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HOMOTOPY PERTURBATION METHOD AND VARIATIONAL ITERATION METHOD FOR SOLVING BURGER'S EQUATION

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Abstract: *Perturbation methods depend on a small parameter which is difficult to be found for real-life nonlinear and linear problems. To overcome this shortcoming, two new powerful analytical methods are introduced to solve Burger's equation in this work. One is He's variational iteration method (VIM) and the other is He's homotopy-perturbation method (HPM). VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory, and the initial approximations can be freely chosen with unknown constants. HPM deforms a difficult problem into a simpler one which can be easily solved. In this article, we have used HPM and VIM to solve nonlinear Burger's equations, which are functions of time and space. This type of equation governs on numerous scientific and engineering experimentations.*

Keywords: *Burger's equations; Homotopy perturbation method; Variational iteration method.*

1. INTRODUCTION

Up to now more and more nonlinear equations have been presented, which describe the motion of the isolated waves, localized in a small part of space, in many fields such as hydrodynamic, plasma physics, nonlinear optic, etc. The investigation of exact solutions of these nonlinear equations is interesting and important. In the past decades, many authors mainly have paid attention to the study of solutions of nonlinear equations by using various methods, such as [1], Darboux transformation [2], inverse scattering method [3], Hirota's bilinear method [4], the sine-cosine method [5,6], the homogeneous balance method [7].

Recently, an extended tanh-function method and symbolic computation have been suggested [8] for solving the new coupled modified KDV equations to obtain four kinds of Soliton solutions. This method has some merits compared to the tanh-function method. It not only uses a simpler algorithm to produce an algebric system, but also can pick up singular Soliton solutions with no extra effort [9-11].

The numerical solution of Burger's equation is of great importance due to the equation's application in the approximate theory of flow through a shock wave traveling in a viscous fluid and in the Burger's model of turbulence [12]. Finite element methods have been applied to fluid

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problems among them, Galerkin and Petrov–Galerkin finite element methods involve a time-dependent grid solution [13, 14]. Numerical solution using cubic spline global trial functions have been developed [15] to obtain two systems or diagonally dominant equations which are solved to determine the evolution of the system. Ali et al. [16] applied B-spline finite element method to the solution of Burger's equation. If the B-spline finite element approach is merged with collocation and applied on constant grid problem the results will be as highly accurate as those of Cubic B-spline elements. Cubic B-spline had a resulting matrix system which is tri-diagonal and then solved by the Thomas algorithm. Soliman [17] used the similarity reductions for the partial differential equations to develop a scheme for solving the Burger's equation. This scheme is based on similarity reductions of Burger's equation on small sub-domains. The resulting similarity equation is integrated analytically. The analytical solution is then used to approximate the flux vector in Burger's equation.

In the numerical method, stability and convergence should be considered to avoid divergence or inappropriate results. Therefore, approximate analytical solutions were introduced, among which VIM [18-24] and HPM [25-27] are the most effective and convenient ones for heat equations. Developing the perturbation method for different usage is very difficult because this method has some limitations and is based on the existence of a small parameter. Therefore, many different new methods have recently been introduced to eliminate the small parameter such as artificial parameter method introduced by Liu [28]. One of the semi-exact methods is HPM. The HPM is one of the well-known methods to solve nonlinear equations that are established in 1999 by He[29]. The references therein to handle a wide variety of scientific and engineering applications: linear and nonlinear, homogeneous and inhomogeneous as well. It was shown by many authors that this method provides improvements over existing numerical techniques [30].

2. HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique. To explain the basic idea of the HPM for solving nonlinear differential equations we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, r \in \Omega, \tag{1}$$

subject to boundary conditions we have:

$$B(u, \partial u/\partial n) = 0, r \in \Gamma, \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain Ω and $\partial u/\partial n$ denotes differentiation along the normal drawn outwards from Ω. The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. Eq. (1) therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

In case that the nonlinear Eq. (1) has no "small parameter", we can construct the following homotopy:

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p(N(v) - f(r)) = 0, \tag{4}$$

where,

$$v(r, p) : w \times [0, 1] \to R, \tag{5}$$

In Eq. (4), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary conditions. We can assume that the solution of Eq. (4) can be written as a power series in p, as following:

$$v = v_0 + pv_1 + p^2 v_2 + ..., \tag{6}$$

And the best approximation for solution is:

$$u = \lim_{p\to 1} \nu = \nu_0 + \nu_1 + \nu_2 + ..., \tag{7}$$

When, Eq. (4) considering to Eq. (1) and Eq. (7) becomes the approximate solution of Eq. (1) some interesting results are attained using this method.

In this study, the Burger's equation is in the form of:

$$u_t + uu_x - \nu u_{xx} = 0 \tag{8}$$

with the initial condition of

$$u(x, 0) = \frac{\alpha + \beta + (\beta - \alpha)e^{(\frac{\alpha(x-\lambda)}{\nu})}}{1 + e^{(\frac{\alpha(x-\lambda)}{\nu})}} \tag{9}$$

Substituting Eq. (6) into Eq. (4) and rearranging based on powers of p-terms, we have:

$$p^0 : \frac{\partial}{\partial t} u_0(x,t) = 0, \tag{10}$$

the initial condition is defined as

$$u_0(x, 0) = \frac{\alpha + \beta + (\beta - \alpha)e^{(\frac{\alpha(x-\lambda)}{\nu})}}{1 + e^{(\frac{\alpha(x-\lambda)}{\nu})}}. \tag{11}$$

in the same way, we have

$$\begin{aligned} p^1 : &\frac{1}{\nu(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})^2}((\frac{\partial}{\partial t}u_1(x,t))\nu e^{(\frac{2\alpha(x-\lambda)}{\nu})} \\ &+2(\frac{\partial}{\partial t}u_1(x,t))\nu e^{(\frac{\alpha(x-\lambda)}{\nu})}) - 2e^{(\frac{\alpha(x-\lambda)}{\nu})}\alpha^2\beta + (\frac{\partial}{\partial t}u_1(x,t))\nu) = 0, \end{aligned} \tag{12}$$

the initial condition is defined as

$$u_1(x, 0) = 0. \tag{13}$$

and

$$\begin{aligned} p^2 : &\frac{1}{\nu^2(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})^3}((\frac{\partial}{\partial t}u_2(x,t))\nu^2 e^{(\frac{3\alpha(x-\lambda)}{\nu})} - 2\alpha^3\beta^2 e^{(\frac{2\alpha(x-\lambda)}{\nu})}t \\ &+3(\frac{\partial}{\partial t}u_2(x,t))\nu^2 e^{(\frac{2\alpha(x-\lambda)}{\nu})} + 3(\frac{\partial}{\partial t}u_2(x,t))\nu^2 e^{(\frac{\alpha(x-\lambda)}{\nu})} \\ &+2\alpha^3\beta^2 e^{(\frac{\alpha(x-\lambda)}{\nu})}t + (\frac{\partial}{\partial t}u_2(x,t))\nu^2) = 0, \end{aligned} \tag{14}$$

$$u_2(x,\ 0) \ = \ 0. \tag{15}$$

and

$$p^3 : \frac{1}{\nu^3(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})(3e^{(\frac{2\alpha(x-\lambda)}{\nu})}+3e^{(\frac{\alpha(x-\lambda)}{\nu})}+e^{(\frac{3\alpha(x-\lambda)}{\nu})}+1)}$$

$$(\frac{\partial}{\partial t}u_3(x,t))\nu^3 e^{(\frac{4\alpha(x-\lambda)}{\nu})} - \alpha^4\beta^3 e^{(\frac{3\alpha(x-\lambda)}{\nu})}t^2 + 4(\frac{\partial}{\partial t}u_3(x,t))\nu^3 e^{(\frac{3\alpha(x-\lambda)}{\nu})}$$

$$+4\alpha^4\beta^3 e^{(\frac{2\alpha(x-\lambda)}{\nu})}t^2 + 6(\frac{\partial}{\partial t}u_3(x,t))\nu^3 e^{(\frac{2\alpha(x-\lambda)}{\nu})} - \alpha^4\beta^3 e^{(\frac{\alpha(x-\lambda)}{\nu})}t^2 \tag{16}$$

$$+4(\frac{\partial}{\partial t}u_3(x,t))\nu^3 e^{(\frac{\alpha(x-\lambda)}{\nu})} + (\frac{\partial}{\partial t}u_3(x,t))\nu^3),$$

$$u_3(x,\ 0) \ = \ 0. \tag{17}$$

and

Solving the above equations (Eqs. (10) - (17)) and when $p \to 1$, the solution may be written in the form:

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)e^{(\frac{\alpha(x-\lambda)}{\nu})}}{1+e^{(\frac{\alpha(x-\lambda)}{\nu})}} + \frac{2\alpha^2\beta e^{(\frac{\alpha(x-\lambda)}{\nu})}}{\nu(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})}t$$

$$+\frac{\alpha^3\beta^2 e^{(\frac{\alpha(x-\lambda)}{\nu})}(e^{(\frac{\alpha(x-\lambda)}{\nu})}-1)}{\nu^2(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})^3}t^2 + \frac{\alpha^4\beta^3 e^{(\frac{\alpha(x-\lambda)}{\nu})}(1-4e^{(\frac{\alpha(x-\lambda)}{\nu})}+e^{(\frac{\alpha(x-\lambda)}{\nu})^2})}{\nu^2(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})^3}t^3 \tag{18}$$

3. VARIATIONAL ITERATION METHOD

To clarify the basic ideas of He's VIM, we consider the following differential equation:

$$Lu + Nu \ = \ g(t), \tag{19}$$

where L is a linear operator, N a nonlinear operator and $g(t)$ an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) \ = \ u_n(t) + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))d\xi. \tag{20}$$

where λ is a general Lagrangian multiplier [31-32] which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

First we construct a correction functional which reads

$$u_{n+1}(x,y,t) \ = \ u_n(x,t) + \int_0^t \lambda(\frac{\partial}{\partial\tau}u_n(x,\tau) - u_n(x,y,\tau)\frac{\partial}{\partial x}u_n(x,\tau) + \upsilon\frac{\partial}{\partial x^2}u_n(x,\tau))d\tau, \tag{21}$$

its stationary conditions can be obtained as follows:

$$\lambda'(\tau) \ = \ 0, \tag{22}$$

$$1 + \ \lambda(\tau)\big|_{\tau=t} \ = \ -1, \tag{23}$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = -1. \tag{24}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,y,t) = u_n(x,t) - \int_0^t \left(\frac{\partial}{\partial\tau}u_n(x,\tau) - u_n(x,y,\tau)\frac{\partial}{\partial x}u_n(x,\tau) + \upsilon\frac{\partial}{\partial x^2}u_n(x,\tau)\right)d\tau, \tag{25}$$

Since this method does not depend on the relation between the initial approximation and the target solution, it converges to the solution with any arbitrary approximation made by the user. Therefore, we start with an arbitrary initial approximation that satisfies the initial condition

$$u_0(x,\ t) = \frac{\alpha+\beta+(\beta-\alpha)e^{(\frac{\alpha(x-\lambda)}{\nu})}}{1+e^{(\frac{\alpha(x-\lambda)}{\nu})}} \tag{26}$$

By the variational formula (21), we can obtain the following result:

$$u_1(x,y,t) = u_0(x,t) - \int_0^t \left(\frac{\partial}{\partial\tau}u_0(x,\tau) - u_0(x,y,\tau)\frac{\partial}{\partial x}u_0(x,\tau) + \upsilon\frac{\partial}{\partial x^2}u_0(x,\tau)\right)d\tau, \tag{27}$$

Substituting Eq.(26) into Eq. (27)and after simplifications, we have

$$u_1(x,\ t) = \frac{\alpha+\beta+(\beta-\alpha)e^{(\frac{\alpha(x-\lambda)}{\nu})}}{1+e^{(\frac{\alpha(x-\lambda)}{\nu})}} + \frac{2\alpha^2\beta e^{(\frac{\alpha(x-\lambda)}{\nu})}}{\nu(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})}t \tag{28}$$

In the same way, we obtain $u_2(x,\ t)$ as follows:

$$u_2(x,\ t) = \frac{\alpha+\beta+(\beta-\alpha)e^{(\frac{\alpha(x-\lambda)}{\nu})}}{1+e^{(\frac{\alpha(x-\lambda)}{\nu})}} + \frac{2\alpha^2\beta e^{(\frac{\alpha(x-\lambda)}{\nu})}}{\nu(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})}t + \frac{\alpha^3\beta^2 e^{(\frac{\alpha(x-\lambda)}{\nu})}(e^{(\frac{\alpha(x-\lambda)}{\nu})}-1)}{\nu^2(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})^3}t^2 \tag{29}$$

In the same way, we obtain $u_3(x,\ t)$ as follows:

$$u(x,t) = u_3(x,t) = \frac{\alpha+\beta+(\beta-\alpha)e^{(\frac{\alpha(x-\lambda)}{\nu})}}{1+e^{(\frac{\alpha(x-\lambda)}{\nu})}} + \frac{2\alpha^2\beta e^{(\frac{\alpha(x-\lambda)}{\nu})}}{\nu(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})}t +$$
$$\frac{\alpha^3\beta^2 e^{(\frac{\alpha(x-\lambda)}{\nu})}(e^{(\frac{\alpha(x-\lambda)}{\nu})}-1)}{\nu^2(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})^3}t^2 + \frac{\alpha^4\beta^3 e^{(\frac{\alpha(x-\lambda)}{\nu})}(1-4e^{(\frac{\alpha(x-\lambda)}{\nu})}+e^{(\frac{\alpha(x-\lambda)}{\nu})^2})}{\nu^2(1+e^{(\frac{\alpha(x-\lambda)}{\nu})})^3}t^3 \tag{30}$$

The answers show that VIM and HPM results are completely similar to each other.

The solution of $u(x,\ t)$ in a series form is given by:

$$u(x,\ t) = \frac{\alpha+\beta+(\beta-\alpha)e^{(\frac{\alpha(x-\beta t-\lambda)}{\nu})}}{1+e^{(\frac{\alpha(x-\beta t-\lambda)}{\nu})}} \tag{31}$$

This can be easily verified (see Fig 1).

4. CONCLUSIONS

In this paper, the authors have studied Burger's equation through variational iteration method (VIM) and homotopy perturbation method (HPM). VIM does not require small parameters, whereas the perturbation technique does. The solution obtained by the variational iteration method is an infinite power series for appropriate initial condition, which can, in turn, be expressed in a form close to the exact solution. The results show that the variational iteration method and homotopy perturbation method are powerful mathematical tools to solve Burger's equations, they are also promising methods to solve other nonlinear equations. The solutions obtained are shown in Fig1.

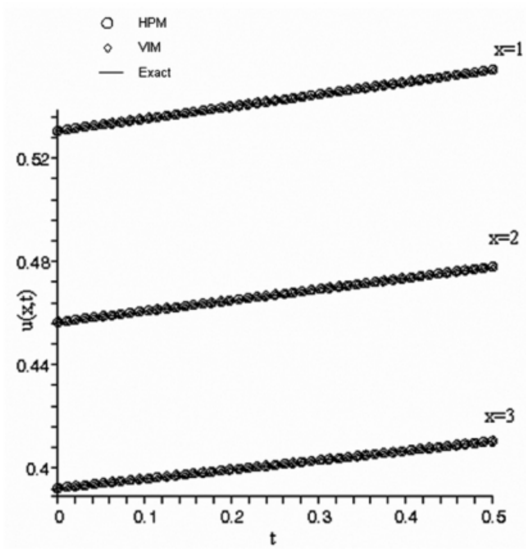

Figure. 1: The HPM and VIM Results for $u(x, t)$ for the First Five Approximations, in Comparison with the Exact Solution.

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