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DIFFERENTIAL TRANSFORMATION METHOD TO DETERMINE TEMPERATURE DISTRIBUTION OF HEAT RADIATING FINS WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

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Abstract: *Radiating extended surfaces are widely used to enhance heat transfer between primary surface and the environment. In this paper, temperature distribution of heat radiating fins with temperature dependent thermal conductivity is solved using a simulation method called the Differential Transformation Method (DTM). The concept of differential transformation is briefly introduced, and then employed to derive solutions of nonlinear equation. The obtained results from DTM are compared with those from the exact and numerical solution to verify the accuracy of the proposed method. The results reveal that the Differential Transformation Method can achieve suitable results in predicting the solution of such problems. After this verification, we analyze the effects of some physical applicable parameters in this problem such as thermo-geometric fin parameter and thermal conductivity parameter.*

1. INTRODUCTION

A growing number of engineering applications are concerned with energy transport by requiring the rapid movement of heat. To increase the heat transfer rate on a surface, fin assembly is commonly used. The heat transfer mechanism of fin is to conduct heat from heat source to the fin surface by its thermal conduction, and then dissipate heat to the air by the effect of thermal convection or thermal radiating. These extended surfaces are extensively used in various industrial applications. Also a considerable amount of research has been conducted about the variable thermal parameters which are associated with fins operating in practical situations [1-8]. In the present paper, the resulting nonlinear differential equation is solved by DTM to evaluate the temperature distribution within the fin and compared with numerical solution. Using the temperature distribution, the efficiency of the fins is expressed through a term called thermo-geometric fin parameter (ψ) and thermal conductivity parameter (β), describing the variation of the thermal conductivity.

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All these problems and phenomena are modeled by ordinary or partial differential equations. In this case study, similarity transformation has been used to reduce the governing differential equations into an ordinary non-linear differential equation. In most cases, these problems do not admit analytical solution, so these equations should be solved using special techniques. In recent years some researchers used new methods to solve these kinds of problem [9-14]. Integral transform methods such as the Laplace and the Fourier transform methods are widely used in engineering problems. These methods transform differential equations into algebraic equations which are easier to deal with. However, integral transform methods are more complex and difficult when applying to nonlinear problems. The Differential Transformation Method was first applied in the engineering domain by Zhou [15]. The differential transform method is based on Taylor expansion. It constructs an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Differential transform has the inherent ability to deal with nonlinear problems, and consequently Chiou [16] applied the Taylor transform to solve non-linear vibration problems Furthermore, the method may be employed for the solution of both ordinary and partial differential equations. Jang *et al*., [17] applied the two dimensional differential transform method to the solution of partial differential equations. Finally, Hassan [18] adopted the Differential Transformation Method to solve some problems. The method was successfully applied to various application problems [19-21].

Recently this kind of problem has been analyzed by some researchers using different methods [22-31]. In this letter, analytical solution distribution of temperature of the heat radiating fin with temperature dependent thermal conductivity has been studied by Differential Transformation Method. For this purpose, after description of the problem and brief introduction for DTM, we applied DTM to find the approximate solution. Obtaining the analytical solution of the model and comparing numerical results reveal the capability, effectiveness, convenience and high accuracy of this method.

2. DESCRIPTION OF THE PROBLEM

A typical heat pipe/fin space radiator is shown in Fig. 1. In the design, parallel pipes are joined by webs, which act as radiator fins. Heat flows by conduction from the pipes down the fin and radiates from both surfaces. Both surfaces of the fin are radiating to the vacuum of outer space at a very low temperature, which is assumed equal to zero absolute. The fin is diffuse-grey with emissivity e, and has temperature-dependent thermal conductivity k, which depends on temperature linearly. The base temperature T_b of the fin and tube surfaces temperature is constant; the radiative exchange between the fin and heat pipe is neglected. Since the fin is assumed to be

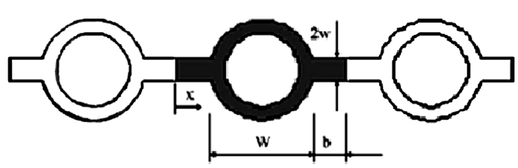

Figure 1: Schematic of a Heat Pipe/Fin Radiating Element

thin, the temperature distribution within the fin is assumed to be one-dimensional. The energy balance equation for a differential element of the fin is given as [32]:

$$2w\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right]-2\varepsilon\sigma T^4=0\,. \tag{2.1}$$

Where $k(T)$ and r are the thermal conductivity and the Stefan–Boltzmann constant, respectively. The thermal conductivity of the fin material is assumed to be a linear function of temperature according to

$$K(T)=K_b[1+\lambda(T-T_b)]\,. \tag{2.2}$$

Where k_b is the thermal conductivity at the base temperature of the fin and λ is the slope of the thermal conductivity temperature curve.

Employing the following dimensionless parameters

$$\theta=\frac{T}{T_b}\qquad \psi=\frac{\varepsilon\sigma b^2T_b^3}{kw}\qquad \xi=\frac{x}{b}\qquad \beta=\lambda T_b\,. \tag{2.3}$$

The formulation of the fin problem reduces to

$$\frac{d^2\theta}{d\xi^2}+\beta\left(\frac{d\theta}{d\xi}\right)^2+\beta\theta\frac{d^2\theta}{d\xi^2}-\psi\theta^4=0\,. \tag{2.4}$$

With boundary conditions

$$\frac{d\theta}{d\xi}=0\qquad \text{at}\quad \xi=0 \tag{2.5}$$

$$\theta=1\qquad \text{at}\quad \xi=1. \tag{2.6}$$

3.FUNDAMENTALS OF DIFFERENTIAL TRANSFORMATION METHOD

We suppose $x(t)$ to be analytic function in a domain D and $t = t_i$ represent any point in D. The function $x(t)$ is then represented by one power series whose center is located at t_i. The Taylor series expansion function of $x(t)$ is of the form [19, 20]

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i},\qquad \forall t\in D\,. \tag{3.1}$$

The particular case of Eq. (3.1) when $t_i = 0$ is referred to as the Maclaurin series of $x(t)$ and is expressed as:

$$X(k)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0},\qquad \forall t\in D\,. \tag{3.2}$$

As explained in [11] the differential transformation of the function x(t) is defined as follows:

$$X(k)=\sum_{k=0}^{\infty}\frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}, \tag{3.3}$$

where $x(t)$ is the original function and $X(k)$ is the transformed function. The differential spectrum of $X(k)$ is confined within the interval $t \in [0, H]$, where H is a constant. The differential inverse transform of $X(k)$ is defined as follows:

$$x(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H}\right)^k X(k). \tag{3.4}$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The values of function $X(k)$ at values of argument k are referred to as discrete, i.e. $X(0)$ is known as the zero discrete, $X(1)$ as the first discrete, etc. The more discrete available, the more precise it is possible to restore the unknown function. The function $x(t)$ consists of the T-function $X(k)$, and its value is given by the sum of the T-function with $(t/H)^k$ as its coefficient. In real applications, at the right choice of constant H, the larger values of argument k the discrete of spectrum reduce rapidly.

The function $x(t)$ is expressed by a finite series and Eq. (12) can be written as:

Mathematical operations performed by differential transform method are listed in Table 1.

Table 1
The Fundamental Operations of Differential Transform Method

Orginal function	*Transformed function*
$x(t) = y(t) \pm z(t)$	$X(k) = Y(k) \pm Z(k)$
$x(t) = \alpha y(t)$	$X(k) = \alpha Y(k)$
$x(t) = \dfrac{dy(t)}{dt}$	$X(k) = \dfrac{k+1}{H} Y(k+1)$
$x(t) = y(t)\, z(t)$	$X(k) = \sum_{\lambda=0}^{k} Y(\lambda) Z(k-\lambda)$
$x(t) = \dfrac{y(t)}{z(t)}$	$X(k) = \dfrac{Y(k) - \sum_{\lambda=0}^{k-1} X(\lambda) Z(k-\lambda)}{Z(0)}$
$x(t) = t^m$	$X(k) = \delta(k-m) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$

4. SOLUTION WITH DIFFERENTIAL TRANSFORMATION METHOD

Now we apply Differential Transformation Method into Eq. (4). Taking the differential transform of Eq. (2-4) with respect to y, and considering $H = 1$ gives:

$$\begin{aligned} &(k+1)(k+2)\,\theta(k+2) \\ &+ \beta \sum_{i=0}^{k} [\theta(i)(k-i+1)(k-i+2) + (i+1)\,\theta(i+1)(k-i+1)\,\theta(k-i+1)] \\ &- \psi \left[\sum_{i=0}^{k} \theta(k-i) \left(\sum_{p=0}^{i} \theta(i-p) \right) \left(\sum_{q=0}^{p} \theta(p-q)\,\alpha(q) \right) \right]. \end{aligned} \tag{3.6}$$

From boundary conditions in Eq. (2.5), that we have it in point y = 0, and exerting transformation

$$\theta(1) = 0. \tag{3.7}$$

The other boundary conditions are considered as follow:

$$\theta(1) = a. \tag{3.8}$$

Where a is constant, and we will calculate it with considering another boundary condition in Eq. (2.6) at point y = 1.

We will have:

$$\begin{aligned}
&\theta(2) = \frac{1}{2}\frac{\psi a^4}{1+\beta a} \\
&\theta(3) = 0 \\
&\theta(4) = \frac{1}{24}\frac{\psi^2 a^7(\beta a + 4)}{(1+\beta a)^3} \\
&\theta(5) = 0 \\
&\theta(6) = \frac{1}{720}\frac{\psi^3 a^{10}(25\beta^2 a^2 + 32\beta a + 52)}{(1+\beta a)^5} \\
&\theta(7) = 0 \\
&\theta(8) = \frac{1}{40320}\frac{\psi^4 a^{13}(95\beta^3 a^3 - 1212\beta^2 a^2 - 1020\beta a - 1288)}{(1+\beta a)^7} \\
&\theta(9) = 0 \\
&. \\
&. \\
&.
\end{aligned} \tag{3.9}$$

The above process is continuous. Substituting Eq. (3.9) into the main equation based on DTM, it can be obtained that the closed form of the solutions is:

$$\begin{aligned}
\theta(\xi) = a + \frac{1}{2}\frac{\psi^2 a^4 \xi^2}{1+\beta a} + \frac{1}{24}\frac{\psi^2 a^7(\beta a + 4)\xi^4}{(1+\beta a)^3} + \frac{1}{720}\frac{\psi^3 a^{10}(25\beta^2 a^2 + 32\beta a + 52)\xi^6}{(1+\beta a)^5} \\
- \frac{1}{40320}\frac{\psi^4 a^{13}(95\beta^3 a^3 - 1212\beta^2 a^2 - 1020\beta a - 1288)\xi^8}{(1+\beta a)^7} + \ldots
\end{aligned} \tag{3.10}$$

To obtain the value of a, we substitute the boundary condition from Eq. (2.6) into Eq. (3.10) in point y = 1. So, we have:

$$\theta(1) = a + \frac{1}{2}\frac{\psi^2 a^4}{1+\beta a} + \frac{1}{24}\frac{\psi^2 a^7(\beta a+4)}{(1+\beta a)^3} + \frac{1}{720}\frac{\psi^3 a^{10}(25\beta^2 a^2 + 32\beta a + 52)}{(1+\beta a)^5}$$
$$- \frac{1}{40320}\frac{\psi^4 a^{13}(95\beta^3 a^3 - 1212\beta^2 a^2 - 1020\beta a - 1288)}{(1+\beta a)^7} + ... = 1 \qquad (3.11)$$

Solving Eq. (3.11), gives the value of a. This value is too long that are not shown in this paper. By substituting obtained a into Eq. (3.10), we can find the expressions of $\theta(\zeta)$.

5. NUMERICAL METHOD

In this part, we will present our numerical results corresponding to various instances mentioned above .The best approximate for solving Eq. (2.4) that can be used is fourth order Runge-Kutta method. It is often utilized to solve differential equation systems. Third order differential equations can be usually changed into second order equations and then first order. After that, it can be solved through Runge-Kutta method. Table 2 illustrate the case of variable thermal conductivity ($\beta = 0.1$ and $\beta = 0.5$), results of the present analysis are tabulated against the numerical solution obtained by fourth-order Runge–Kutta. In this case, a very interesting agreement between the results is observed too, which confirms the excellent validity of the DTM. Then in Fig. 2 the comparison of the solutions between DTM and numerical results is shown.

Table 2
The Results of DTM and NS for $\theta(\xi)$

	$\beta = 0.1$, $\psi = 0.3$			$\beta = 0.5$, $\psi = 1$		
ξ	*DTM*	*NS*	*Error*	*DTM*	*NS*	*Error*
0	0.9024500251	0.902450026423	0.0000000013	0.8203106204	0.820310630414	0.0000000100
0.05	0.9026782003	0.902678200845	0.0000000005	0.8207121034	0.820712111382	0.0000000079
0.1	0.9033631592	0.903363160392	0.0000000011	0.8219177832	0.821917784329	0.0000000011
0.15	0.9045062039	0.904506205731	0.0000000018	0.8239313651	0.823931375476	0.0000000103
0.2	0.9061095142	0.906109515597	0.0000000013	0.8267590703	0.826759074832	0.0000000045
0.25	0.9081761604	0.908176161361	0.0000000009	0.8304097035	0.830409711255	0.0000000077
0.3	0.9107101236	0.910710125529	0.0000000019	0.8348947520	0.834894760462	0.0000000084
0.35	0.9137163207	0.913716321496	0.0000000007	0.8402285139	0.840228520007	0.0000000061
0.4	0.9172006355	0.917200637194	0.0000000016	0.8464282615	0.846428264417	0.0000000029
0.45	0.9211699579	0.921169959567	0.0000000016	0.8535144414	0.853514445013	0.0000000036
0.5	0.9256322287	0.925632230352	0.0000000016	0.8615109156	0.861510917341	0.0000000017
0.55	0.9305964928	0.930596494272	0.0000000014	0.8704452488	0.870445256147	0.0000000073
0.6	0.9360729600	0.936072960483	0.0000000004	0.8803490500	0.880349050972	0.0000000009
0.65	0.9420730770	0.942073077721	0.0000000007	0.8912583719	0.891258378834	0.0000000069
0.7	0.9486096066	0.948609607971	0.0000000013	0.9032141824	0.903214183179	0.0000000014
0.75	0.9556967193	0.955696720529	0.0000000012	0.9162629154	0.916262916810	0.0000000014
0.8	0.9633500996	0.963350101053	0.0000000014	0.9304571188	0.930457121954	0.0000000031
0.85	0.9715870624	0.971587064247	0.0000000018	0.9458562124	0.945856220233	0.0000000078
0.9	0.9804266871	0.980426687989	0.0000000008	0.9625273811	0.962527385029	0.0000000039
0.95	0.9898899710	0.989889971861	0.0000000007	0.9805466247	0.980546626869	0.0000000021
1	0.9999999998	1.000000000000	0.0000000002	0.9999999999	1.00000000000	0.0000000001

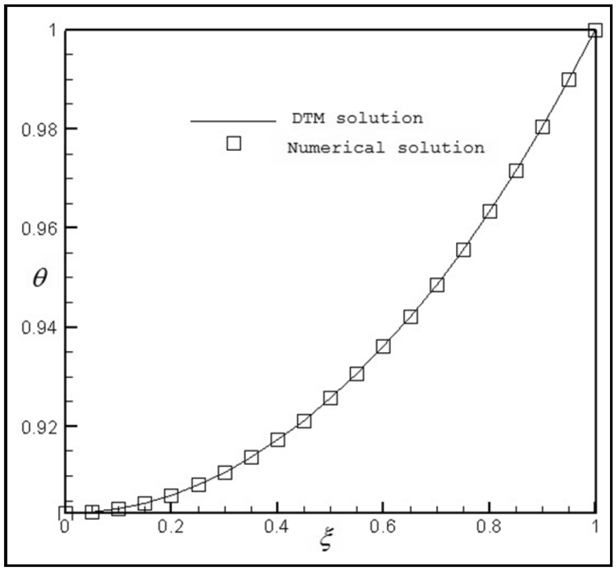

Figure 2(a): Comparison of the Solutions via DTM and Numerical Solution for $\theta(\xi)$ For $\beta = 0.1$, $\psi = 0.3$

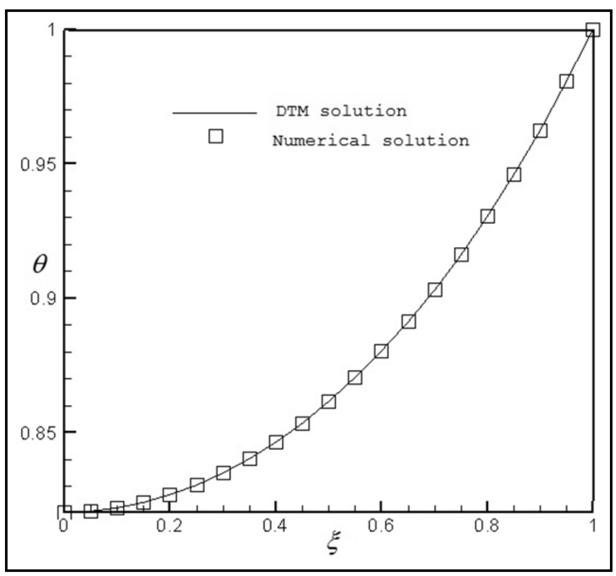

Figure 2(b): Comparison of the Solutions via DTM and Numerical Solution for $\theta(\xi)$ For $\beta = 0.5$, $\psi = 1$

Fig. 3 depicts the effect temperature difference with various values of thermo-geometric fin parameter (ψ) from 0.3 to 0.8 when the thermal conductivity is constant ($\zeta = 0$). This figure display that increasing in the values of thermo-geometric fin parameter produce decrease in values of dimensionless temperature.

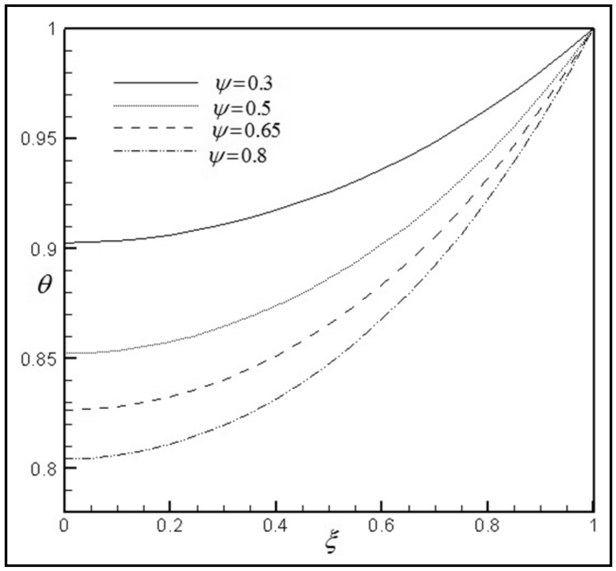

Figure 3: Temperature Distribution in Heat Radiatig Fins with Variable Thermal Conductivity. For Various ψ when $\beta = 0$.

The dimensionless temperature distributions along the fin surface with β varying from 0 to 0.8 are depicted in Fig. 4 for different values of $\psi = 0.6$ and $\psi = 1$, respectively. If the thermal conductivity of the fin's material increases with the temperature, so the dimensionless temperature increases, too.

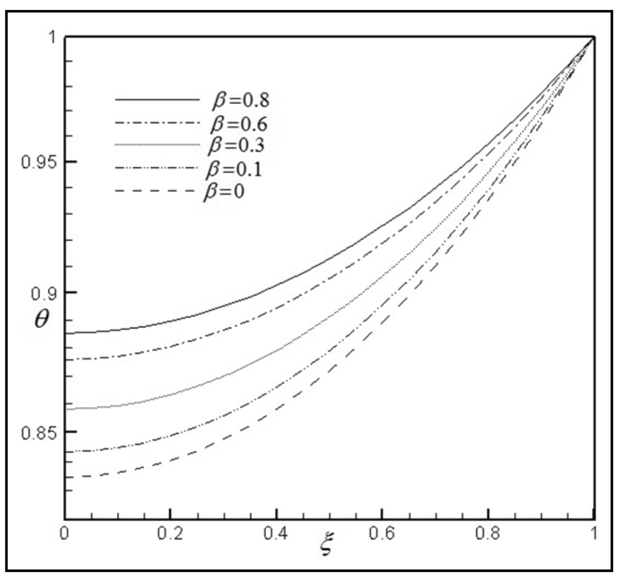

Figure 4(a): Temperature Distribution in Heat Radiating Fins with Variable Thermal Conductivity When $\psi = 0.6$

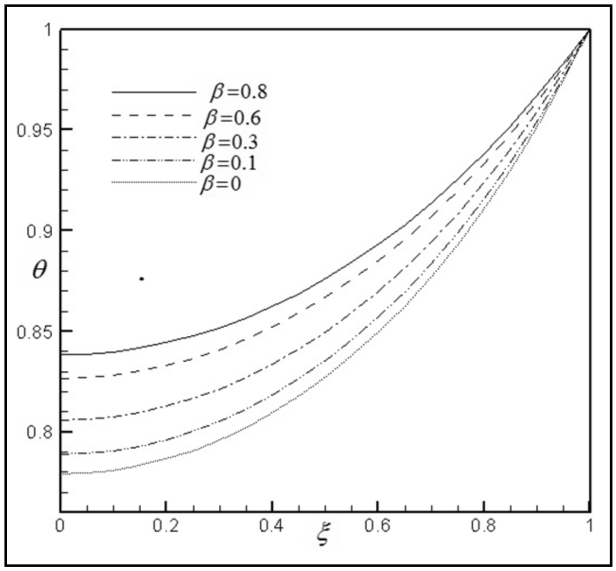

Figure 4(b): Temperature Distribution in Heat Radiating Fins with Variable Thermal Conductivity When $\psi = 1$

6. CONCLUSIONS

In this study, we presented the definition and operation of one dimensional Differential Transformation Method (DTM). Using the differential transform, differential equations can be transformed to algebraic equations in the K domain. This method has applied to solve Nonlinear differential equation arising in heat radiating fins with temperature dependent thermal conductivity problem. This exerting of DTM is compared to fourth order Runge–Kutta Numerical solution. The figures and tables clearly show high accuracy of DTM to solve heat transfer problems in engineering.

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