2\pi$ seconds the fuzzy mass is given a sharp blow.

Solution. From the Laplace transform of the differential equation (2.6) is

$$s^2 Y(s) - s + y(s) = c\,.\,e^{-2\pi s}.\tilde{1}$$

or

$$Y(s) = \frac{s}{s^2+1} + \frac{ce^{-2\pi s}}{s^2+1}\tilde{1}. \tag{2.7}$$

Utilizing the inverse form of the translation, we find the solution

$$y(t) = \cos t + c\,.\sin(t - 2\pi)\, u(t - 2\pi)\ \tilde{1} \tag{2.8}$$

Put

$$[y(t)]^\alpha = [\cos t + c\sin(t - 2\pi)u(t - 2\pi)\frac{2}{3-\alpha}, \tag{2.9}$$

$$\cos\ t + c\sin(t - 2\pi)u(t - 2\pi)\frac{2}{\alpha+1}],$$

for $\alpha \in [0, 1]$. Let $T > 0$. Consider the following solution set

$$X^\alpha = \{[y(t)]^\alpha : [y(t)]^\alpha \text{ satisfies eq. (2.9) for } \ t \in [0, T] \text{ and } \alpha \in [0, 1]\}$$

Nonempty is obvious since we can select $\alpha \in [0, 1]$. Let $[y(t)]^\alpha \in X^\alpha$, then there is $\alpha \in [0, 1]$ such that

$$|[y(t)]^\alpha| \leq \sqrt{c^2 \left| \frac{2}{\alpha+1} - \frac{2}{3-\alpha} \right|} \leq \frac{4}{3}c. \tag{2.10}$$

Thus X^α is bounded. Let $[y]^{\alpha_k} \in X^\alpha$ for each $[y]^{\alpha_k}$, then there is $\alpha_k \in [0, 1]$ such that $\alpha_k \to \alpha \in [0, 1]$ and

$$\lim_{k\to\infty}[y]^{\alpha_k} \quad = \quad \lim_{k\to\infty}[\cos t + c\sin(t - 2\pi)u(t - 2\pi)\frac{2}{3-\alpha_k},$$

$$\cos t + c\sin(t-2\pi)u(t-2\pi)\frac{2}{\alpha_k+1}],$$

$$= [\cos t + c\sin(t-2\pi)u(t-2\pi)\frac{2}{3-\alpha},$$

$$\cos t + c\sin(t-2\pi)u(t-2\pi)\frac{2}{\alpha+1}] = [y(t)]^\alpha$$

because $[\tilde{1}]^\alpha = [\frac{2}{3-\alpha}, \frac{2}{\alpha+1}]$ is closed. Thus X^α is compact. From the (2.8) the solution can be written as

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + c\,.\,\tilde{1}\,.\sin t & t \ge 2\pi. \end{cases}$$

We see from the solution $y(t)$ that the mass exhibiting simple harmonic motion it is struck at $t = 2\pi$. The influence of the fuzzy unit impulse is to increase the amplitude of vibration to $\sqrt{(c\tilde{1})^2+1}$ for $t > 2\pi$.

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