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EVALUATION OF SINGULAR INTEGRALS OF THE CAUCHY TYPE

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ABSTRACT: Two different representations of the same sectionally analytic function of a single complex variable are utilized, along with the corresponding Plemelj-type formulae for the limiting values of the function, to derive an alternative formula to evaluate Cauchy Principal Value integrals involving a strong singularity in terms of integrals with only logarithmically singular integrands, helping numerical computations make simpler.

1. INTRODUCTION

The Cauchy type singular integrals of the type

$$I(x) = \int_{\alpha}^{\beta} \frac{\phi(t)dt}{(t-x)} \quad (\alpha < x < \beta) \tag{1.1}$$

occur in varieties of studies involving solutions of boundary value problems of many branches of physics and engineering (See [1], [2], [3], [4], [5], [6] etc.).

The major disadvantage that exists with the integrals of the type (1.1) is that standard numerical quadrature formulae can not be implemented to evaluate these integrals numerically and artificial methods of avoiding the singularity in the integrand become essential to draw any physical conclusion for problems whose solution involves such singular integrals.

A simple analysis is employed in this note to obtain an equivalent alternative formula for the integral (1.1), which is in terms of an integral involving a weaker logarithmic singularity, enabling numerical computations being performed more easily.

The detailed analysis, presented in the next section, involves the use of two different representations of the same sectionally analytic function ¦(z), which is analytic in the complex z-plane cut along the segment (α, β) of the real axis and the corresponding Plemelj-type formulae (see Gakhov [7]).

The equivalence of the alternative formula with the original form (1.1) is tested through simple examples.

2. THE DETAILED ANALYSIS

Let us start with two different representations of the same sectionally analytic function $\Phi(z)$, as given by the following two relations:

$$\Phi(z) \equiv \Phi^{(1)}(z) = \int_{\alpha}^{\beta} \frac{[\phi(t) - \phi(\beta)]}{(t-z)} dt \tag{2.1}$$

and

$$\Phi(z) \equiv \Phi^{(2)}(z) = \int_{\alpha}^{\beta} \psi(t) \log(t-z) dt - A \log(\alpha - z), \tag{2.2}$$

$$\text{with } A = \int_{\alpha}^{\beta} \psi(t) dt .$$

Then, by using the following limiting values (see Gakhov [7]):

$$\frac{1}{t-z} \to \pm\pi i\delta(t-x)+\frac{1}{t-x},\ z \to x \pm i0\,, \alpha < x < \beta \tag{2.3}$$

and

$$\begin{aligned}&\log(t-z) \to \log|x-t|, z \to x \pm i0, t > x > \alpha\\&\log(t-z) \to \log|x-t| \mp \pi i, z \to x \pm i0, t < x < \beta,\end{aligned} \tag{2.4}$$

where $\delta(x)$ denotes Dirac's delta function (see Jones [8]), we obtain the following Plemelj-type formulae:

$$\Phi^{(1)}(z) \to \Phi^{(1)}_{\pm}(x) = \pm\pi i[\phi(x)-\phi(\beta)] + \int_{\alpha}^{\beta}\frac{[\phi(t)-\phi(\beta)]}{(t-x)}dt, z \to x \pm i0 \tag{2.5}$$

and

$$\Phi^{(2)}(z) \to \Phi^{(2)}_{\pm}(x) = \mp\pi i\int_{\alpha}^{x}\psi(t)dt + \int_{\alpha}^{\beta}\psi(t)\log|x-t|dt - A\log(x-\alpha) \pm \pi i A, z \to x \pm i0. \tag{2.6}$$

Now, by using the relations (2.5) we find that the integral $I(x)$, as given by the relation (1.1), can be expressed in the following two equivalent forms: [since $\Phi^{(1)}(z) \equiv \Phi^{(2)}(z)$.]

(A)
$$2I(x) = \Phi^{(1)}_{+}(x) + \Phi^{(1)}_{-}(x) + 2\phi(\beta)\log(\frac{\beta-x}{x-\alpha}) \tag{2.7}$$

and

(B)
$$2I(x) = \Phi^{(2)}_{+}(x) + \Phi^{(2)}_{-}(x) + 2\phi(\beta)\log(\frac{\beta-x}{x-\alpha}) \tag{2.8}$$

Of the above two equivalent relations (2.7) and (2.8), the relation (2.7) is same as the relation (1.1), whereas the relation (2.8) gives the alternative representation, by using the results (2.6), as given by the formula:

$$I(x) = \int_{\alpha}^{\beta}\psi(t)\log|t-x|dt - A\log(x-\alpha) + \phi(\beta)\log(\frac{\beta-x}{x-\alpha}), \tag{2.9}$$

along with the following relation which also holds good:

$$\phi(x)-\phi(\beta) = -\int_{\alpha}^{x}\psi(t)dt + A = \Phi^{(1)}_{+}(x) - \Phi^{(1)}_{-}(x) \equiv \Phi^{(2)}_{+}(x) - \Phi^{(2)}_{-}(x). \tag{2.10}$$

It is clear that the new formula (2.9), for the evaluation of the integral $I(x)$ involves an integration with only a logarithmically singular integrand , as opposed to the original integral (1.1), possessing a strongly singular integrand.

It is interesting to observe that in the circumstances when $\phi(x)$ is a differentiable function , the relation (2.10) gives

$$\phi^{'}(x) = -\psi(x), A = \phi(\alpha) - \phi(\beta), \tag{2.11}$$

and the evaluation of $I(x)$ can be completed by using the formula (2.9), which turns out to be the same as the one obtainable by integrating the right side of the relation (1.1) by parts, as is to be expected.

A simple example of the utility of the above formulae is provided by the function

$$\phi(t) = [(t-\alpha)(\beta-t)]^{1/2}, \tag{2.12}$$

giving $\phi(\alpha) = 0 = \phi(\beta)$, and the following result is derived by utilizing either of the two formulae (1.1) (see Gakhov[7] for the evaluation of the Cauchy type singular integral) and (2.9) , along with the relations (2.11):

$$I(x) = \pi[\frac{(\alpha+\beta)}{2} - x]. \tag{2.13}$$

The values of the following integrals (see [9]) have been used in deriving the result (2.13), by means of the formula (2.9):

$$\begin{aligned}
\int_0^{\pi/2} \log\left|a\sin^2\theta - 1\right| \cos^2\theta d\theta &= \frac{\pi}{4}\log a + \frac{\pi}{2a} - \frac{\pi}{4}(1+\log 4), \\
\int_0^{\pi/2} \log\left|a\sin^2\theta - 1\right| \sin^2\theta d\theta &= \frac{\pi}{4}\log a - \frac{\pi}{2a} + \frac{\pi}{4}(1-\log 4),
\end{aligned} \tag{2.14}$$

where
$$a = \frac{(\beta-\alpha)}{(x-\alpha)} > 1.$$

Similarly, many other simple results can be derived by utilizing the above idea. We observe that for the special value of $\phi(x)$, as given by $\phi(x) = \frac{(x-\alpha)^{1/2}}{(\beta-x)^{1/2}}$, the idea explained above cannot be applied straightaway. But, we can rewrite the given integral (1.1), in this case, as

$$I(x) = \int_\alpha^\beta \frac{[(t-\alpha)(\beta-t)^{1/2}}{(\beta-t)(t-x)} dt = \frac{1}{(\beta-x)} \int_\alpha^\beta \{[\frac{t-\alpha}{\beta-t}]^{1/2} + \frac{[(t-\alpha)(\beta-t)]^{1/2}}{(t-x)}\} dt \tag{2.15}$$

and evaluate the second integral by utilizing the formula (2.9), while the evaluation of the first integral is standard.

The cases when $\phi(x) = [\frac{\beta-t}{t-\alpha}]^{1/2}$ and when $\phi(x) = [(\beta-t)(t-\alpha)]^{1/2}$ can also be dealt with similarly, after rewriting the corresponding integrals in suitable forms.

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REFERENCES

[1] Chakrabarti, A. "A Note on Singular Integral Equations", Int.J. Math. Educ. Sci. Technol., 26, pp. 737-742, (1995).

[2] Chakrabarti, A. "On the Solution of the Problem of Surface Water Waves by a Sharp Discontinuity in the Surface Boundary Conditions", ANZIAM J, 42, pp. 277-286, (2000).

[3] Chakrabarti, A. "On the Solution of the Problem of Scattering of Surface Water Waves by the Edge of an ice-cover", Proc. R. Soc., London, Series A (Mathematical,Physical and Engineering Sciences), 456, pp. 1087-1100, (2000).

[4] Chakrabarti, A. and Manam, S.R. “Solution of a Logarithmic Singular Integral Equation”, Appl. Math.Letters, 16, pp. 369-373, (2003).

[5] Chakrabarti, A. “Solution of Certain Weakly Singular Integral equations” , IMA Jl. of Appl.Maths, 71, pp.534-543, (2006).

[6] Evans, D.V. “Diffraction of Water Waves by a Submerged Vertical plate”, J.Fluid Mech., 40, pp. 433-451, (1970).

[7] Gakhov, F.D. *Boundary Value Problems*, Pergamon Press, Oxford, (1966).

[8] Jones, D.S. *The Theory of Generalized Functions*, Cambridge University Press, Cambridge, (1982).

[9] Muskhelishvili, N.I. Singular Integral Equations, Groningen, Noordhoff, (1977).

[10] Gradshteyn, I.S. and Ryzhik, I.M. *Table of Integrals, Series and Products*, Academic Press, New York, (2000).

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