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ANALYTICAL SOLUTION OF SECOND ORDER WAVE FORCES ON A SUBMERGED SPHERE IN REGULAR WAVES

S. H. Mousavizadegan & M. Rahman

***ABSTRACT*:** An analytical benchmark solution is presented for the second-order steady force acting on a submerged sphere in a fluid of infinite depth. The analytical solution is obtained by using the multipole expansion method and expanding the velocity potentials into a series of associated Legendre functions. The far field method is applied to find the horizontal drift force. The horizontal drift force is computed taking into account the effect of the radiation velocity potential using the far field method. The contribution of components of the radiation velocity potential on the total horizontal drift force is studied along with the effect of the position of the center of mass.

Keywords and phrases: wave forces, drift forces, submerged sphere, diffraction, multipole expansion, Legendre functions, radiation, gravity waves.

AMS Mathematics Subject Classication: 74J05, 76B15, 78A40.

1. INTRODUCTION

Marine structures are usually designed to operate in a wave environment. Structural loading of the body surface under the water and unsteady motions of the body are two of the principal resulting problems. When the characteristic body dimension is comparable to the wave length, the potential effects dominate. The presence of the body alters the pattern of wave propagation in the vicinity of the structure and causes wave scattering. The body may also oscillate and cause the radiation of waves if the constraints are not suciently rigid. As a consequence, the body experiences reacting forces from the surrounding fluid and constraints. Due to the complexity of the associated boundary value problems with the wave-body interactions in the frame of the potential flow theory, analytical solutions can be obtained for a few special geometries. In general, a numerical solution of Laplace's equation along with the associated boundary conditions is imperative. The objective of this research paper is to obtain an analytical bench mark solution for the effect of the radiation velocity potential on the second-order horizontal drift force acting upon a submerged sphere in infinite fluid depth.

A number of notable works have been carried out over the last few decades on analytical solutions of spherical structures in waves. Havelock [3] gave a solution of the fluid motion, due to a heaving hemisphere in the surface of a fluid of infinite depth. Following Havelock [3], Hulme [4] obtained the added mass and damping coefficients associated with the periodic motions of a floating hemisphere in the surge and heave motions. They applied the multipole expansion method. This method was originally used by Ursell [11] for the case of a half-immersed horizontal circular cylinder and then was extended to three dimensional problems by Thorne [10]. They expressed the velocity potential as a summation of an infinite series of wave-free potentials together with a suitable source at the center of the sphere. The velocity potential satisfied all the boundary conditions except the body surface boundary condition. The body surface boundary condition was used to derive an infinite linear system of equations for the infinite number of unknowns appearing in the expansion of the potentials.

The problem of scattering of a surface wave by a fully submerged, rigid and stationary sphere was discussed by Gray [2]. Using the general solution of the Laplace equation in spherical coordinates, Gray [2] expressed the diffraction potential in a series solution of associated Legendre functions with unknown coefficients. Applying source distributions around the body, the diffraction problem was formulated as an integral equation with the aid of the free surface Green function and imposing the body surface boundary condition. The Green function and the incident velocity potential were expanded in spherical harmonics centered at the origin. Carrying out the integrations, an infinite system of linear algebraic equations was set up to obtain the unknown coefficients of the diffraction potential expansion.

Srokosz [9], in his study on a submerged sphere as an absorber of wave power, applied the multipole potentials derived by Thorne [10] to obtain the non-dimensional parameters of the surge and heave radiation problems. Wang [12] expressed the velocity potentials as a summation of a wave source at the center of a sphere with an infinite series of wave free potentials to discuss the free motions of a spherical submarine in infinite depth. Linton [5] studied the radiation and diffraction of water waves by a submerged sphere in finite depth by the method of multipole expansion. He also discussed the elevation of the free surface due to the presence of the sphere.

Wu *et al.* [15] presented a solution for the drift forces along with the analysis of the exciting forces acting on a submerged sphere in finite water depth using the multipole expansion method. They applied both the far field and near field methods to obtain the drift forces. They only considered the contribution of the diffraction problem velocity potentials to the drift force.

Rahman [8] studied the fields of the hydrodynamic diffraction pressure and fluid velocity around a submerged sphere in finite depth. He used the multipole expansion method to obtain the fluid velocity potential. The spatial potential was expressed in the form of a double series of zonal harmonics with the unknown coefficients of the infinite set of linear algebraic equations. He presented the free surface elevation, the hydrodynamic diffraction pressure, the fluid velocity around the sphere and the total exciting forces acting on the body in graphical form for a wide range of the body submergences, fluid depth and wave spectrum.

2. FORMULATION OF THE PROBLEM

Two sets of coordinate systems are considered. One is a right-handed Cartesian coordinate system (x, y, z) fixed in the fluid with oz opposing the direction of gravity and $o - xy$ lying in the undisturbed free surface. The other set is the spherical coordinate system (r, θ, ψ) with the origin at the center of the sphere, as shown in Fig. 1. Two coordinate systems are related by

$$x = r \sin\theta \cos\psi,\ y = r \sin\theta \sin\psi,\ z = -r \cos\theta - h$$

$$\text{and } \tan\theta = -\frac{R}{z+h} \qquad 0 \le \theta \le \pi,$$

$$\tan\psi = \frac{y}{x} \qquad 0 \le \psi \le 2\pi,$$

$$r = \sqrt{R^2 + (z+h)^2}, \tag{1}$$

where $R = \sqrt{x^2 + y^2}$ is the horizontal distance.

It is considered that the amplitudes of both the incident wave and the motion of the sphere are small. Therefore, the usage of linearized potential theory is justified and the whole fluid flow can be characterized by a scalar function called the velocity potential. The symmetrical geometry of the sphere indicates that there are only three modes of motion responding to disturbance in any given direction. These are the surge, heave and pitch motions, as the incident wave propagates along the x-axis. The hydrodynamic reactions in pitch motion vanish due to the symmetrical geometry of the sphere. The total potential can be written as

$$\phi(r, \theta, \psi, t) = [(\eta_1 \phi_1 + \eta_3 \phi_3) + A(\phi_I + \phi_S)]e^{-i\omega t}, \tag{2}$$

where η_1 and η_3 are the amplitudes of the surge and heave motion, respectively, and ϕ_1 and ϕ_3 are the spatial velocity potential for the surge and heave motions, respectively. The notation A is the amplitude of the incoming wave and ϕ_I and ϕ_S are the incoming and scattering wave time independent velocity potentials, respectively. The frequency of the incoming wave is given by ω.

The radiation and scattering velocity potentials are subjected to the Laplace equation in the fluid domain, the linearized free surface boundary condition, a bottom boundary condition that indicates no flux through the bottom of the fluid, the radiation condition at innity and the Neumann body surface boundary condition at the mean position of the body. The hydrodynamic forces and moments acting on the sphere due to the reaction of the fluid are

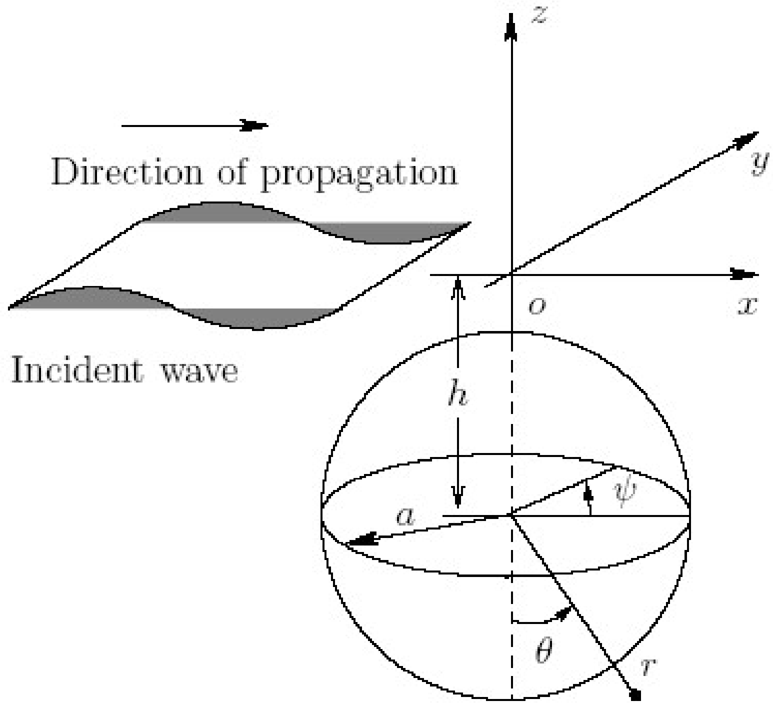


Figure 1: Sketch of the Submerged Sphere Geometry and the Coordinates Definition

$$\begin{Bmatrix} \mathbf{F} \\ \mathbf{M} \end{Bmatrix} = \int_0^{2\pi} \int_0^{\pi} P \begin{Bmatrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} \end{Bmatrix} a^2 \sin\theta \, d\theta \, d\psi, \tag{3}$$

where P is the pressure around the surface of the sphere and the unit normal vector $\mathbf{n}$ is positive when pointing out of the fluid domain from the surface of the sphere. In the linearized potential theory, the forces and moments result from the transient pressure

$$P(r,\theta,\psi,t) = \Re[p(r,\theta,\psi)e^{-i\omega t}] = -\rho\Re\left[\frac{\partial\Phi}{\partial t}\right], \tag{4}$$

where ρ is the fluid density and p is the time-independent pressure.

The dynamic pressure due to the fluid velocity contributes to the second-order forces on the sphere.

$$P_d(r,\theta,\psi,t) = -\frac{\rho}{2}\Re(\nabla\phi e^{-i\omega t})\Re(\nabla\phi e^{-i\omega t})$$

$$= -\frac{\rho}{4}\Re[(\nabla\phi)^2 e^{-i2\omega t} + \nabla\phi\nabla\phi^*] \tag{5}$$

There is an oscillatory contribution and a time average part. The forces and moments resulting from the time average part are drift forces and moments. For detail explanation refer to Mousavizadegan [6].

3. DIFFRACTION PROBLEM

The presence of a body in the fluid results in the diffraction of the incident wave system. If the sphere is held fixed in the fluid, the total disturbance is due to the incoming wave and the scattering effect of the sphere. The velocity potentials for this problem are constructed by means of superimposing the various orders of the multipole potentials.

3.1 Incident Wave

It is assumed that the incident wave is propagating in the positive *x*-direction with the amplitude A and the frequency ω. The time-independent linear velocity potential of the incoming wave can be given as

$$\phi_I = -\frac{iAg}{\omega} e^{-vz} e^{ivR\cos\psi}. \tag{6}$$

This potential can be expressed in terms of the associated Legendre function in the form (see Appendix *B*)

$$\phi_I = \frac{Ag}{\omega} e^{-vh} \sum_{m=0}^{\infty} \varepsilon_m i^{m+1} \cos m\psi \sum_{u=m}^{\infty} (-1)^{u+m-1} \frac{(vr)^u}{(u+m)!} P_u^m(\cos\theta), \tag{7}$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ for $m \geq 1$. The associated Legendre function given by $P_u^m(\cos\theta)$ is dened by

$$P_u^m(\cos\theta) = (\sin\theta)^m \frac{d^m P_u(\cos\theta)}{d(\cos\theta)^m}, \tag{8}$$

where P_u(cos) is the Legendre polynomial.

The Froude-Krylov force is the effect of the incident wave on a body without considering the diffraction and radiation effect of the body. It can be calculated by replacing (7) in (3) and considering the orthogonality properties of associated Legendre functions and cosine functions. The time-independent part of the Froude-Krylov force is (see Appendix *C*)

$$f_i = C_i \frac{4}{3} \rho\pi a^3 A\omega^2 e^{-vh}, \tag{9}$$

where $C_1 = -i$ for surge motion and $C_3 = -1$ for heave motion.

3.2 Scattering Potential

The time-independent scattering velocity potential ϕ_S is a solution of the Laplace equation and must satisfy the boundary conditions at the fluid boundaries. It can be expressed in virtue of the body surface boundary condition and (7) as

$$\phi_S(r,\theta,\psi) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+2} A_n^m \widehat{\phi}_n^m \cos m\psi, \tag{10}$$

where $\widehat{\phi}_n^m$ is the multipole potential. The function $\widehat{\phi}_n^m \cos m\psi$ cosm is the effect of a *n*th-order singularity at the origin of the sphere. The multipole potential can be written in the form (Thorne [10])

$$\widehat{\phi}_n^m \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{m+n-1}}{(n-m)!} P.V.\int_0^{\infty} \frac{v+k}{v-k} k^n e^{k(z-h)} J_m(kR)dk$$

$$-\frac{(-1)^{m+n}}{(n-m)!}2\pi i v^{n+1}e^{v(z-h)}J_m(KR), \tag{11}$$

where J_m is the Bessel function of first kind of order m. Taking into account the identity (Linton [5])

$$e^{\pm kr\cos\theta}J_m(kr\sin\theta) = (\pm 1)^m \sum_{u=m}^{\infty}\frac{(\pm kr)^u}{(u+m)!}P_u^m(\cos\theta), \tag{12}$$

the second and third terms in (11) can be expanded into a series of the associated Legendre function in the region near the body surface. It may be written

$$\widehat{\phi}_n^m(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} = +\sum_{u=m}^{\infty}(C_{un}^m + iD_{un}^m)r^u P_u^m(\cos\theta), \tag{13}$$

where

$$C_{un}^m = \frac{(-1)^{n+u-1}}{(n-m)!(u+m)!}P.V.\int_0^{\infty}\frac{v+k}{v-k}k^{n+u}e^{-2kh}dk$$

$$D_{un}^m = \frac{(-1)^{n+u-1}}{(n-m)!(u+m)!}2\pi v^{n+u+1}e^{-2vh}. \tag{14}$$

This is valid for $r < 2h$. The coefficients C_{un}^m can be obtained by the method of integrating by parts and applying the exponential integrals. The spatial scattering velocity potential can be obtained by replacing (13) in (10).

$$\phi_S(r,\theta,\psi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}a^{n+2}A_n^m\left[\frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{u=m}^{\infty}(C_{um}^m + iD_{un}^m)r^u P_u^m(\cos\theta)\right]\cos m\psi \tag{15}$$

The coefficients A_n^m are calculated by imposing the body surface boundary condition for the diffraction problem.

$$\left.\frac{\partial\phi_I^m}{\partial r}\right|_{r=a} = -\left.\frac{\partial\phi_S^m}{\partial r}\right|_{r=a} \quad \text{for } m = 0, 1, \ldots \tag{16}$$

Substituting (7) and (15) in (16), the body surface boundary condition takes the following form.

$$\sum_{n=m}^{\infty}A_n^m\left[(n+1)P_n^m(\cos\theta) - \sum_{u=m}^{\infty}(C_{un}^m + iD_{un}^m)a^{n+u+1}P_u^m(\cos\theta)\right] =$$

$$A\omega e^{-vh}\varepsilon_m i^{m+1}\cos m\psi\sum_{u=m}^{\infty}(-1)^{m+u-1}\frac{u}{(u+m)!}(va)^{u-1}P_u^m(\cos\theta) \quad \text{for } m = 0,1,\ldots \tag{17}$$

It can be simplied by application of the orthogonality property of the associated Legendre functions in the form

$$\sum_{n=m}^{\infty}A_n^m\left\{(n+1)\delta_{un} - (C_{un}^m + iD_{un}^m)ua^{n+u+1}\right\} = A\omega e^{-vh}u(va)^{u-1}\varepsilon_m i^{m+1}\frac{(-1)^{u+m-1}}{(u+m)!}$$

$$\text{for } m = 0, 1, \ldots \quad \text{and } u = m, m+1, m+2, \ldots, \tag{18}$$

where δ_{nu} is Kronecker delta function. This is a system of linear algebraic equations of the unknowns A_n^m. The hydrodynamic coefficients attributed to the diffraction problem can be computed when the coefficients A_n^m are obtained.

3.3 Exciting Forces

The exciting force in y-direction and all exciting moments vanish due to the geometrical symmetry of the sphere and the direction of the incident wave propagation. The wave forces in x– and z–directions can be found by replacing (4) in (3).

$$\begin{Bmatrix} F_{ex} \\ F_{ez} \end{Bmatrix} = \Re\left[\begin{Bmatrix} f_{ex} \\ f_{ez} \end{Bmatrix} e^{-i\omega t}\right]$$

$$= \Re\left\{\left[i\rho\omega a^2 \int_0^{\pi}\int_0^{2\pi} \phi_D(a,\theta\psi)\begin{Bmatrix} n_x \\ n_x \end{Bmatrix}\sin\theta d\theta d\psi\right] e^{-i\omega t}\right\} \tag{19}$$

The normals n_x and n_z are dened as

$$n_x = -P_1^1(\cos\theta)\cos\psi \qquad n_z = P_1(\cos\theta). \tag{20}$$

The time-independent diffraction velocity potential ϕ_D may be obtained by (see Appendix D)

$$\phi_D(a,\theta,\psi) = a\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{2n+1}{n}A_n^m P_n^m(\cos\theta)\cos m\psi \tag{21}$$

at the surface of the sphere. Substituting (20) and (21) in (19) yield

$$f_{ex} = -i\rho\omega a^3\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{2n+1}{n}A_n^m \times \int_0^{\pi} P_n^m(\cos\theta)P_1^1(\cos\theta)\sin\theta d\theta\int_0^{2\pi}\cos m\psi\cos\psi d\psi$$

$$f_{ez} = -i\rho\omega a^3\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{2n+1}{n}A_n^m \times \int_0^{\pi} P_n^m(\cos\theta)P_1(\cos\theta)\sin\theta d\theta\int_0^{2\pi}\cos m\psi d\psi \tag{22}$$

Taking into account the orthogonal property of the cosine and associated Legendre functions, the force coefficients are obtained in simple forms as

$$f_{ex} = -4i\pi\rho\omega a^3 A_1^1 \qquad f_{ez} = 4i\pi\rho\omega a^3 A_1 \tag{23}$$

where the coefficients A_1^1 and A_1 are obtained through the solution of the complex matrix equations (18)). Dividing by $-\frac{4}{3}i\rho\pi a^3 A\omega^2 e^{-vh}$

$$\hat{f}_{ex} = 3e^{vh}\frac{A_1^1}{A\omega} \qquad \hat{f}_{ez} = -3e^{vh}\frac{A_1}{A\omega}, \tag{24}$$

where $\hat{f}_{ex}$ and $\hat{f}_{ez}$ are the non-dimensional exciting force coefficients in the x– and z-direction, respectively.

4. Radiation Problem

The radiation potentials are subjected to the same set of boundary conditions as the scattering potential except for the body surface boundary condition. This condition may be expressed in the form

$$\left.\frac{\partial \phi_1}{\partial r}\right|_{r=a} = i\omega P_1^1(\cos\theta)\cos\psi$$

$$\left.\frac{\partial \phi_3}{\partial r}\right|_{r=a} = -i\omega P_1(\cos\theta) \tag{25}$$

for the surge and heave motions, respectively. These imply that the time-independent radiation velocity potentials are written as

$$\phi_j^m(r,\theta,\psi) = \sum_{n=m}^{\infty} a^{n+2} B_n^m \hat{\phi}_n^m \cos m\psi, \tag{26}$$

where $m = 1$ and $j = 1$ for the surge motion and $m = 0$ and $j = 3$ for the heave motion. The multipole potential $\hat{\phi}_n^m$ is the same as (11) due to the same set of boundary conditions. Therefore, the radiation velocity potentials may be given in the form

$$\phi_j^m = \sum_{n=m}^{\infty} a^{n+2} B_n^m \left[\frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{u=m}^{\infty}(C_{un}^m + iD_{un}^m) r^u P_u^m(\cos\theta)\right]\cos m\psi \quad \text{for } m = 0,1, \tag{27}$$

where C_{un}^m and D_{un}^m were expressed in (14).

The complex coefficients B_n^m are obtained by the application of the body surface boundary condition for each mode of motion. Replacing (27) in (25), and using the orthogonal property of the cosine and associated Legendre function, leads to the following infinite sets of linear algebraic equations for the surge and heave motions, respectively.

$$\sum_{n=1}^{\infty} B_n^1\left[(u+1)\delta_{un} - (C_{un}^1 + iD_{un}^1)ua^{n+u+1}\right] = i\omega\delta_{u1} \quad \text{and} \quad u = 1,2,3,\ldots$$

$$\sum_{n=0}^{\infty} B_n\left[(u+1)\delta_{un} - (C_{un} + iD_{un})sa^{u+s+1}\right] = i\omega\delta_{u1} \quad \text{and} \quad u = 0,1,2,\ldots \tag{28}$$

Once the coefficients B_n^1 and B_n are found, the radiation problem is completely solved.

The components of force associated with the radiated wave velocity potentials are

$$F_x = \Re\{\eta_1 e^{-i\omega t} f_{11}\} \text{ and } F_z = \Re\{\eta_3 e^{-i\omega t} f_{33}\}, \tag{29}$$

where f_{11} and f_{33} are complex force coefficients of the surge and heave motions, respectively. The force component in the y-direction and all moment components vanish due to the symmetry of the sphere. The force coefficients can be calculated by making use of (3) and (4).

$$f_{11} = i\omega\rho\int_0^{2\pi}\int_0^{\pi} \phi_1(a,\theta,\psi) n_1 a^2 \sin\theta d\theta d\psi$$

$$f_{33} = i\omega\rho\int_0^{2\pi}\int_0^{\pi} \phi_3(a,\theta,\psi) n_3 a^2 \sin\theta d\theta d\psi \tag{30}$$

The radiation velocity potentials are

$$\phi_1\,|_{r=a} = a\cos\psi\left[\sum_{n=1}^{\infty} B_n^1 P_n^1(\cos\theta)\frac{2n+1}{n} + i\omega P_1^1(\cos\theta)\right]$$

$$\phi_3 \big|_{r=a} = a\left[\sum_{n=1}^{\infty} B_n P_n(\cos\theta)\frac{2n+1}{n} - i\omega P_1(\cos\theta)\right] \tag{31}$$

at the surface of the sphere, as shown in Appendix E. Substituting (20) and (31) in (30) yields

$$f_{11} = i\rho\omega a^3 \int_0^{2\pi} \cos^2 \psi d\psi \int_0^{\pi} \left[\sum_{n=1}^{\infty} B_n^1 P_n^1(\cos\theta)\frac{2n+1}{n} + i\omega P_1^1(\cos\theta)\right] P_1^1(\cos\theta)\sin\theta d\theta$$

$$f_{33} = -i\rho\omega a^3 \int_0^{2\pi} d\psi \int_0^{\pi} \left[\sum_{n=1}^{\infty} B_n P_n(\cos\theta)\frac{2n+1}{n} - i\omega P_1(\cos\theta)\right] P_1(\cos\theta)\sin\theta d\theta. \tag{32}$$

Using the orthogonal property of the associated Legendre functions, the force coefficients may be given in the form

$$f_{11} = -\frac{4}{3}\pi a^3 \rho\omega^2 (1 - \frac{3iB_1^1}{\omega}) \qquad f_{33} = -\frac{4}{3}\pi a^3 \rho\omega^2 (1 + \frac{3iB_1}{\omega}), \tag{33}$$

where the coefficients B_1^1 and B_1 are computed through the solution of the complex matrix equations (28).

The radiation force coefficients are usually expressed as

$$f_{ij} = \omega^2 \alpha_{ij} + i\omega\beta_{ij}, \tag{34}$$

where α_{ij} are the added mass coefficients and β_{ij} are the damping coefficients. The added mass and damping coefficients are made non-dimensional by dividing by $\frac{4}{3}\pi a^3 \rho$ and $\frac{4}{3}\pi a^3 \rho\omega$, respectively. Therefore, it can be written that

$$\mu_{11} = -1 - \frac{3\Im(B_1^1)}{\omega} \qquad \lambda_{11} = \frac{3\Re(B_1^1)}{\omega}$$

$$\mu_{33} = -1 + \frac{3\Im(B_1)}{\omega} \qquad \lambda_{33} = -\frac{3\Re(B_1)}{\omega}, \tag{35}$$

where μ_{11} and μ_{33} are the non-dimensional added mass coefficients and 11 and 33 are the non-dimensional damping coefficients for the surge and heave motions, respectively.

5. MOTION OF THE SPHERE

It is supposed that the sphere is hydrostatically stable, suggesting that the center of mass is located under the center of buoyancy. The mass matrix elements of the non-zero motions are $m_{11} = m_{33} = M$, $m_{15} = m_{51} = MZ_G$ and $m_{55} = Mk^2$, where $M = \frac{4}{3}\rho\pi a^3$ and $k = \frac{2}{5}a^2 + Z_G^2$ is the radius of gyration of the sphere. The term denoted by Z_G is the vertical position of the center of mass. It is measured in respect to the center of the sphere. The only restoring force coefficient is $c_{55} = MgZ_G$. The equations of motion of a submerged sphere in regular wave may be written

$$\begin{cases} -\omega^2 (M + \alpha_{11})_{\eta_1} - i\omega\beta_{11}\eta_1 - \omega^2 MZ_{G\eta_5} = f_{ex} & \text{for surge motion} \\ -\omega^2 (M + \alpha_{33})_{\eta_3} - i\omega\beta_{33}\eta_3 = f_{ez} & \text{for heave motion} \\ -\omega^2 Z_{G\eta_1} - \omega^2 k^2 \eta_5 + gZ_{G\eta_5} = 0 & \text{for pitch motion,} \end{cases} \tag{36}$$

where η_1, η_3 and η_5 are the surge, heave and pitch amplitudes, respectively. These may be expressed in non-dimensional form as

$$\begin{cases} (1+\mu_{11}+i\lambda_{11})\frac{\eta_1}{A}+Z_G\frac{\eta_5}{A}=i\hat{f}_{ex}e^{-vh} & \text{for surge motion} \\ (1+\mu_{33}+i\lambda_{33})\frac{\eta_3}{A}=i\hat{f}_{ez}e^{-vh} & \text{for heave motion} \\ \frac{\eta_1}{A}+\left(\frac{k^2}{Z_G}-\frac{1}{v}\right)\frac{\eta_5}{A}=0 & \text{for pitch motion.} \end{cases} \tag{37}$$

The linear motion along x-direction and the rotational motion about y-axis are coupled with each other.

6. DRIFT FORCE

The sphere is also acted upon by a steady force in the direction of the incident wave propagation. The far field method is applied to find this component of the drift force. The total velocity potential at large distance from the sphere may be defined by

$$\Phi=\left\{-\frac{igA}{\omega}e^{vz}e^{ivR\cos(\theta-\beta)}+\left(\frac{2}{\pi vR}\right)^{\frac{1}{2}}e^{vz}e^{i(vR+\frac{\pi}{4})}H(v,\psi)\right\}e^{-i\omega t} \tag{38}$$

as shown in (59) (see Appendix A). The main task is to obtain an expression for H-function to find the drift force. This function results from the scattering and radiating effects of the sphere in fluid domain. It may be written that

$$H(v,\psi) = H_S(v,\psi) + H_1(v,\psi) + H_3(v,\psi), \tag{39}$$

where $H_S(v,\psi)$ is the scattering H-function, $H_1(v,\psi)$ is the surge H-function and $H_3(v,\psi)$ is the heave H-function. The scattering potential and the potentials due to the motion of the sphere in x- and z-directions are expressed in (10) and (26), respectively. The expression (13) for multipole potential cannot be used because it is valid near the surface of the sphere. The expression (11) is valid for the whole domain of the fluid. The integral part of this equation may be computed with the application of the contour integral rules. By adopting a contour in the first half of the complex space and deforming it at the point $v = k$, the multipole potential can be finally expressed in the form

$$\hat{\phi}_n^m=\frac{P_n^m\cos\theta}{r^{n+1}}+\frac{(-1)^{m+n}}{(n-m)!}\Re\left\{\frac{2}{\pi}\int_0^\infty\frac{v+ik}{v-ik}(ik)^n e^{ik(z-h)}e^{-im\frac{\pi}{2}}K_m(kR)dk\right\}$$
$$-\frac{(-1)^{m+n}}{(n-m)!}2\pi i v^{n+1}e^{v(z-h)}H_m^{(1)}(vR) \tag{40}$$

as shown in Appendix F. The function $K_m(kR)$ is the second kind of modied Bessel function of order m and argument kR. The $H_m^1(vR)$ is the Hankel function of the first kind of order m and argument vR. The asymptotic value of the second kind of modied Bessel function for large argument is (Thorne [10])

$$K_m(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}e^{-x}\left\{1+\frac{4m^2-1^2}{1!8x}+O\left(\frac{1}{x^2}\right)\right\}. \tag{41}$$

Hence, the first and second parts of (40) tend to zero as $R\to\infty$. Taking into account the asymptotic value of the Hankel function of the first kind, the multipole potential tends to

$$\hat{\phi}_n^m \to 2\pi i \frac{(-1)^{m+n}}{(n-m)!} v^{n+1} e^{v(z-h)} \left(\frac{2}{\pi v R}\right)^{\frac{1}{2}} e^{i(vR - \frac{m\pi}{2} - \frac{\pi}{4})} \tag{42}$$

at a large distance from the sphere.

Substituting (42) in (10) and (26), the velocity potentials take the form

$$\phi_S = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+2} A_n^m \left[-2\pi i \frac{(-1)^{m+n}}{(n-m)!} v^{n+1} e^{v(z-h)} \left(\frac{2}{\pi v R}\right)^{\frac{1}{2}} e^{i(vR - \frac{m\pi}{2} - \frac{\pi}{4})} \right] \cos m\psi$$

$$\phi_1 = \sum_{n=1}^{\infty} a^{n+2} B_n^1 \left[-2\pi i \frac{(-1)^{n+1}}{(n-1)!} v^{n+1} e^{v(z-h)} \left(\frac{2}{\pi v R}\right)^{\frac{1}{2}} e^{i(vR - \frac{3\pi}{4})} \right] \cos \psi$$

$$\phi_3 = \sum_{n=0}^{\infty} a^{n+2} B_n \left[-2\pi i \frac{(-1)^{n}}{n!} v^{n+1} e^{v(z-h)} \left(\frac{2}{\pi v R}\right)^{\frac{1}{2}} e^{i(vR - \frac{\pi}{4})} \right] \tag{43}$$

at a large distance from the sphere. Taking into account (43), and comparing with (38), the H-functions are given as

$$H_S(v,\psi) = -2\pi i e^{-vh} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(-1)^{m+n}}{(n-m)!} a^{n+2} A_n^m v^{n+1} e^{-i\left(\frac{m+1}{2}\right)} \cos m\psi$$

$$H_1(v,\psi) = 2\pi i e^{-vh} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n-m)!} a^{n+2} B_n^1 v^{n+1} \cos \psi$$

$$H_3(v,\psi) = -2\pi e^{-vh} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} a^{n+2} B_n v^{n+1}, \tag{44}$$

where the $H_1(v, \psi)$ and $H_3(v, \psi)$ are for unit motion of the sphere in x- and z-directions. The horizontal drift forces and moment are obtained using, according to Mousavizadegan [6]

$$\bar{F}_x = -\frac{\rho}{2\pi} \int_0^{2\pi} H(v,\theta) H^*(v,\theta) \cos\theta \, d\theta + \frac{\rho\omega A}{v} \Re\left[H(v,\beta)\right] \cos\beta$$

$$\bar{F}_y = -\frac{\rho}{2\pi} \int_0^{2\pi} H(v,\theta) H^*(v,\theta) \cos\theta \, d\theta + \frac{\rho\omega A}{v} \Re\left[H(v,\beta)\right] \sin\beta$$

$$\bar{M}_z = -\frac{\rho}{2\pi v} \Im\left[\int_0^{2\pi} H^*(v,\theta) H'(v,\theta) d\theta \right] - \frac{\rho g A}{\omega v} \Im\left[H'(v,\theta)\right]. \tag{45}$$

The horizontal force in y-direction and the yaw drift moment are zero due to the orthogonal property of cosine and sine functions and the direction of the incident wave propagation along x-direction. Replacing (44) in (45), the horizontal drift force acting on the sphere in the x-direction may be given by (as shown in Appendix G).

$$\bar{F}_x = -2\pi\rho e^{-vh} \left\{ \sum_{m=0}^{\infty} \left[e^{-vh} w_m \Im(C_m C_{m+1}^*) + \frac{\omega A}{v} \Re\left(e^{-i\frac{m\pi}{2}} C_m \right) \right] \right.$$

$$+e^{-vh}\Im[C_{R1}(w_0C_0^* - w_2C_2^*] + \frac{\omega A}{v}\Re\left(e^{-i\frac{\pi}{2}}C_{R1}\right)$$

$$-e^{-vh}w_0\Im(C_{R3}C_1^*) + \frac{\omega A}{v}\Re\left(C_{R3}\right)$$

$$+e^{-vh}w_0\Im(C_{R1}C_{R3}^*)\Big\}\,, \tag{46}$$

where

$$C_m = \sum_{n=m}^{\infty}\frac{(-1)^{m+n}}{(n-m)!}a^{n+2}A_n^m v^{n+1},$$

$$C_{R1} = \frac{\eta_1}{A}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-1)!}a^{n+2}B_n^1 v^{n+1},$$

$$C_{R3} = \frac{\eta_3}{A}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}a^{n+2}B_n v^{n+1}, \tag{47}$$

$w_0 = 2\pi$ and $w_m = \pi$ for $m > 0$. The first bracket in (46) under the summation sign is due to the components of the diffraction potential. The first part in this bracket results from the scattering potential, and the second part is due to cross effect of the incoming and scattering waves. The second line in (46) shows the effect of the surge motion potential together with the scattering and the incident wave potentials. The third line indicates the effect of the heave potential together with the components of diffraction potential. The last term is due to the cross eect of the surge and heave potentials.

7. RESULTS AND DISCUSSION

The calculations were performed on a sphere of radius a at different immersion depths. The immersion depth is the distance from the undisturbed free surface to the center of the sphere and is denoted by *h*. The ratio of *h*/a is set to be equal to 1.50, 1.75, 2.00 and 3.00. All computations are carried out using a PC with an Intel(R) 4 CPU 1.80 *Ghz* and total memory of 384 *MB*.

The results are obtained by calculating the infinite systems of linear algebraic equations (18) for A_n^m, which is related to the diffraction problem, and the infinite systems of linear algebraic equations (28) for B_n^m , which is related to the radiation problem for the surge and heave motions. The solution of the systems of equations are obtained with the FORTRAN routine ZGESV from the LAPACK libraries. ZGESV is a complex solver in which the LU decomposition with partial pivoting and row exchanging is applied to factor the coefficient matrix. Using this factored form of the coefficient matrix, the solution of the complex system of equations is computed.

It is necessary to compute the coefficients A_n^m for different values of *m* and *n*. The infinite systems of equations (18) are truncated to compute the coefficients A_n^m. It is considered that $m = 0, 1, 2, \ldots, M$. For each system of equations (m = constant), the value of *n* is set to be $n = m, m + 1, \ldots, N$. The systems of equations show a very good convergence property with increasing the value of *N*. A value of $N = 5$ is sufficient to find a set of stable solutions for the non-dimensional parameters associated with the diffraction problem up to Ka = 1.4. To find stable solutions for higher values of diffraction parameters, the value of *N* should be increased. A value of $N = 18$ gives a

Table 1
Exciting Force Coefficients of the Surge Motion

	real part				*imaginary part*			
va	*1.5*	*1.75*	*2.0*	*3.0 h/a*	*1.5*	*1.75*	*2.0*	*3.0*
0.0	1.528618	1.517783	1.511838	1.503481	0.000000	0.000000	0.000000	0.000000
0.1	1.541914	1.527286	1.519038	1.506641	0.001854	0.001730	0.001627	0.001310
0.2	1.558703	1.538561	1.526921	1.508624	0.011397	0.010037	0.008940	0.005847
0.3	1.573404	1.546571	1.531073	1.507365	0.029739	0.024683	0.020802	0.011053
0.4	1.582000	1.548675	1.529957	1.503744	0.054585	0.042690	0.034058	0.014756
0.5	1.582691	1.544568	1.524166	1.499289	0.082491	0.060878	0.046054	0.016347
0.6	1.575549	1.535354	1.515205	1.495113	0.110182	0.076929	0.055305	0.016155
0.7	1.561835	1.522671	1.504641	1.491756	0.135222	0.089607	0.061358	0.014798
0.8	1.543316	1.508106	1.493718	1.489351	0.156185	0.098556	0.064407	0.012850
0.9	1.521767	1.492929	1.483274	1.487812	0.172521	0.103977	0.064952	0.010727
1.0	1.498697	1.478035	1.473791	1.486965	0.184289	0.106353	0.063574	0.008690
1.2	1.452248	1.451096	1.458433	1.486628	0.195942	0.104249	0.057161	0.005337
1.4	1.409365	1.429211	1.447786	1.487173	0.195709	0.096403	0.048497	0.003077
1.6	1.371919	1.412335	1.440969	1.487937	0.187918	0.085820	0.039560	0.001697
1.8	1.340009	1.399704	1.436929	1.488627	0.175770	0.074393	0.031361	0.000906
2.0	1.313015	1.390400	1.434742	1.489148	0.161418	0.063223	0.024321	0.000472
2.5	1.261585	1.376398	1.433114	1.489756	0.123732	0.039560	0.012005	0.000086
3.0	1.223898	1.368408	1.432436	1.489713	0.090191	0.023231	0.005516	0.000014
3.5	1.191451	1.361017	1.430354	1.489319	0.063414	0.013034	0.002414	0.000002
4.0	1.158517	1.351582	1.426342	1.488711	0.043274	0.007064	0.001020	0.000000
4.5	1.121135	1.338920	1.420437	1.487942	0.028746	0.003727	0.000420	0.000000
5.0	1.076396	1.322506	1.412759	1.487037	0.018615	0.001925	0.000170	0.000000

set of stable solutions for $Ka \leq 10$. These solutions do not change with an increase in the value of N. The number of systems of equations is set to be $M = 11$ in computation of the horizontal drift force. This provides a set of stable solutions for the associated hydrodynamic coefficients with six decimal points for $0 \leq Ka \leq 10$.

The coefficients B_n and B_n^1 are obtained by solving the infinite systems of equations (28). The coefficient B_n is related to the motion of the sphere along vertical axis z. The coefficient B_n^1 pertains to the oscillatory motion of the sphere in x-direction. These infinite systems of equations should also be truncated to find B_n and B_n^1. The value of n is set to be $n = m, m + 1, \ldots, N$, where $m = 0$ for the heave motion and $m = 1$ for the surge motion. A set of stable solutions are obtained with six decimal points by assigning a value of $N = 18$ for both coefficients concerned with the heave and surge motions of the sphere.

The solutions for the non-dimensional spatial exciting forces are given in Tables 1

Table 2
Exciting force coefficients of the heave motion

	real part				*imaginary part*			
va	*1.5*	*1.75*	*2.0*	*3.0 h/a*	*1.5*	*1.75*	*2.0*	*3.0*
0.0	0.000000	0.000000	0.000000	0.000000	-1.558657	-1.536073	-1.523891	-1.506980
0.1	0.003921	0.003585	0.003335	0.002641	-1.585797	-1.555228	-1.538323	-1.513278
0.2	0.024553	0.021035	0.018447	0.011774	-1.620267	-1.577893	-1.553985	-1.517116
0.3	0.064907	0.051953	0.042895	0.022099	-1.649250	-1.592948	-1.561398	-1.514310
0.4	0.119293	0.089280	0.069531	0.029140	-1.662516	-1.594301	-1.557285	-1.506823
0.5	0.178042	0.125185	0.092342	0.031801	-1.655694	-1.581731	-1.543442	-1.497909
0.6	0.231915	0.154295	0.108337	0.030927	-1.630397	-1.558783	-1.523760	-1.489815
0.7	0.274980	0.174429	0.117116	0.027875	-1.591941	-1.530100	-1.501975	-1.483530
0.8	0.305148	0.185786	0.119703	0.023821	-1.546474	-1.499665	-1.480729	-1.479213
0.9	0.323100	0.189773	0.117607	0.019574	-1.499081	-1.470184	-1.461524	-1.476609
1.0	0.330888	0.188137	0.112287	0.015610	-1.453160	-1.443174	-1.445015	-1.475329
1.2	0.325284	0.174221	0.096510	0.009293	-1.372168	-1.398689	-1.420306	-1.475240
1.4	0.303852	0.153486	0.078744	0.005195	-1.307969	-1.366510	-1.405021	-1.476641
1.6	0.276016	0.131176	0.062087	0.002779	-1.258452	-1.344226	-1.396526	-1.478308
1.8	0.246671	0.109858	0.047766	0.001441	-1.220222	-1.329206	-1.392492	-1.479747
2.0	0.218168	0.090656	0.036065	0.000730	-1.190260	-1.319274	-1.391162	-1.480830
2.5	0.155992	0.053529	0.016823	0.000124	-1.137598	-1.306835	-1.392552	-1.482208
3.0	0.108344	0.030088	0.007385	0.000020	-1.100429	-1.300899	-1.394029	-1.482423
3.5	0.073386	0.016296	0.003112	0.000003	-1.067114	-1.294287	-1.392727	-1.482046
4.0	0.048494	0.008572	0.001274	0.000000	-1.031111	-1.284043	-1.388353	-1.481324
4.5	0.031246	0.004405	0.000511	0.000000	-0.988276	-1.269097	-1.381239	-1.480357
5.0	0.019610	0.002222	0.000202	0.000000	-0.935678	-1.249105	-1.371704	-1.479191

and 2 as a function of the diffraction parameter and immersion depth to radius ratio for the surge and heave motions, respectively. The solutions show a strong agreement with the results obtained by Wang [12]. The only substantial differences are found in the spatial exciting force for the heave motion for $Ka = 5$ and $h / a = 1.5$. The calculated value at this condition complies with the calculated value by *Wu* [14].

The solutions of the non-dimensional parameters of the radiation problem are presented in Tables 3 and 4 as a function of the diffraction parameter and immersion depth to radius ratio for the surge and heave motions, respectively. These are in agreement with those obtained by Wang [12]. The differences occur at the fourth decimal digit. Here, the solutions are given with six decimal digits, whereas Wang's solutions are tabulated with four decimal points.

The ratios of η_1/A, η_3/A and (η_5/KA) are depicted in Fig. 2 for various submergence depths and different positions of the center of mass. The position of the center

Table 3
Radiation Problem Coefficients of the Surge Motion

	added mass coefficient				*damping coefficient*			
va	*1.5*	*1.75*	*2.0*	*3.0* *h/a*	*1.5*	*1.75*	*2.0*	*3.0*
0.0	0.528618	0.517782	0.511838	0.503481	0.000000	0.000000	0.000000	0.000000
0.1	0.540286	0.526392	0.518499	0.506518	0.001844	0.001721	0.001620	0.001305
0.2	0.554455	0.536151	0.525421	0.508254	0.011171	0.009848	0.008777	0.005743
0.3	0.565245	0.541897	0.528160	0.506707	0.028468	0.023672	0.019967	0.010620
0.4	0.568805	0.541252	0.525462	0.502941	0.050581	0.039669	0.031689	0.013750
0.5	0.564026	0.534491	0.518388	0.498617	0.073361	0.054351	0.041192	0.014651
0.6	0.551976	0.523382	0.508871	0.494858	0.093268	0.065459	0.047173	0.013817
0.7	0.534910	0.510089	0.498739	0.492133	0.108098	0.072113	0.049532	0.011987
0.8	0.515357	0.496488	0.489301	0.490464	0.117038	0.074471	0.048854	0.009788
0.9	0.495510	0.483899	0.481295	0.489666	0.120338	0.073267	0.045980	0.007633
1.0	0.476964	0.473081	0.475009	0.489494	0.118875	0.069440	0.041738	0.005740
1.2	0.447156	0.457704	0.467391	0.490168	0.106177	0.057433	0.031725	0.002986
1.4	0.428376	0.449910	0.464868	0.491276	0.087240	0.043907	0.022298	0.001429
1.6	0.419013	0.447563	0.465403	0.492298	0.067687	0.031755	0.014810	0.000643
1.8	0.416313	0.448512	0.467412	0.493096	0.050383	0.022034	0.009419	0.000276
2.0	0.417733	0.451137	0.469902	0.493684	0.036342	0.014799	0.005787	0.000114
2.5	0.428335	0.459222	0.475462	0.494561	0.014542	0.004916	0.001526	0.000011
3.0	0.439265	0.465537	0.479024	0.495018	0.005255	0.001458	0.000356	0.000001
3.5	0.447218	0.469613	0.481147	0.495298	0.001760	0.000398	0.000076	0.000000
4.0	0.452541	0.472205	0.482479	0.495490	0.000553	0.000102	0.000015	0.000000
4.5	0.456117	0.473933	0.483383	0.495629	0.000165	0.000025	0.000003	0.000000
5.0	0.458615	0.475157	0.484038	0.495735	0.000046	0.000006	0.000001	0.000000

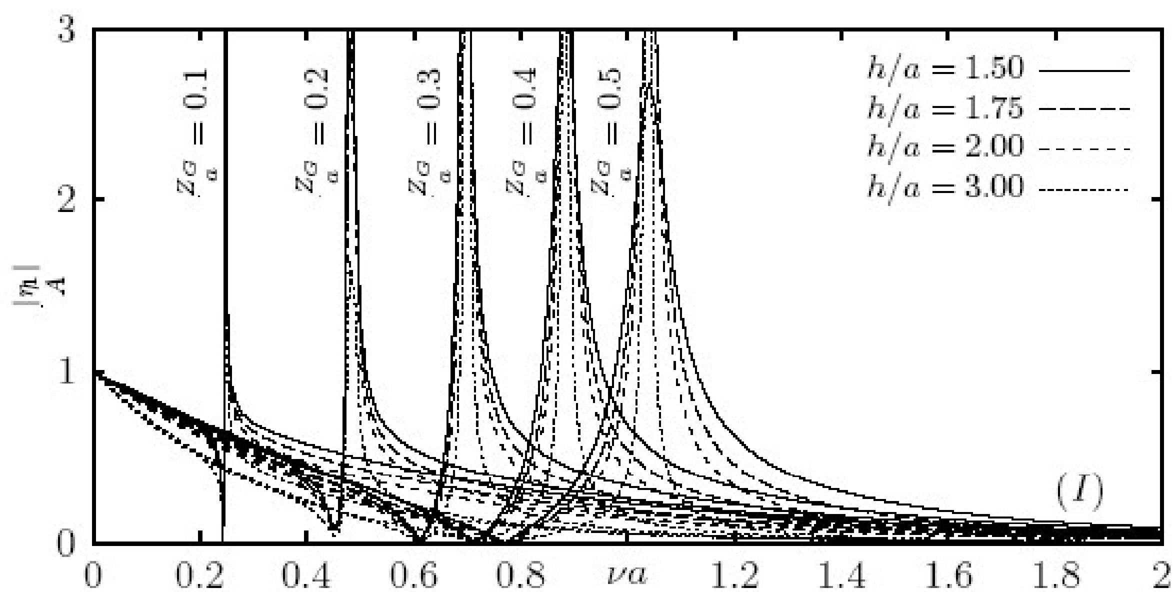
of mass is set to be $Z_G / a = 0.1, 0.2, 0.3, 0.4$ and 0.5. The location of the center of mass does not produce any effect on the heave RAO. However, the surge and pitch RAOs are affected significantly near the resonance frequency. A broader range of frequency spectrum is affected by the resonance frequency if the stability of the sphere is increased. The motion of the structure has no effect on the first order exciting force. But, the drift forces are affected by the motion of the structure.

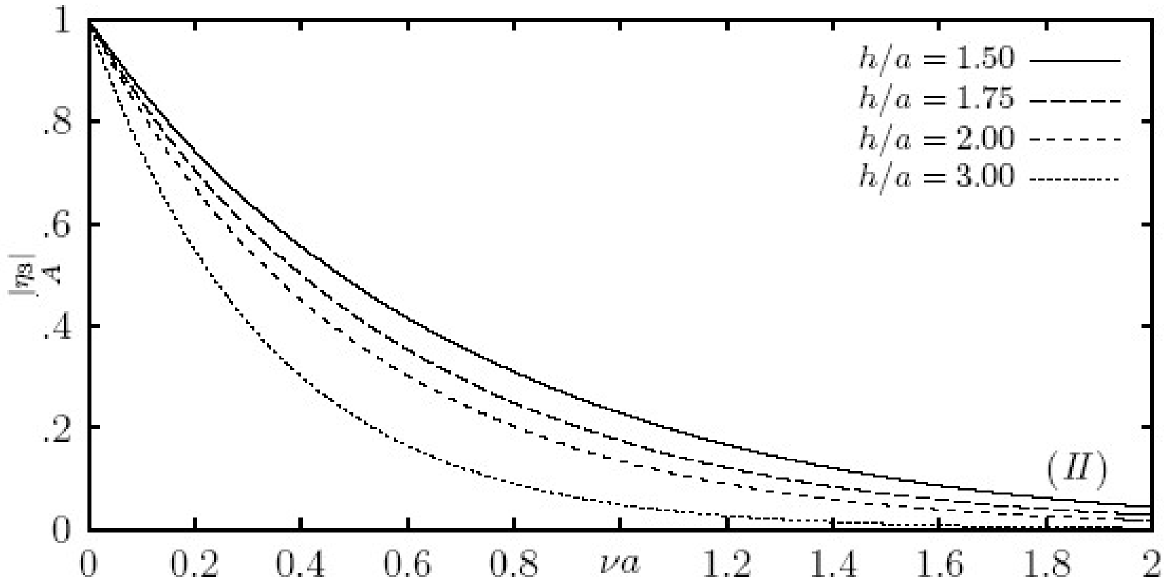
The horizontal drift force is computed using (46). In this equation, the first bracket under the summation sign is the contribution of the components of the diffraction velocity potential. The rest are the effect of the radiation velocity potentials. The drift force is made non-dimensional by dividing by $\frac{1}{2}\rho g A^2 a$. The computations are carried out where h/a = 1.25, 1.5, 1.75 and 2.0 for a set of diffraction parameters in the range $0 \leq a \leq 5$.

Table 5 shows the calculated values for the non-dimensional horizontal drift force acting on a sphere, due to the effect of the components of the diffraction potential and the radiation potentials, separately. The values of the non-dimensional

Table 4
Radiation Problem Coefficients of the Heave Motion

	added mass coefficient				*damping coecient*			
va	*1.5*	*1.75*	*2.0*	*3.0 h/a*	*1.5*	*1.75*	*2.0*	*3.0*
0.0	0.558657	0.536072	0.523891	0.506980	0.000000	0.000000	0.000000	0.000000
0.1	0.583271	0.553860	0.537504	0.513093	0.003902	0.003570	0.003322	0.002632
0.2	0.613616	0.574186	0.551699	0.516559	0.024146	0.020719	0.018183	0.011616
0.3	0.636414	0.585756	0.556968	0.513325	0.062633	0.050266	0.041555	0.021439
0.4	0.641880	0.582986	0.550525	0.505636	0.112162	0.084280	0.065761	0.027620
0.5	0.627096	0.566689	0.534925	0.496936	0.161987	0.114533	0.084706	0.029258
0.6	0.595567	0.541479	0.514702	0.489479	0.202800	0.135925	0.095770	0.027448
0.7	0.554194	0.512718	0.493904	0.484138	0.229592	0.147021	0.099149	0.023717
0.8	0.509984	0.484637	0.475171	0.480907	0.241710	0.148904	0.096464	0.019315
0.9	0.468144	0.459814	0.459785	0.479389	0.241302	0.143773	0.089689	0.015038
1.0	0.431680	0.439413	0.448073	0.479089	0.231609	0.133963	0.080582	0.011301
1.2	0.378610	0.412177	0.434520	0.480458	0.196653	0.107819	0.060361	0.005881
1.4	0.349632	0.399747	0.430576	0.482658	0.155392	0.080923	0.042084	0.002818
1.6	0.338033	0.397083	0.432139	0.484681	0.117188	0.057867	0.027854	0.001270
1.8	0.337335	0.399973	0.436261	0.486260	0.085498	0.039899	0.017703	0.000546
2.0	0.342731	0.405519	0.441157	0.487425	0.060816	0.026718	0.010885	0.000226
2.5	0.364921	0.421191	0.451897	0.489164	0.023864	0.008871	0.002882	0.000022
3.0	0.385243	0.433115	0.458759	0.490072	0.008533	0.002637	0.000675	0.000002
3.5	0.399646	0.440799	0.462860	0.490628	0.002829	0.000720	0.000145	0.000000
4.0	0.409252	0.445704	0.465441	0.491008	0.000878	0.000184	0.000029	0.000000
4.5	0.415733	0.448988	0.467195	0.491285	0.000256	0.000044	0.000006	0.000000
5.0	0.420290	0.451324	0.468471	0.491496	0.000070	0.000010	0.000001	0.000000





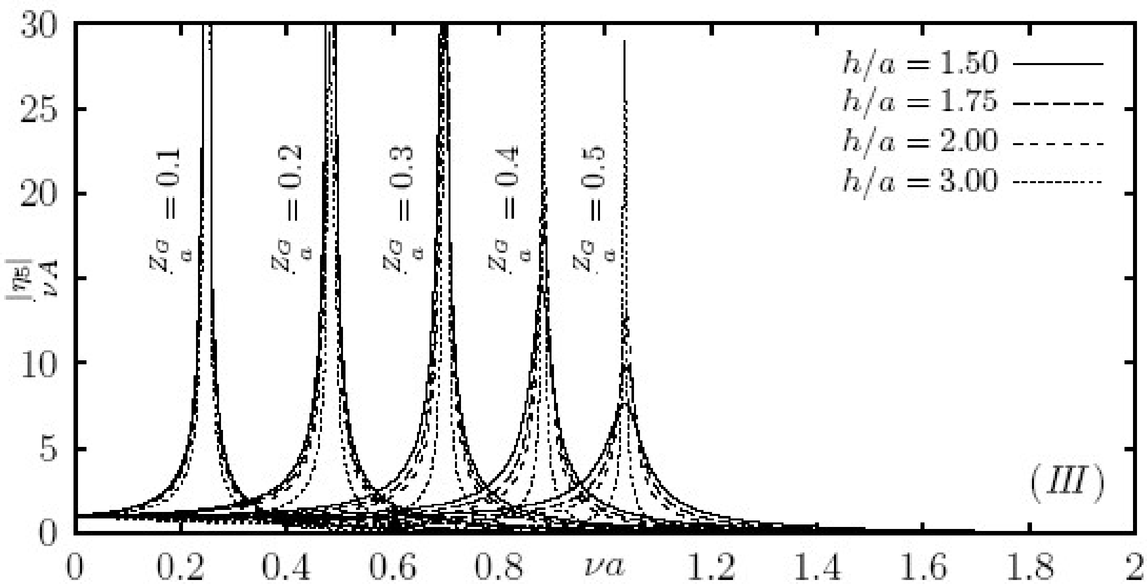


Figure 2: Amplitudes of the Sphere Motions at Various Positions of Center of Mass and Immersion Depths: *(I)* surge *(II)* heave *(III)* pitch

drift force at $h/a = 1.25$, due to the contribution of the diffraction velocity potential components, are almost the same as the results obtained by Wu. et. al. [15] at $d/a = 11$. The differences occur at the fourth decimal digit in all diffraction parameters. However, Wu et. al. [15] gave their results with four decimal points.

The contribution of the diffraction velocity potential components to the horizontal drift force is more pronounced than the contribution of the radiation velocity potentials. The effect of the radiation velocity potentials to the horizontal drift force arises from their cross product either with each other or with the components of the diffraction velocity potential. The contribution of the diffraction problem velocity potentials to the horizontal drift force is about four times the effect of the radiation potentials in $0.7 \le \nu a \le 2.5$ at $h/a = 1.25$ and $Z_G/a = 0.1$. The effect of the radiation velocity potentials increases for $\nu a < 0.7$, but is less than the effect of the diffraction potential components at $h/a = 1.25$ and $Z_G/a = 0.1$. The inuence of the radiation velocity potentials is impaired for $\nu a > 2.5$. The contribution of the radiation potentials decreases with increasing immersion depth.

The effect of each velocity potential on the horizontal drift force is depicted in Fig. 3 at h/a = 1.25 and Z_G/a = 0.1. The inuence of the motion of the sphere may be separated into the cross effect resulting from the surge motion with the scattering and incoming waves, the contribution of heave motion with the scattering and incident waves and the surge motion with the heave motion. As illustrated in Fig. 3, the effect of the heave motion is almost canceled out by the effect of the two other components. Therefore, the contribution of the motion of the sphere is much less than the effect of the components of the diffraction velocity potentials.

The horizontal drift force is also aected by the position of the center of mass, as illustrated in Fig. 4. The ratio of the center of the mass to the radius of the sphere is set to vary from 0.1 to 0.5. The sphere is more stable when Z_G/a is increased. As shown in Fig. 4, a more stable sphere experiences a higher drift force. The effect of the radiation velocity potentials near the resonance frequency is signicant. It may cause a rapid movement of the sphere near the resonance frequency. This is more pronounced when the stability of the sphere is increased. A wider range of wave spectrum is affected by the resonance frequency if Z_G/a is increased. It seems that the proper position of the center of mass for a spherical structure is less than 0.2a in respect of the horizontal drift force.

8. CONCLUSIONS

An analytical solution is obtained for the first-order problems and the horizontal drift force of a sphere in time-harmonic waves in fluid of infinite depth. The multipole expansion method is used to derive analytical solutions for the diffraction and radiation velocity potentials in a series of associated Legendre functions. The associated hydrodynamic coefficients are obtained, and the response amplitude operators of the surge, heave and pitch motions are computed, taking into account the effect of the position of

Table 5
The Non-Dimensional Horizontal Drift Force Due to the Diffraction and Radiation Velocity Potentials at Z_G/a = 0.1

	diffraction problem effect				*radiation problem effect*			
va	*1.25*	*1.5*	*1.75*	*2.0 h/a*	*1.25*	*1.5*	*1.75*	*2.0*
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.000074	0.000061	0.000053	0.000047	0.000055	0.000051	0.000046	0.000042
0.2	0.001576	0.001145	0.000885	0.000702	0.000703	0.000622	0.000522	0.000433
0.3	0.008127	0.005121	0.003512	0.002491	0.007158	0.005373	0.003972	0.002934
0.4	0.023255	0.012649	0.007699	0.004894	0.013770	0.009894	0.006713	0.004511
0.5	0.047610	0.022419	0.012160	0.006956	0.021574	0.014924	0.009263	0.005652
0.6	0.078044	0.032124	0.015633	0.008087	0.027849	0.018577	0.010522	0.005822
0.7	0.109122	0.039798	0.017503	0.008224	0.031778	0.020036	0.010311	0.005166
0.8	0.135768	0.044493	0.017799	0.007623	0.033977	0.019336	0.008986	0.004068
0.9	0.155044	0.046203	0.016896	0.006611	0.035434	0.017044	0.007091	0.002890
1.0	0.166323	0.045473	0.015253	0.005460	0.036821	0.013902	0.005112	0.001863
1.2	0.169133	0.039530	0.011216	0.003363	0.039943	0.007540	0.001986	0.000546
1.4	0.155876	0.031411	0.007551	0.001893	0.041861	0.003127	0.000361	0.000031
1.6	0.136245	0.023697	0.004817	0.001006	0.040098	0.000877	-0.000215	-0.000090
1.8	0.115664	0.017332	0.002968	0.000514	0.034549	0.000007	-0.000301	-0.000079
2.0	0.096621	0.012436	0.001786	0.000255	0.027197	-0.000210	-0.000230	-0.000048
2.5	0.059841	0.005183	0.000470	0.000041	0.011675	-0.000089	-0.000057	-0.000007
3.0	0.036705	0.002088	0.000117	0.000006	0.004178	-0.000007	-0.000009	-0.000001
3.5	0.022571	0.000824	0.000028	0.000001	0.001349	0.000004	-0.000001	0.000000
4.0	0.013933	0.000320	0.000006	0.000000	0.000396	0.000002	0.000000	0.000000
4.5	0.008624	0.000123	0.000001	0.000000	0.000102	0.000001	0.000000	0.000000
5.0	0.005346	0.000046	0.000000	0.000000	0.000021	0.000000	0.000000	0.000000

the center of mass. The second-order steady force is obtained by the far field method. The effect of all velocity potentials is taken into account in derivation of the horizontal drift force. The results of the computations are presented in tabular form to provide a very precise benchmark solution to validate the numerical schemes.

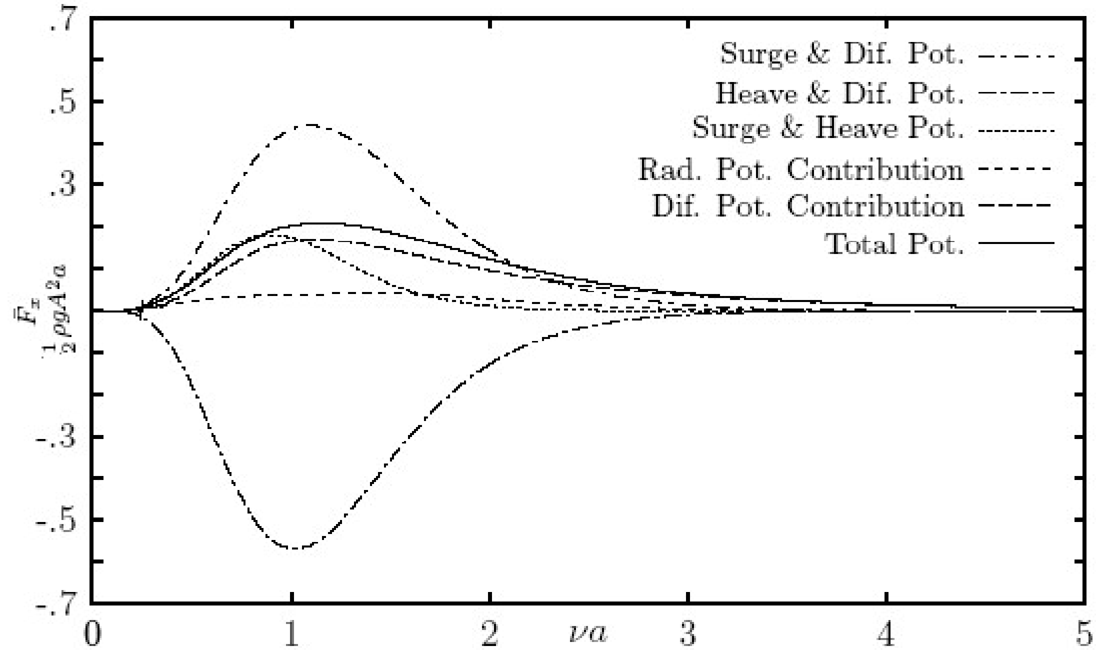


Figure 3: The Effect of Different Velocity Potentials on the Horizontal Drift Force at *h/a* = 1.25 and Z_G/a = 0.1

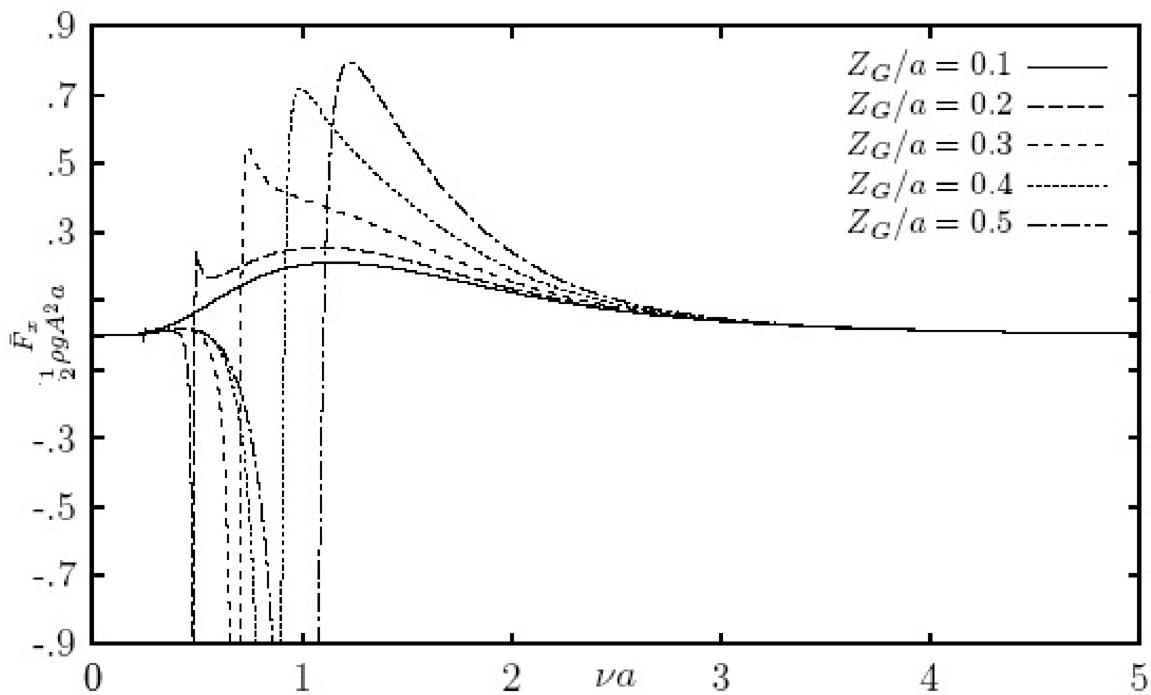


Figure 4: The Effect of the Location of the Center Mass on the Horizontal Drift Force of a Sphere at *h/a* = 1.25

The results of the analytical solutions for a submerged sphere indicate that:

- The horizontal drift force acting on the spherical structure is mostly due to the effect of the components of the diffraction velocity potential;
- The individual components of the radiation velocity potential alone have no effect on the horizontal drift force;
- The effect of components of the radiation velocity potential results from the cross-product either with each other or with the scattering and incoming velocity potentials;
- The effect of the heave velocity potential almost cancels out the contribution of the surge velocity potential;
- The total contribution of the radiation velocity potential is minimal to the horizontal drift force if the center of mass is at a distance less than twenty percent of the radius from the center of the sphere;
- The effect of the radiation velocity potential in vicinity of the resonant frequency is augmented and may create a relatively large steady force; and
- It is desirable to set the center of mass at a distance less than twenty percent of the radius from the center of the sphere.

This problem can be extended to a more general case and to study the effect of fluid depth on the drift force, taking into account the contribution of the radiation velocity potentials. It can be more generalized to derive the total drift force using the near field method, which is the integration of the dynamic pressure stemming from the fluid velocity around the body surface. This also provides the solution for the vertical drift force which may be useful in the study of the motion of spherical structures under the free surface of a fluid.

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APPENDICES

A Horizontal Drift Forces and Moment

The mean horizontal forces and moment about the vertical axis may be calculated by the far field method. A control volume is taken into account so that it is bounded by the body surface S_B, the free surface S_F , a bottom surface and a control surface at innity S_∞. If the zero flux of the fluid through S_B, S_F and the bottom surface, the constant atmospheric pressure at the free surface and the periodicity of the motions are taken into account, it can be written according to Newman [7]

$$\bar{F}_x = -\left\langle \iint_{S_\infty} \left[p\cos\theta + \rho V_R (V_R \cos\theta - V_\theta \sin\theta) \right] R \, d\theta \, dz \right\rangle$$

$$\bar{F}_y = -\left\langle \iint_{S_\infty} \left[p\sin\theta + \rho V_R (V_R \sin\theta - V_\theta \cos\theta) \right] R \, d\theta \, dz \right\rangle$$

$$\bar{M}_z = -\left\langle \iint_{S_\infty} V_R V_\theta R^2 \, d\theta \, dz \right\rangle, \tag{48}$$

where $\bar{F}_x$ and $\bar{F}_y$ are the mean horizontal forces on the body and $\bar{M}_z$ is the mean moment about the vertical axis. It is considered that the control surface S_∞ is a circular cylinder with large radius R about the z-axis.

The equations (48) result from the application of the conservation of the linear and angular momentum to this control volume. The pressure is obtained by using the Bernoulli equation

$$\frac{p}{\rho} = -\Re\left(\frac{\partial \Phi}{\partial t}\right) - gz - \frac{1}{2}|\mathbf{V}|^2, \tag{49}$$

where

$$|\mathbf{V}|^2 = V_R^2 + V_\theta^2 + V_z^2, \tag{50}$$

and components of the velocity vector $\mathbf{V}$ may be given by

$$V_R = \Re\left(\frac{\partial \Phi}{\partial R}\right), \quad V_\theta = \frac{1}{R}\Re\left(\frac{\partial \Phi}{\partial \theta}\right) \quad \text{and} \quad V_z = \Re\left(\frac{\partial \Phi}{\partial z}\right). \tag{51}$$

The total velocity potential is the sum of the first-order velocity potentials for a large R.

$$\Phi = \Phi_I + \Phi_S + \sum_{j=1}^{6} \Phi_j \tag{52}$$

The mean value of the integration of the pressure around the control surface S_∞ is

$$\left\langle \iint_{S_\infty} p\cos\theta\, R\, d\theta\, dz \right\rangle = \frac{\rho\omega^2}{4g}\int_0^{2\pi} |\phi|^2_{z=0}\, R\cos\theta\; d\theta$$

$$-\frac{\rho}{2}\left\langle \int_0^{2\pi}\left[\int_{-d}^{0} |\mathbf{V}|^2\, dz\right] R\cos\theta\, d\theta \right\rangle + O(\varepsilon^3). \tag{53}$$

Making use of (49), (50) and (53), the horizontal drift forces and moment may be rewritten as

$$\bar{F}_x = --\frac{\rho}{2}\left\langle \int_0^{2\pi}\left[\int_{-d}^{0} (V_R^2 - V_\theta^2 - V_z^2)dz\right] R\cos\theta d\theta \right\rangle$$

$$+\rho\left\langle \int_0^{2\pi}\left[\int_{-d}^{0} V_R V_\theta dz\right] R\sin\theta\, d\theta \right\rangle - \frac{\rho\omega^2}{4g}\int_0^{2\pi} |\phi|^2_{z=0} R\cos\theta d\theta$$

$$\bar{F}_y = --\frac{\rho}{2}\left\langle \int_0^{2\pi}\left[\int_{-d}^{0} (V_R^2 - V_\theta^2 - V_z^2)dz\right] R\sin\theta d\theta \right\rangle$$

$$+\rho\left\langle \int_0^{2\pi}\left[\int_{-d}^{0} V_R V_\theta dz\right] R\cos\theta d\theta \right\rangle - \frac{\rho\omega^2}{4g}\int_0^{2\pi} |\phi|^2_{z=0} R\sin\theta\, d\theta$$

$$\bar{M}_z = -\rho\left\langle \int_0^{2\pi}\left[\int_{-d}^{0} V_R V_\theta dz\right] R^2 d\theta \right\rangle. \tag{54}$$

The incident wave velocity potential for infinite fluid depth can be obtained by

$$\Phi_I(x,y,z,t) = \Re\left\{-\frac{igA}{\omega} e^{vz} e^{i(vx\cos\beta + vy\sin\beta - \omega t)}\right\}. \tag{55}$$

The time-independent velocity potentials, due to the scattering and radiating effects of the body at any point in the fluid domain, are calculated from the knowledge of each velocity potential at the body surface. It can be written by the application of Green's second identity that

$$\phi = \frac{1}{4\pi}\int_{S_B}\left(G\frac{\partial\phi_B}{\partial_n} - \phi_B\frac{\partial G}{\partial_n}\right) ds, \tag{56}$$

where G is the Green function. The asymptotic form of Green's function in the fluid of infinite depth can be written according to Wehausen and Laitone [13] as

$$G(x_p, y_p, z_p, y_q, z_q) = 2\pi v\left(\frac{2}{\pi v R_0}\right)^{1/2} e^{v(z_p + z_p) + i(vR_0 + \pi/4)} + O(R^{-3/2}). \tag{57}$$

Substituting (57) in (56), the velocity potential ϕ_{RS}, due to the scattering and radiating effect of the body at large distance, may be written as

$$\phi_{RS} = \frac{v}{2}\left(\frac{2}{gpvR}\right)^{1/2} e^{i(vR_0+\pi/4)} e^{vz_p}\int_{S_B}\left(\frac{\partial\phi_q}{\partial n_q} - \phi_q\frac{\partial}{\partial n_q}\right) e^{-iv(x_q\cos\theta + y_q\sin\theta)} e^{Kz_q} ds. \tag{58}$$

Making use of (52), the total velocity potential can be given in the form

$$\Phi \simeq \Re\left\{\left[-\frac{igA}{\omega} e^{vz_p} e^{ivR\cos(\theta-\beta)} + \left(\frac{2}{\pi v R}\right)^{1/2} e^{vz_p} e^{i(vR+\pi/4)} H(v,\theta)\right] e^{-i\omega t}\right\} \tag{59}$$

for infinite fluid depth. The term denoted by $H(v, \theta)$ is the Kochin H-function and is

$$H(v,\theta) = \frac{v}{2}\int_{S_B}\left(\frac{\partial \phi_q}{\partial n_q} - \phi_q \frac{\partial}{\partial n_q}\right) e^{-iv(x_q\cos\theta + y_q\sin\theta)} e^{vz_q} ds. \tag{60}$$

The components of the fluid velocity at a large distance R can be obtained by replacing (58) in (51).

B. Incident wave

Using Abramowitz and Stegun [1], it can be written that

$$e^{iz\cos\theta} = J_0(z) + 2\sum_{m=1}^{\infty} i^m J_m(z)\cos m\psi, \tag{61}$$

where $J_m(z)$ is the Bessel function of first kind and order m. The spherical coordinate system is related to the cartesian coordinate system by

$$z + h = -r\cos\theta,\ \ x = r\sin\theta\cos\psi,\ \ y = r\sin\theta\ \sin\psi. \tag{62}$$

Using (61) and (62), it can be written

$$e^{vz} e^{ivR\cos\psi} = e^{-vh} e^{-vr\cos\theta} \sum_{m=0}^{\infty} \varepsilon_m i^m J_m(vR)\cos m\psi, \tag{63}$$

where $R = r\sin\theta$ is the horizontal distance from the center of sphere. Using the identity (Linton [5])

$$e^{\pm vr\cos\theta} J_m(vr\sin\theta) = (\pm)^m \sum_{u=m}^{\infty} \frac{(\pm vr)^2}{(u+m)!} P_u^m(\cos\theta), \tag{64}$$

it can be written

$$e^{vz} e^{ivR\cos\theta} = e^{-vh} \sum_{m=0}^{\infty} \varepsilon_m i^m \cos m\psi \sum_{u=m}^{\infty} (-1)^{u+m} \frac{vr}{(u+m)!} P_u^m(\cos\theta). \tag{65}$$

C. Froude-Krylov Force

The Froude-Krylov force acting on the sphere may be written in the form

$$F_i = \Re\left\{\left[i\rho\omega a^2 \int_0^{2\pi}\left(\int_0^{\pi} \phi_I n_i \sin\theta d\theta\right) d\psi\right] e^{-i\omega t}\right\}, \tag{66}$$

where n_i is the inward normal to the sphere

$$n_1 = n_x = -P_1^1(\cos\theta)\cos\psi,\ \ n_3 = n_z = P_1(\cos\theta), \tag{67}$$

and ϕ_i is the time-independent incident wave velocity potential as given in (7). The surge and heave forces due to the incident wave can be obtained by replacing (67) and (7) in (66).

$$f_x = \rho g A e^{-vh} a^2 \sum_{m=0}^{\infty}\left\{\varepsilon_m i^m \int_0^{2\pi} \cos m\psi \cos\psi d\psi\right.$$

$$\left.\sum_{m=0}^{\infty}\left[(-1)^{m+u-1} \frac{(va)^u}{(u+m)!} \int_0^{\pi} P_u^m(\cos) P_1^1(\cos\theta)\sin\theta d\theta\right]\right\}$$

$$f_z = -\rho g A e^{-vh} a^2 \sum_{m=0}^{\infty} \left\{ \varepsilon_m i^m \int_0^{2\pi} \cos m\psi d\psi \right.$$

$$\left. \sum_{u=m}^{\infty} \left[(-1)^{m+u-1} \frac{(va)^u}{(u+m)!} \int_0^{\pi} P_u^m(\cos\theta) P_1(\cos\theta) \sin\theta d\theta \right] \right\}. \tag{68}$$

Taking into account the orthogonal properties of the cosine and Legendre functions (Abramowitz and Stegun [1]), it may be written that

$$\int_0^{2\pi} \cos u\psi \cos s\psi d\psi = \delta_{us}\pi$$

$$\int_0^{\pi} P_u^m(\cos\theta) P_u^m(\cos\theta) \sin\theta d\theta = \delta_{us} \frac{2}{2u+1} \frac{(u+m)!}{(u-m)!}, \tag{69}$$

where δ_{us} is the kronecker delta function. In view of (69), the Froude-Krylov force components may be simplified as given in (9).

D. Diraction velocity potential at the surface of the sphere

The diffraction velocity potential is the summation of the incident and scattering velocity potentials.

$$\phi_D = \phi_I + \phi_S$$

$$= \frac{Ag}{\omega} e^{-vh} \sum_{m=0}^{\infty} \varepsilon_m i^{m+1} \cos m\psi \sum_{u=m}^{\infty} (-1)^{u+m-1} \frac{(vr)^u}{(u+m)!} P_u^m(\cos\theta)$$

$$+ \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+2} \left[\frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{u=m}^{\infty} (C_{un}^m + iD_{un}^m) r^u P_u^m(\cos\theta) \right] \cos m\psi \tag{70}$$

It can be written at the surface of the sphere that

$$\phi_D |_{r=a} = \phi_I |_{r=a} + \phi_S |_{r=a}$$

$$= A\omega e^{-vh} \sum_{m=0}^{\infty} \varepsilon_m i^{m+1} a \cos m\psi \sum_{u=m}^{\infty} (-1)^{u+m-1} \frac{(va)^{u-1}}{(u+m)!} P_u^m(\cos\theta)$$

$$+ \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} A_n^m \left[P_n^m(\cos\theta) + \sum_{u=m}^{\infty} (C_{un}^m + iD_{un}^m) a^{u+n+1} P_u^m(\cos\theta) \right] a \cos m\psi. \tag{71}$$

Using the body surface boundary condition, it can be written that

$$\phi_D |_{r=a} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} A_n^m P_n^m(\cos\theta) a \cos m\psi +$$

$$A\omega e^{-vh} \sum_{m=0}^{\infty} \sum_{u=m}^{\infty} \varepsilon_m i^{m+1} a \cos m\psi (-1)^{u+m-1} \frac{(va)^{n-1}}{(u+m)!} P_u^m(\cos\theta) +$$

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} P_n^m(\cos\theta) \left[\frac{n+1}{n} A_n^m - \right.$$

$$A\omega e^{-vh}\varepsilon_m i^m (-1)^{u+m-1}\frac{(va)^{n-1}}{(u+m)!}\Bigg] a\cos m\psi. \tag{72}$$

The second and last terms in the bracket cancel out and the final solution is obtained as given in (21).

E. Radiation Velocity Potential

The radiation wave velocity potential may be written in the form

$$\phi_R^m = a\cos m\psi\Bigg[\sum_{n=m}^{\infty} B_n^m P_n^m(\cos\theta)\left(\frac{a}{r}\right)^{n+1}$$

$$+\sum_{u=m}^{\infty}\sum_{u=m}^{\infty}(C_{un}^m + iD_{un}^m)a^{u+n+1}\left(\frac{r}{a}\right)^u P_u^m(\cos\theta)\Bigg]$$

$$\text{for } m = 0,\ 1. \tag{73}$$

From (73), it can be written that

$$\sum_{u=1}^{\infty} B_u^1 (C_{un}^1 + iD_{un}^1)a^{n+u+1} = \frac{(u+1)B_u^1 + i\omega\delta_{u1}}{u} \text{and} \quad u = 1,2,3,\ldots$$

$$\sum_{u=0}^{\infty} B_u (C_{un} + iD_{un})a^{n+u+1} = \frac{(u+1)B_u + i\omega\delta_{u1}}{u} \text{and} \quad u = 0,1,2,\ldots \tag{74}$$

for the surge and heave motions, respectively. Replacing (74) in (73) and carrying out some simplifications, the radiation velocity potentials may be expressed in the form

$$\phi_1 = a\left[\sum_{n=1}^{\infty} B_n^1 P_n^1(\cos\theta)\left[\left(\frac{r}{a}\right)^{n+1} + \frac{n+1}{n}\left(\frac{r}{a}\right)^n\right] + i\omega\left(\frac{r}{\omega}\right)P_1^1(\cos\theta)\right]\cos\psi$$

$$\phi_3 = a\left[\sum_{n=0}^{\infty} B_n P_n(\cos\theta)\left[\left(\frac{r}{a}\right)^{n+1} + \frac{n+1}{n}\left(\frac{r}{a}\right)^n\right] - i\omega\left(\frac{r}{\omega}\right)P_1(\cos\theta)\right]. \tag{75}$$

The velocity potentials at the surface of the sphere are obtained by considering $r = a$, as given in (31).

F. Multipole Potential in an Alternative form

The multipole potential is rewritten here, as given in (11).

$$\hat{\phi}_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{m+n-1}}{(n-m)!} P.V.\int_0^\infty \frac{v+k}{v-k} k^n e^{-k(z+h)} J_m(kR)\, dk$$

$$-\frac{(-1)^{m+n}}{(n-m)!} 2\pi i v^{n+1} e^{-v(z+h)} J_m(vR) \tag{76}$$

The integral can be calculated with the application of the contour integral rules. A contour in the first half of the complex space is selected. It is deformed at the point

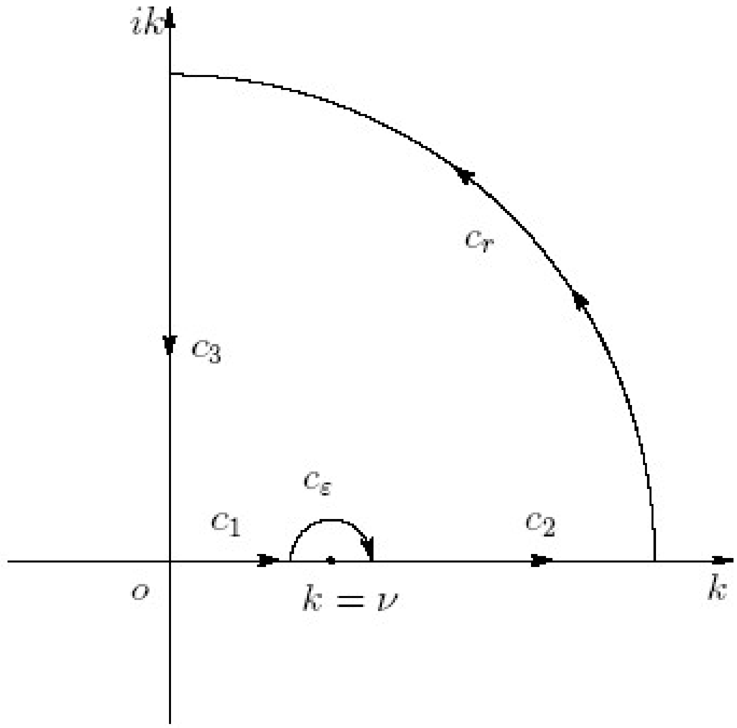

Figure 5: Sketch of the Contour of the Integration

$\nu = k$, as shown in Fig. 5. It may be written

$$I_R = P.V.\int_0^\infty \frac{\nu + k}{\nu - k} k^n e^{-k(z+h)} J_m(kR)\, dk$$

$$= \Re\left\{ P.V.\int_0^\infty \frac{\nu + k}{\nu - k} k^n e^{-k(z+h)} H_m^{(1)}(kR)\, dk \right\}, \tag{77}$$

where $H_m^1(kR) = J_m(kR) + iY_m(kR)$ is the Hankel function of the first kind and order m. If the radius of the contour tends to infinity, then

$$I = P.V.\int_0^\infty \frac{\nu + k}{\nu - k} k^n e^{-k(z+h)} H_m^1(kR)\, dk$$

$$= 2\pi i \nu^{n+1} e^{-\nu(z+h)} H_m^{(1)}(\nu R) + \int_{i\infty}^0 \frac{\nu + k}{\nu - k} k^n e^{-k(z+h)} H_m^{(1)}(kR)\, dk, \tag{78}$$

where the residue rule in contour integration is applied. Using the Bessel identity (Abramowitz and Stegun [1])

$$K_m(x) = \frac{1}{2}\pi i e^{\frac{1}{2}m\pi i} H_m^{(1)}(ix) \tag{79}$$

and changing the variable $k = ik$ in (78), it may be written

$$I = 2\pi i \nu^{n+1} e^{-\nu(z+h)} H_m^{(1)}(\nu R)$$

$$-\frac{2}{\pi}\int_0^\infty \frac{\nu + ik}{\nu - ik} (ik)^n e^{-ik(z+h)} e^{-im\frac{\pi}{2}} K_m(kR)\, dk, \tag{80}$$

where $K_m(kR)$ is the modified Bessel function of the second kind and order m. Now it can be written

$$I_R = -2\pi v^{n+1} e^{-v(z+h)} Y_m(vR)$$

$$-\frac{2}{\pi}\Re\left\{\int_0^\infty \frac{v+ik}{v-ik}(ik)^n e^{-ik(z+h)} e^{-im\frac{\pi}{2}} K_m(kR)\,dk,\right\}. \tag{81}$$

Therefore, the multipole potential may be expressed in the form

$$\hat{\phi}_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{m+n}}{(n-m)!}\Re\left\{\frac{2}{\pi}\int_0^\infty \frac{v+ik}{v-ik}(ik)^n e^{-ik(z+h)} e^{-im\frac{\pi}{2}} K_m(kR)\,dk\right\}$$

$$+\frac{(-1)^{m+n}}{(n-m)!} 2\pi v^{n+1} e^{-v(z+h)}[Y_m(vR) - iJ_m(vR)]. \tag{82}$$

This can be finally expressed as given in (40).

G. Horizontal drift force on the sphere

The horizontal drift force in x-direction acting on a body may be written

$$\overline{F}_x = -\frac{\rho}{2\pi}\int_0^{2\pi} H(v,\psi)H^*(v,\psi)\cos\psi d\psi + \frac{\rho\omega A}{v}\Re[H(v,\beta)]\cos\beta. \tag{83}$$

It is assumed that the angle of attack of the wave to the body is zero. The Kochin H-function is expressed in (39). The effect of each radiation potential by itself on the drift force is zero, due to the orthogonal property of the cosine function. The only effects are due to their cross product with each other or with the incident and scattering potentials. Using the general relation in complex calculus

$$f(z)g^*(z) + g(z)f^*(z) = 2\Re[(f(z)g^*(z)], \tag{84}$$

the horizontal drift force acting on the sphere in x-direction takes the following form.

$$\overline{F}_x = -\frac{\rho}{2\pi}\left\{\int_0^{2\pi} H_S(v,\psi)H_S^*(v,\psi)\cos\psi d\psi\right.$$

$$-\int_0^{2\pi} 2\Re[H_1(v,\psi)H_S^*(v,\psi)]\cos\psi d\psi$$

$$-\int_0^{2\pi} 2\Re[H_3(v,\psi)H_S^*(v,\psi)]\cos\psi d\psi$$

$$\left.-\int_0^{2\pi} 2\Re[H_1(v,\psi)H_3^*(v,\psi)]\cos\psi d\psi\right\}$$

$$+\frac{\rho\omega A}{v}\Re[H_S(v,\beta)H_1(v,\beta) + H_3(v,\beta)] \tag{85}$$

The integration can be carrying out by making use of the orthogonal property of cosine functions.

$$I_1 = \int_0^{2\pi} H_S(v,\psi)H_S^*(v,\psi)\cos\psi d\psi = 4\pi^2 e^{-2vh}\sum_{m=0}^{\infty} w_m \Im(C_m C_{m+1}^*)$$

$$I_2 = \int_0^{2\pi} 2\Re[H_1(v,\psi)H_S^*(v,\psi)]\cos\psi d\psi = 4\pi^2 e^{-2vh}\Im[C_{R1}(w_0 C_0^* - w_2 C_2^*)]$$

$$I_3 = \int_0^{2\pi} 2\Re[H_3(v,\psi)H_S^*(v,\psi)]\cos\psi d\psi = 4\pi^2 e^{-2vh} w_0 \Im(C_{R3}C_1^*)$$

$$I_4 = \int_0^{2\pi} 2\Re[H_1(v,\psi)H_3^*(v,\psi)]\cos\psi d\psi = 4\pi^2 e^{-2vh} w_0 \Im(C_{R1}C_{R3}^*) \tag{86}$$

The H-functions are expressed in the form

$$H_S(v,0) = -2\pi^{-vh}\sum_{m=0}^{\infty} e^{-i\frac{m\pi}{2}} C_m$$

$$H_1(v,0) = -2\pi e^{-vh} e^{-i\frac{\pi}{2}} C_{R1}$$

$$H_3(v,0) = -2\pi e^{-vh} C_{R3}, \tag{87}$$

where C_m, C_{R1} and C_{R3} are defined in (47). The horizontal drift force on the sphere can be found by

$$\overline{F}_x = -2\pi\rho e^{-vh}\left\{e^{-vh}\left[\sum_{m=0}^{\infty} w_m \Im(C_m C_{m+1}^*) + \Im[C_{R1}(w_0 C_0^* - w_2 C_2^*)]\right.\right.$$

$$\left. -w_0\Im(C_{R3}C_1^*) + w_0\Im(C_{R1} + C_{R3}^*)\right]$$

$$\left. + \frac{\omega A}{v}\Re\left[\sum_{m=0}^{\infty} e^{-i\frac{m\pi}{2}} C_m + e^{-i\frac{\pi}{2}} C_{R1} + C_{R3}\right]\right\}. \tag{88}$$

This may be rewritten an alternative form as given in (46).