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ON ANTIFUZZY NORMAL CLOSED b-IDEALS OF *BCI*-ALGEBRAS

ABSTRACT: *The antifuzzification of normal b-ideal is considered and some related properties are investigated. A characterization of an antifuzzy normal closed b-ideal is given.*

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1. INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras : BCKalgebras and BCI-algebras ([4, 5]). It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra. Jun [6] defined a doubt fuzzy subalgebra, doubt fuzzy ideal and doubt fuzzy prime ideal in BCI-algebra. In this note, we define an antifuzzy normal closed b-ideal and normal b-ideal and consider the antifuzzification of (normal) b-ideals in BCI-algebras and investigate some related properties. We give a characterization of an antifuzzy normal closed b-ideal.

2. PRELIMINARIES

We recall that a fuzzy subset μ of a set X is a function μ from X into [0, 1]. Let Im(μ) denote the image set of μ. We will write $a \wedge b$ for min$\{a, b\}$, and $a \vee b$ for max $\{a, b\}$, where a and b are any real numbers. Given a fuzzy set μ and $t \in [0, 1]$ we define

$L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$, which is called lower level cut of μ. Let $(X, *, 0)$ be an algebraic structure. We call it a BCI-algebra when it satisfies the conditions for any $x, y, z \in$ X,

(1) $(x * y) * (x * z) \leq z * y$,

(2) $x * (x * y) \leq y$,

(3) $x * x = 0$,

(4) if $x * y = y * x = 0$ then $x = y$.

The relation "≤" is defined as follows: $x \leq y \Leftrightarrow x * y = 0$. It is easy to show that $(X, \leq)$ is a partially ordered set and the following proposition holds.

(a) $x * 0 = x$,

(b) $0 * (x * y) = (0 * x)\ \ (0 * y)$,

(c) $(x * y) * \ z = (x * z) * y$,

(d) $x \leq y$ implies $x * z \leq y * z$ and $z * x \leq z * y$.

Definition 2.1: Any subset I of X is called closed b-ideal if

(i) $0 * x \in I$, for all $x \in X$

(ii) $x, y \in I$ implies $x * (0 * y) \in I$.

Definition 2.2: A non-empty subset N of BCI-algebra X is said to be normal if $(x\ * a) * (y * b) \in N$ whenever $x * y \in N$ and $a * b \in N$.

Definition 2.3: A fuzzy set μ in X is called antifuzzy subalgebra of X if it satisfies the inequality: $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Example 2.4: Let $X = \{0, a, b, c, d, e\}$ be a set with the following table

*	0	*a*	*b*	*c*	*d*	*e*
0	0	*b*	*a*	*c*	*d*	*e*
a	*a*	0	*b*	*d*	*e*	*c*
b	*b*	*a*	0	*e*	*c*	*d*
c	*c*	*d*	*e*	0	*b*	*a*
d	*d*	*e*	*c*	*a*	0	*b*
e	*e*	*c*	*d*	*b*	*a*	0

Then $(X, *, 0)$ is a BCI-algebra. Define $\mu : X \to [0, 1]$ by

$\mu(0) = \mu(c) = 0.1 < 0.5 = \mu(x)$ for all $x \in X \setminus 0, c\}$. Then μ is an antifuzzy subalgebra of X.

3. MAIN RESULTS

Proposition 3.1: Every antifuzzy subalgebra μ satisfies the inequality

$$\mu(0) \leq \mu(x), \text{ for all } x \leq \boldsymbol{X}.$$

Proof: Since $x * x = 0$ for all $x \in X$, we have

$$\mu(0) = \mu(x * x) \leq \max \mu(x), \mu(x)\} = \mu(x).$$

For any elements x, y of X, let us write $\prod^{n} x * y$ for $x * (\cdots * (x * (x * y)))$, where x occurs n times.

Proposition 3.2: Let a fuzzy set μ in X be an antifuzzy subalgebra and let $n \in \mathbb{N}$, then

(i) $\mu(\prod^{n} x * x) \leq \mu(x)$, whenever n is odd.

(ii) $\mu(\prod^{n} x * x) = \mu(x)$, whenever n is even.

for all $x, y \in X$.

Proof: Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k. Observe that $\mu(x * x) = \mu(0) \leq \mu(x)$. Suppose that $\mu(\prod^{2k-1} x * x) \leq \mu(x)$ for positive integer k. Then

$$\mu\left(\prod^{2(k+1)-1} x * x\right) = \mu\left(\prod^{2k+1} x * x\right) = \mu\left(\prod^{2k-1} x * x(x * x))\right) = \mu\left(\prod^{2k-1} x * x\right) \leq \mu(x)$$

which proves (i). Similarly we obtain the second part.

Definition 3.3: A fuzzy set μ in X is called antifuzzy closed b-ideal of X if,

(i) $\mu(0 * x) \leq \mu(x)$,

(ii) $\mu(x * (0 * y)) \leq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in X$.

Definition 3.4: A fuzzy set μ in X is said to be antifuzzy normal if it satisfies the inequality

$$\mu((x * a) * (y * b)) \leq \max\{\mu(x * y), \mu(a * b)\}.$$

Example 3.5: If we define a fuzzy set $\mu : X \to [0, 1]$ by

$\mu(0) = \mu(a) = \mu(b) = 0.1$ and $\mu(c) = \mu(d) = \mu(e) = 0.4$ in Example 2.4, then μ is an antifuzzy normal set in X.

Example 3.6: Let $X = \{0, a, b, c\}$ be a set with the following table.

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then $(X, *, 0)$ is a BCI-algebra. If we define $\mu : X \to [0, 1]$ by

$\mu(0) < \mu(b) < \mu(a) = \mu(c)$, then μ is an antifuzzy normal set in X. Moreover, if we define $\vee : X \to [0, 1]$ by $\nu(0) = \nu(b) < \nu(a) = \nu(c)$, then ν is also an antifuzzy normal set in X.

Theorem 3.7: *Every antifuzzy normal set μ in X is an antifuzzy subalgebra of X.*

Proof: $\mu(x * y) = \mu((x * x) * (0 * 0)) \leq \max\{\mu(x * 0), \mu(y * 0)\} = \max\{\mu(x), \mu(y)\}$, then $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$. Hence μ is an antifuzzy subalgebra of X.

Remark: The converse of Theorem 3.7 is not true. For example, an antifuzzy subalgebra μ in Example 2.4 is not an antifuzzy normal since

$$\mu((b * e)\ \ (d * a)) = \mu(b) > \mu(c) = \max\{\mu(b * d), \mu(e * a)\}.$$

Definition 3.8: A fuzzy set μ in X is called an antifuzzy normal closed b-ideal if it is an antifuzzy closed b-ideal which is antifuzzy normal.

Example 3.9: The fuzzy set discussed in Example 3.5 and 3.6 are indeed an antifuzzy normal closed b-ideal.

Proposition 3.10: If a fuzzy set μ in X is an antifuzzy normal closed b-ideal then $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$.

Proof: Let $x, y \in X$. Then

$\mu(x * y) = \mu((x * y) * (x * x)) \leq \max\{\mu(x * x), \mu(y * x)\} = \mu(y * x)$.
Interchanging x with y, we obtain $\mu(y * x) \leq \mu(x * y)$, which proves the Proposition.

Theorem 3.11: Let μ be an antifuzzy normal closed b-ideal. Then the set $X_\mu = \{x \in \mid \mu(x) = \mu(0\}$ is a normal closed b-ideal of X.

Proof: It is sufficient to show that X_μ is antifuzzy normal. let $a, b, x, y \in X$ be such that $x * y \in X_\mu$ and $a * b \in X_\mu$. Then $\mu(x * y) = \mu(0) = \mu(a * b)$. Since μ is antifuzzy normal it follows that

$\mu((x * a)(y * b)) \leq \max\{\mu(x * y), \mu(a * b)\} = \mu(0)$. Applying Proposition 3.10, we conclude that $\mu((x * a)(y * b)) = \mu(0)$, which shows that $(x * a)(y * b) \in X_\mu$. This completes the proof.

Theorem 3.12: The intersection of any antifuzzy normal closed b-ideal is also an antifuzzy normal closed b-ideal.

Proof: Let $\{\mu_\alpha \mid \alpha \in \Omega\}$ be a family of antifuzzy normal closed b-ideals and let $x, y, a, b \in X$. Then

$$(\bigcap_{\alpha\in\Omega} \mu_\alpha)(x * a) * (y * b)) = \inf_{\alpha\in\Omega} \mu_\alpha((x * a) * (y * b))$$

$$\leq \inf_{\alpha\in\Omega} \max\{\mu_\alpha(x * y), \mu_\alpha(a * b)\}$$

$$\max\{\inf_{\alpha\in\Omega} \mu_\alpha(x * y), \inf_{\alpha\in\Omega} \mu_\alpha(a * b)\}$$

$$\max\{(\bigcap_{\alpha\in\Omega} \mu_\alpha)(x * y), (\bigcap_{\alpha\in\Omega} \mu_\alpha)(a * b)\}$$

which shows that $\bigcap_{\alpha\in\Omega} \mu_\alpha$ is an antifuzzy normal set in X, using Theorem 3.7, we conclude that $\bigcap_{\alpha\in\Omega} \mu_\alpha$ is an antifuzzy normal closed b-ideal.

The union of any set of antifuzzy normal closed b-ideals need not be antifuzzy normal b-ideal. For example if we define a fuzzy set

$\delta : X \to [0, 1]$ by $\delta(0) = \delta(d) = 0, 1 < 0.3 = \delta(a) = \delta(b) = \delta(c) = \delta(e)$ in Example 2.4, then it is also an antifuzzy closed b-ideal. Since $(\mu \cup \delta)(c * d) = 0.3$ and $\max\{(\mu \cup \delta)(c), (\mu \cup \delta)(d)\} = 0.1$, $\mu \cup \delta$ is not an antifuzzy closed b-ideals.

4. ANTIFUZZY NORMAL CLOSED *b*-IDEALS

Theorem 4.1: Let A be a non-empty subset of X and let μ_A be a fuzzy set

$$\mu_A(x) = \begin{cases} \alpha & \text{if } x \in A, \\ \beta & \text{otherwise,} \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Then μ_A is an antifuzzy normal closed b-ideal if and only if A is a normal subalgebra of X. Moreover, in this case $X\mu_A = A$.

Proof: Assume that μ_A is an antifuzzy normal closed b-ideal. Let $a, b, x, y \in X$ be such that $x * y \in A$ and $a * b \in A$. Then

$$\mu_A((x * a)\ \ (y * b)) \leq \max\{\mu_A(x * y), \mu(a * b)\} = \alpha$$

and so $\mu_A((x * a) * (y * b)) = \alpha$, which shows that $(x * a)\ \ (y * b) \in A$. Hence A is a normal subalgebra of X.

Conversely, suppose that A is a normal subalgerba of X and let

$a, b, x, y\ \ X$. If $x * y \in A$ and $a * b \in A$, then $(x * a)\ \ (y * b) \in A$ and so $\mu_A((x * a)\ \ (y * b))\ \alpha = \ = \max\{\mu_A(x * y), \mu_A(a * b)\}$. If $x * y \notin A$ and $a * b \notin A$, then clearly

$$\mu_A((x * a) * (y * b)) \leq \beta = \max\{\mu_A(x * y), \mu_A(a * b)\}.$$

This shows that μ_A is an antifuzzy normal set. It follows from Theorem 3.7 that μ_A is an antifuzzy normal closed b-ideal. Moreover, using Theorem 3.11, we have

$$X_{\mu A} = \{x \in X | \mu_A(x) = \mu_A(0)\} = \{x \in X \mid \mu_A(x) = \alpha\} = A.$$

This completes the proof.

Theorem 4.2: Let μ be a fuzzy set in X. Then μ is an antifuzzy normal closed b-ideal if and only if the set $L(\mu, \alpha) = \{x \in X | \mu(x)\ \alpha\}$, is a normal subalgebra of X for all $\alpha \in [0, 1]$, where $L(\mu, \alpha) \neq \phi$.

Proof: Let μ be an antifuzzy normal closed b-ideal and assume that

$L(\mu, \alpha) \neq \phi$ for all $\alpha \in [0, 1]$. Let $a, b, x, y \in X$ be such that $x * y \in L(\mu,\alpha)$ and $a * b \in L(\mu, \alpha)$. Then

$$\mu((x * a)\ \ (y * b)) \leq \max\{\mu(x * y), \mu(a * b)\} \leq \alpha$$

and thus $(x * a)\ \ (y * b) \in L(\mu, \alpha)$. Hence $L(\mu, \alpha)$ is a normal subalgebra of X.

Conversely, suppose that $L(\mu, \alpha) \neq \phi$ is a normal subalgebra of X for every $\alpha \in [0, 1]$. Using Theorem 3.7, it is sufficient to show that μ is an antifuzzy normal set in X. If there are $a_0, b_0, x_0, y_0 \in X$ such that

$$\mu((x_0 * a_0)\ \ (y_0 * b_0)) > \max\{\mu(x_0 * y_0), \mu(a_0 * b_0)\}.$$

Then by taking

$$\alpha_0 = \frac{1}{2}[\mu((x_0 * a_0)\ \ (y_0 * b_0)) + \max\{\mu(x_0 * y_0), \mu(a_0 * b_0)\}]$$

we have

$$\mu((x_0 * a_0)\ \ (y_0 * b_0)) > \alpha\ 0 > \max\{\mu(x_0 * y_0), \mu(a_0 * b_0)\}.$$

It follows $x_0 * y_0 \in L(\mu, \alpha_0)$ and $a_0 * b_0 \in L(\mu, \alpha_0)$, but $(x_0 * a_0)(y_0 * b_0) \notin L(\mu, \alpha_0)$, a contradiction. Hence μ is an antifuzzy normal, which proves the theorem.

Theorem 4.3: Let μ be an antifuzzy normal closed b-ideal with $\mathrm{Im}(\mu) = \{\alpha_i \mid i \in \Omega\}$ where Ω is an arbitrary index set. Then

(i) there exists a unique $t_0 \in \Omega$ such that $\alpha_i \leq \alpha_{i_0}$ for all $i \in \Omega$,

(ii) $X_\mu = \bigcap_{i\in\Omega} L\mu, \alpha_i) = L(\mu, \alpha_{i_0})$,

(iii) $X = \bigcap_{i\in\Omega} L(\mu, \alpha_i)$.

Proof: (i) Since $\mu(0) \in \mathrm{Im}(\mu)$, there exists a unique $t_0 \in w$ such that $\mu(0) = \alpha_{i_0}$. It follows from Proposition 3.1, that $\mu(0) = \alpha_{i_0} \leq \mu(x)$, for all $x \in X$ so that $\alpha_{i_0} < \alpha_i$ for all $i \in \Omega$.

(ii) We have

$$\begin{aligned} L(\mu, \alpha_{i_0}) &= \{x \in X \mid \mu(x) \leq \alpha_{i_0}\} \\ &= x \in X \mid \mu(x) = \alpha_{i_0}\} \\ &= \{x \in X \mid \mu(x) = \mu(0)\} = X_\mu. \end{aligned}$$

Since $\alpha_i \leq \alpha_{i_0}$ for all $i \in \Omega$. It follows $L(\mu, \alpha_{i_0}) = L(\mu, \alpha_i)$, for all $i \in \Omega$. Hence $L(\mu, \alpha_{i_0}) \subseteq \bigcap_{i\in\Omega} L(\mu, \alpha_i)$ and so $L(\mu, \alpha_{i_0}) = \bigcap_{i\in\Omega} L(\mu, \alpha_i)$ because $L_0 \in \Omega$.

(iii) Clearly $\bigcap_{i\in\Omega} L(\mu, \alpha_i) \subseteq X$. For every $x \in X$ there exists $i(x) \in \Omega$ such that $\mu(x) = \alpha_i(x)$. This implies $x \in L(\mu, \alpha_i(x)) \subseteq \bigcap_{i\in\Omega} L(\mu, \alpha_i)$, which proves (iii).

Theorem 4.4: Let μ be an antifuzzy normal closed *b*-ideal and $A = \{L(\mu, \alpha_i) | i \in \Omega\}$ where Ω is an arbitrary index set. Then *A* contains all lower level cuts of μ if and only if μ attains its supremum on all normal subalgebra of *X*.

Proof: Suppose *A* contains all lower level cuts of μ and let *A* be a normal *b*-ideal of *X*. If μ is constant on *A*, then we are done. Assume μ is not constant on *A*. We distinguish the following two cases : (1) $A = X$ and (2) $A \subset X$. For the case (1), we let $\beta = \sup\{\alpha i \mid i\ \alpha \in \Omega\}$. Then $\alpha_i \leq \beta$ and so $L(\mu, \alpha_i) \subseteq L(\mu, \beta)$ for $i \in \Omega$. Note that $X = L(\mu, 1) \in A$ because *A* contains all lower level cuts of μ. Hence, there exists $j \in \Omega$ such that $\alpha j \in \text{Im}(\mu)$ and $L(\mu, \alpha_j) = X$. It follows that $X = L(\mu, \alpha_j) \subseteq L(\mu,\beta)$ so that $L(\mu,\beta) = \text{L}(\mu, \alpha_j) = X$ because every lower level cut of μ is normal *b*-ideal of *X*. Now it is sufficient to show that $= \beta = \alpha_j$. If $\alpha_j < \beta$, then there exist $k \in \Omega$ such that $\alpha\, k \in \text{Im}(\mu)$ and $\alpha_j < \alpha_k \leq \beta$. This implies that $X = L(\mu, \alpha_j) \subseteq L(\mu, \alpha_k)$, a contradiction. Therefore, $\beta = \alpha_{\text{j}}$. If the case (2) holds, consider the restriction μ*A* of μ to *A*. By Theorem 4.1, μ_A is an antifuzzy normal closed *b*-ideal.

Let $\Omega_A = \{i \in \Omega \mid \mu(y) = \alpha_i$, for some $y \in A\}$ and

$A_A = \{L(\mu_A, \alpha_i) \mid i \in \Omega_A\}$. Noticing that A_A contains all lower level cuts of μ_A, we conclude that there exists $z \in A$ such that $\mu(z) = \sup\{\mu_A(x) \mid x \in A\}$, which implies that $\mu(z) = \sup\{\mu(x) \mid x \in A\}$.

Conversely, assume that μ attains its supremum on all normal *b*-ideal of *X*. Let $L(\mu, \alpha)$ be a lower level cut of μ. If $\alpha = \alpha_i$ for some $i \in \Omega$, then clearly $L(\mu, \alpha) \in A$. Assume that $\alpha \neq \alpha_i$ for all $i \in \Omega$. Then there does not exist $x \in X$ such that $\mu(x) = \alpha$. Let $A = \{x \in X \mid \mu(x) < \alpha\}$. Let $a, b, x, y \in X$ be such that $x * y \in A$ and $a * b \in A$. Then $\mu(x * y) < \alpha$ and $\mu(a * b) < \alpha$. It follows that $\mu((x * a)\ \ (y * b)) \leq \mu\{\mu(x * y)$, $\mu(a * b)\} < \alpha$. So that $(x * a) * (y * b) \in A$. This show that *A* is a normal *b*-ideal of *X*. By hypothesis, there exists $y \in A$ such that $\mu(y) = \sup\{\mu(x) | x \in A\}$. Now $\mu(y) \in \text{Im}(\mu)$ implies $\mu(y) = \alpha_i$ for some $i \in w$. Hence we get $\sup\{\mu(x) \mid x \in A\} = \alpha_i$. Obviously, $\alpha_i \leq \alpha$, and so $\alpha_i < \alpha$ by assumption. Note that there does not exist $z \in X$ such that $\alpha_i < \mu(z) \leq \alpha$. It follows that $L(\mu, \alpha) = L(\mu, \alpha_i) \in A$. This concludes the proof.

Theorem 4.5: Let μ be a fuzzy set in *X* with a finite image $\text{Im}(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$ where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{N_n \mid n = 0, 1, \dots, k\}$ be a family of normal *b*-ideals of *X* such that

(i) $N_0 \subset N_1 \subset \dots \subset N_k = N$,

(ii) $\mu(\tilde{N}_n) = \alpha\, n$ where $\tilde{N}_n = N_n \setminus N_{n-1}$ and $N_{n-1} = \phi$ for $n = 0, 1, \dots k$.

Then μ is an antifuzzy normal closed *b*-ideal.

Proof: According to Theorem 3.7, it is sufficient to show that μ is an antifuzzy normal set in X. Let $a, b, x, y \in X$. If $x * y \in \tilde{N}_n$ and $a * b \in \tilde{N}_n$ for every n, then $(x * a)$ $(y * b) \in N$. Since N_n is a normal b-ideals of X. Hence

$$\mu((x * a) * (y * b)) \leq \alpha_n = \max\{\mu(x * y), \mu(a * b)\}.$$

If $x * y \in \tilde{N}_n$ and $a * b \in \tilde{N}_m$ where $0 \leq m < n \leq k$, then $x * y \in N_n$ and $a * b \in N_m \subseteq N_n$. It follows that $(x * a) * (y * b) \in N_n$. Therefore, $\mu((x * a) * (y * b)) \leq \alpha_n = \mu(x * y)$. Since $m < n$ implies $\alpha_n < \alpha_m$, we have $\mu(a * b) = \alpha_m < \alpha_n$. Consequently,

$\mu((x * a) *(y * b)) \leq \alpha_n = \max\{\mu(x * y), \mu(a * b)\}$. Similarly, for the case $x * y \in \tilde{N}_m$ and $a * b \in \tilde{N}_n$ for $0 \leq m < n \leq k$, proving the result.

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