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AN ANALYTICAL SOLUTION FOR LAMINAR FLOW THROUGH A TUBE WITH POROUS WALLS

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Abstract: *In this paper a laminar and linear flow through a leaky tube by solving the momentum and conservation of energy equations were analytically investigated. With using the Haigen-Poisseuile equations and defining an unknown function, equations of axial and radial velocity, pressure and mass transfer were obtained and their profiles were plotted according to different parameters. The results indicate that the value of coefficient of permeability has an important role on behavior of fluid flow in a tube with porous walls and with zero permeability coefficient the fluid flow has same behavior with well known equation of Haigen-Poisseuile flow.*

Keywords: *Mass Transfer, Leaky Tube, Axial Velocity, Radial Velocity, Leakage, Porous Media.*

NOMENCLATURE

V	Velocity (ms^{-1})
P	Pressure (pa)
p_0	Atmospheric pressure (pa)
U_0	Inlet velocity (ms^{-1})
R	Radius of tube (m)
r	Radius(m)
k	Coefficient of permeability ($m^2 s\ kg^{-1}$)
μ	Dynamic viscosity($kg\ m^{-1}\ s^{-1}$)
j	Leakage mass transfer ($m\ s^{-1}$)
l	Tube length (m)
β	Dimensionless parameter ($\beta = (\frac{k\mu}{R})^{1/2}$)
$f(z)$	An arbitrary intermediate function

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1. INTRODUCTION

Leaky tube is a pipe with leakages around its surface, therefore wall is permeable so that there is a uniform mass transfer from the wall [1] as shown schematically in Fig.1. Flow in leaky tube must supply both the forward flow and the leaks. Since some mass passes through the leaks then the axial mass transfer in direction of pipe length decreases and that is why the velocity and pressure vary in direction of pipe length.

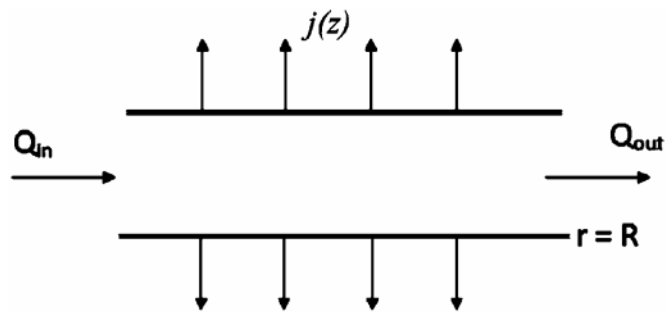

Figure.1: Schematic View of a Typical Laminar Flow through a Leaky Tube

This mechanism is used in nature and industry. For example in plants the transference of sap from land to its leaves and branches is done through a stack of pipes which are the same as leaky tubes[2]. In agriculture industry to spray poison and water the vegetables and in industry for making the expected fluid flow for lubricating and cooling the machines, leaky tubes are used in different forms and sizes [3,4]. So analysis of fluid flow and understanding the quality of velocity, pressure, and mass transfer variations through the leaky tubes is needed.

In this paper, presumptions of flow are as follow:

1. The flow is uniform, steady, fully developed, laminar and symmetric.
2. The fluid is Newtonian and incompressible.
3. The wall is permeable with uniform mass transfer.
4. Inertial and body forces are neglected.

2. PROBLEM FORMULATION

By considering the above presumptions, we have:

$$V = [V_r(r,z), 0, V_z(r,z)] \tag{1}$$

$$P = p(r,z) \tag{2}$$

The final goal of solving this problem is determining of equations of fluid penetration from a leaky tube. In this paper, equations of pressure, radial velocity, axial velocity, mass transit and quantity of leak were determined.

The Navier-Stokes equations in direction of tube axis (z) and also the continuity equation were written, it will be as follow [5, 6]:

$$-\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right] = 0 \tag{3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,v_r\right)+\frac{\partial v_z}{\partial z} = 0 \tag{4}$$

This point must be considered that, as the fluid moves along the tube, the pressure decreases and consequently the amount of leakage around the tube axis is reduces.

It is necessary to define the parameter of mass transfer which is dimensionally equal to the radial component of the velocity. Mass transfer was defined by the following equation [7]:

$$j(z) = k\left[p(z)-p_0\right] \tag{5}$$

Where $j(z)$ is the mass transfer, $p(z)$ tube pressure, p_0 the atmospheric pressure and k is the coefficient of permeability. Considering that the magnitude of leakage is around the radius of tube, velocity vector, in the tube wall will be:

$$j(z) = V_r\left(r,z\right)=V_r\left(z\right) \tag{6}$$

From Haigen-Poisseuile equation the values of V_z will be[8]:

$$V_z = 2U_0\left[1-\left(\frac{r}{R}\right)^2\right]f\left(z\right) \tag{7}$$

In this equation, U_0 is the average velocity in the tube entrance, and is equal to $U_0=\frac{Q_0}{\pi R^2}$, and $f(z)$ is an unknown arbitrary intermediate function that must be determined. Using continuity equation, the relation between V_r and V_z, from Eq. (1) can be determined as follow:

$$V_r = -\frac{1}{r}\int_0^r r\frac{\partial V_z}{\partial z}dr \tag{8}$$

3. PROBLEM SOLUTION

By differentiating Eq. (7) with respect to z, and substituting it in to Eq. (8) and integrating the obtained equation with respect to r, V_r will be:

$$V_r = -RU_0\,f'(z)\left[\left(\frac{r}{R}\right)^2-\frac{1}{2}\left(\frac{r}{R}\right)^4\right] \tag{9}$$

Where

$$f'(z) = \frac{\partial f}{\partial z}$$

On the other hand, in tube surface where $r = R$, V_r will be:

$$V_r\left(R,z\right) = V_R\left(z\right)=-\frac{RU_0}{2}f'\left(z\right) \tag{10}$$

Now, considering Eq. (3) Navier-Stocks equation in tube axial direction, the value of $\frac{\partial p}{\partial z}$ will be:

$$\frac{\partial p}{\partial z} = \mu\left\{-\frac{8U_0}{R^2}f\left(z\right)+2U_0\left[1-\left(\frac{r}{R}\right)^2\right]f''\left(z\right)\right\} \tag{11}$$

Considering Eqs. (3), (4) and (11) :

$$\frac{\partial p}{\partial z} = \frac{1}{k}\frac{dV_R}{dz} = -\frac{RU_0}{2k} f''(z) \tag{12}$$

Substituting Eq. (10) into Eq. (9), $f''(z)$ will be:

$$f''(z) = \frac{16K\mu}{R^3} f(z) - \frac{4K\mu}{R}\left[1-\left(\frac{r}{R}\right)^2\right] f''(z) \tag{13}$$

In Eq. (13), there is a term of *r*, which means that *f* is a function of *r* and *z* but we assumed that *f* is just a function of *z*. Now, since the differential equation is contradictory with the first assumption and also because the latter term in Eq. (13) is negligible in comparison to first one, this term can be omitted, therefore, the equation will be in the form of:

$$f''(z) - \frac{16K\mu}{R^3} f(z) = 0 \tag{14}$$

The above differential equation, is linear and simple to solve. We need two boundary conditions for solving this equation. Considering the assumptions that are applied in [5,6] that related to fully development of the flow, the boundary conditions will be:

at $z = 0 \Rightarrow f(z) = 0$

and

at $z \to \infty \Rightarrow f(z) = \text{const}.$

As a result we have:

$$f(z) = \exp\left(-4\beta\frac{z}{R}\right) \tag{15}$$

Where β is a dimensionless parameter, as follow :

$$\beta = \left(\frac{k\mu}{R}\right)^{\frac{1}{2}}$$

And the absolute pressure $p(z)$ from Eqs. (12) and (14) were determined:

$$\frac{dp}{dz} = -\frac{8\mu U_0}{R^2} f(z) = -\frac{8\mu U_0}{R^2}\exp\left(-4\beta\frac{z}{R}\right) \tag{16}$$

Knowing that at $z = 0$ the pressure is equal to the atmospheric one and by integration from the above equation, we will have:

$$p(z) = p_0 + \frac{2\mu U_0}{\beta R}\left[\exp\left(-4\beta\frac{z}{R}\right) - 1\right] \tag{17}$$

according to atmospheric pressure, pressure was expressed . If that is assumed, the relation about the atmospheric pressure will be:

$$p_0 = \Delta p = \frac{2\mu U_0}{\beta R}\left[1 - \exp\left(-4\beta\frac{L}{R}\right)\right] \tag{18}$$

Substituting Eq. (18) into Eq. (17), it can be obtained:

$$p(z) = \frac{2\mu U_0}{\beta R}\left[\exp\left(-4\beta\frac{z}{R}\right)-\exp\left(-4\beta\frac{L}{R}\right)\right] \tag{19}$$

In this condition mass transfer will be:

$$j(z) = k\,p(z)=2\beta U_0\left[\exp\left(-4\beta\frac{z}{R}\right)-\exp\left(-4\beta\frac{L}{R}\right)\right] \tag{20}$$

And the amount of leakage will be as follows:

$$Q_L = \int_0^L 2\pi R\, j(z)\,dz=\pi R^2 U_0\left[1-\left(1+4\beta\frac{L}{R}\right)\exp\left(-4\beta\frac{L}{R}\right)\right] \tag{21}$$

And

$$\frac{Q_L}{Q_{in}} = \left[1-\left(1+4\beta\frac{L}{R}\right)\exp\left(-4\beta\frac{L}{R}\right)\right] \tag{22}$$

And

$$\frac{Q_z}{Q_l} = 1-\frac{Q_l}{Q_z}=\exp\left(-4\beta\frac{Z}{R}\right)+\frac{4\beta Z}{R}\exp\left(-4\beta\frac{L}{R}\right) \tag{23}$$

The radial velocity is:

$$V_r = 4\beta U_0\exp\left(-4\beta\frac{Z}{R}\right)\left[\left(\frac{r}{R}\right)^2-\frac{1}{2}\left(\frac{r}{R}\right)^4\right] \tag{24}$$

And finally, the axial velocity will be:

$$V_z = 2U_0\exp\left(-4\beta\frac{Z}{R}\right)\left[1-\left(\frac{r}{R}\right)^2\right] \tag{25}$$

These equations can be examined by a simple test. If there is not any leakage around the tube the general equations of Haigen-Poisseuille must be arrived from equations of V_r and V_z.

This condition was investigated and results of this test will be as follow:

If there is not any leakage, the values of coefficient of permeability k and dimensionless parameter β will be zero and hence the value of radial velocity is zero too. In this case axial velocity will be as follows:

$$V_z = 2U_0\left(1-\left(\frac{r}{R}\right)^2\right) \tag{26}$$

It is very obvious that in the case of no leakage (k = 0) the profile of axial velocity and radial velocity will be the well known equation of Haigen-Poisseuille flow.

If the coefficient of permeability be infinite the radial and axial velocities will be as follows;

$V_r = \infty$ and $V_z = 0$

It means that, all of fluid goes out from the walls of tube and therefore the axial velocity become zero and the radial velocity become infinite.

4. RESULTS AND DISCUSSION

In following figures velocity of flow in direction r and z, mass transfer parameter and pressure profile have been shown.

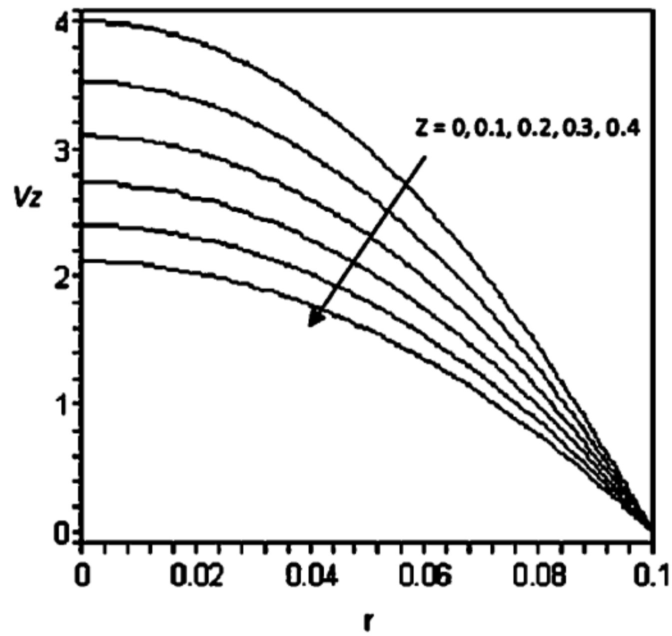

Figure.2: Axial Velocity Profile with Respect to r at Various z and $k = 0.1$

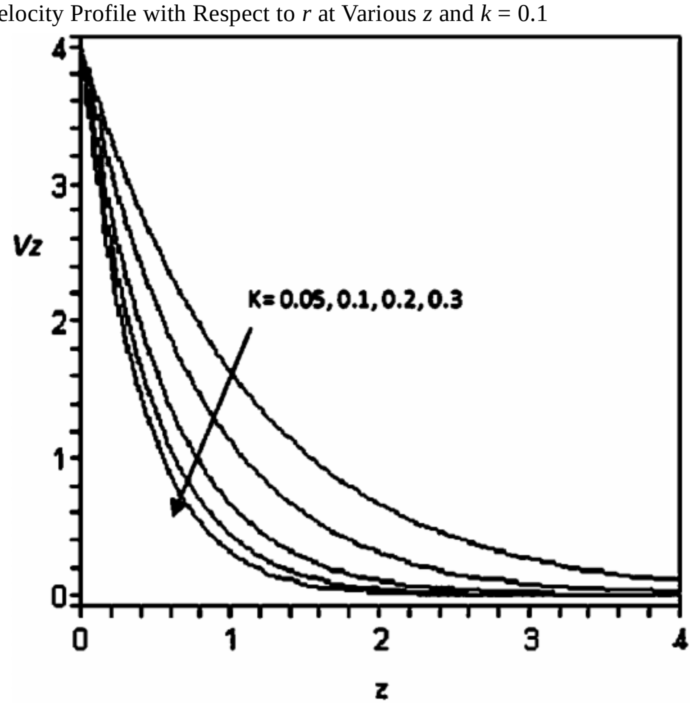

Figure.3: Axial Velocity Profile with Respect to z at Various k and $r = 0.1$

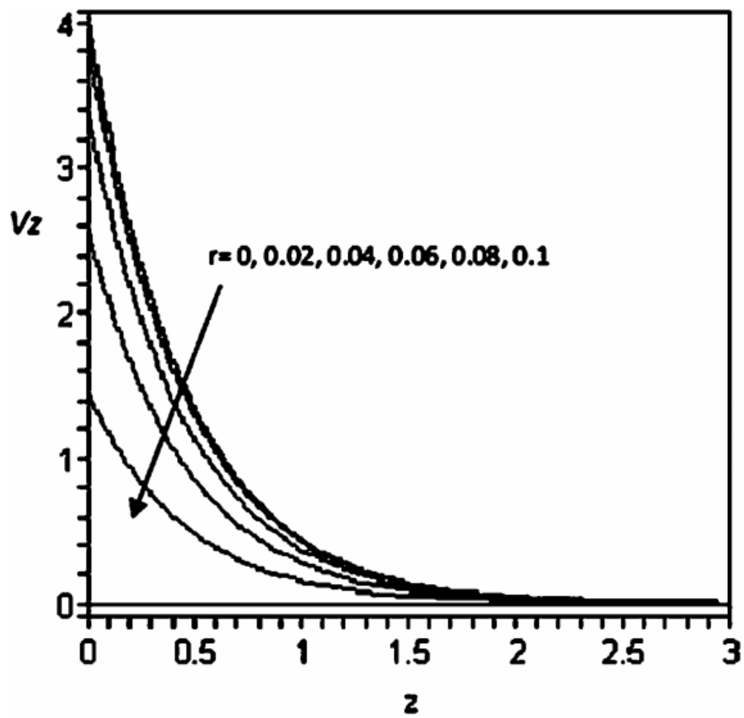

Figure.4: Axial Velocity Profile with Respect to *z* at Various *r* and *k* =0.2

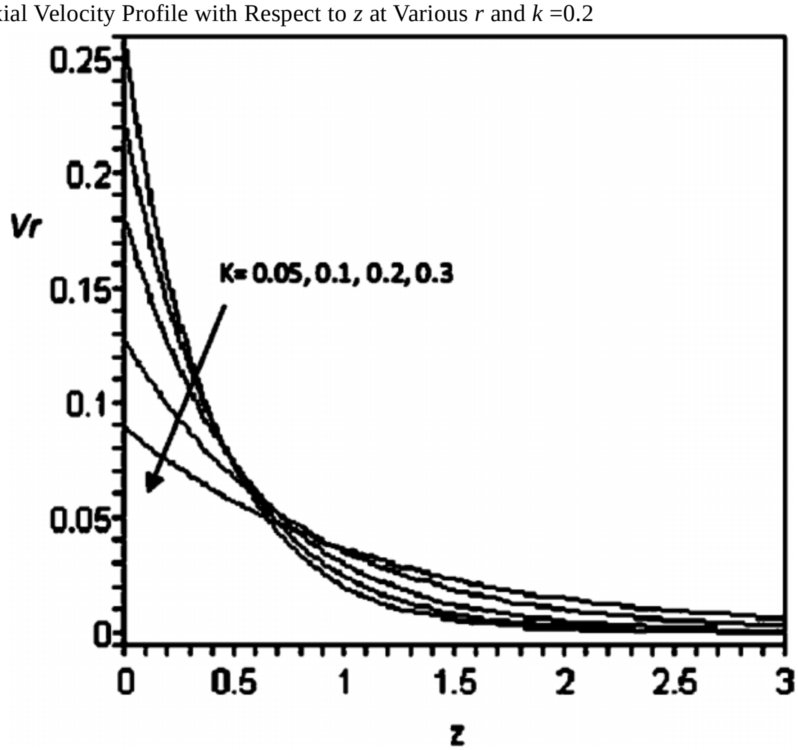

Figure.5: Radial Velocity Profile with Respect to *z* at Various *k* and *r* = 0.1

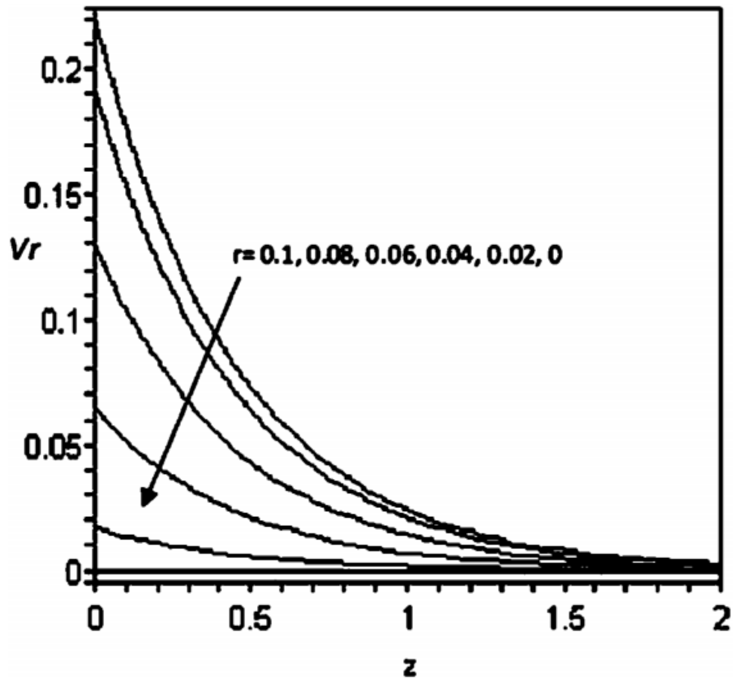

Figure.6: Radial Velocity Profile with Respect to *z* at Various *r* and *k* = 0.2

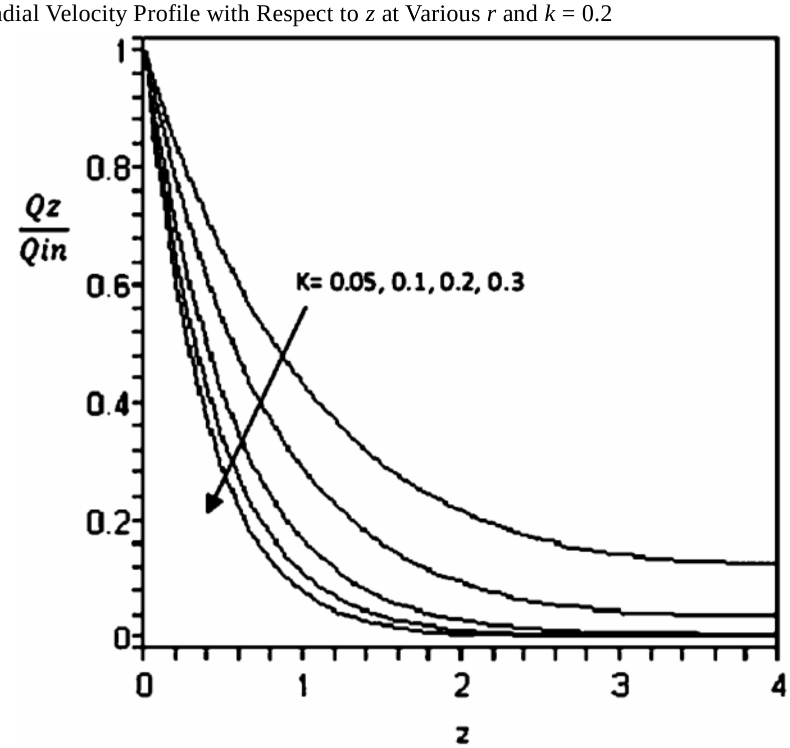

Figure.7: Ratio of Axial Mass Transfer to Entrance Mass Transfer Profile with Respect to *z* at Various *k*

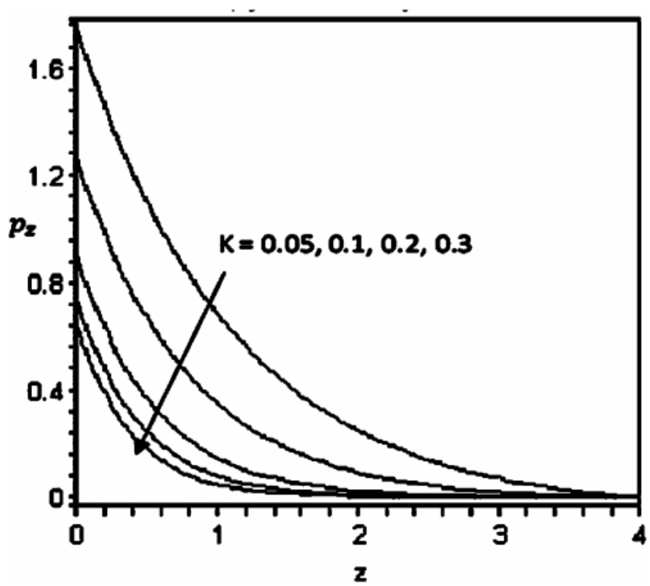

Figure.8: Pressure Profile with Respect to *z* at Various *k*.

By consideration of the figures (2-8) the following results can be identified:

1. Variation of axial velocity with respect to radius has same manner with axial velocity in a pipe without any leakage. The value of velocity is maximum on the pipe axis whereas it value is zero on the pipe wall (Fig.2).
2. When the coefficient of permeability increases the variations of axial velocity in direction of pipe length will increase and the axial velocity will proceed to zero (Fig.3).
3. The axial velocity will have the maximum value on the center of the tube and the minimum value on the wall of tube (Fig.4).
4. On the top of the leaky tube the radial velocity will have a higher value for different values of permeability coefficient but the radial velocity variations in direction of tube's length will have a higher value for small values of permeability coefficient. As result on one of the section of the leaky tube the radial velocity has less value for small values of permeability coefficient than higher values of that (Fig.5).
5. On the axis of leaky tube the value of the radial velocity is zero and when the pipe radius increases the value of radial velocity will increase too. So the radial velocity will have the maximum value on the wall of leaky tube (Fig.7).
6. When the coefficient of permeability increases the variation of relative mass transfer in direction of leaky tube length will rise (Fig.7).
7. The pressure decreases in direction of pipe length and when the permeability coefficient increases the pressure will reduce in each section (Fig.8).

5. CONCLUSIONS

In this paper the laminar flow through a tube with porous walls has been analytically investigated, in addition the velocity profiles in directions of *r* and *z* have been determined. The results indicate that when the permeability coefficient increases, in each section the values of axial velocity and axial mass transfer will decrease. The radial velocity is equal to zero on the axis of leaky tube and it has the maximum value on the wall of leaky tube and also on top of that. It decreases in direction of tube axis. The results also indicate that for zero permeability coefficients the variations of axial velocity and pressure are equal to well known haigen-poisseuile equations.

REFERENCES

[1] A. White, "Flow of Fluid in an Axially Pipe", *J.Mech. Engeg Sci*, **6**, 1964.

[2] R.E. Jahnson al, "Steady-State Coating Flows Inside a Rotating Horizontal Cylindrical Department of Theoretical and Applied Mechanics", University of Illinois, Urbana, IL 61801, USA.

[3] W. Fox Robert, A. Mc Donald, *Introduction to Fluid Mechanics*, **4**, 1934.

[4] H.M. Nagib, Z. LAvan, A. A. Fejer, "L. Wolf Stability of Pipe Flow with Superposed Solid Body Rotion", *Phys. Fluid*, **14,** 1971.

[5] Streeter, Victor Lyle. Wylie E, Benjamin. Bedford, Krith W, *Fluid Mechanics*, **9**, 1909.

[6] P. A Mackrodt, "Stability of Hagen-Poiseuille Flow with Superimposed Rigid Totatio", *J. Fluid Mech*, **73**, 1976.

[7] V. K. Shchukin, "Hydraulic Resistance of Rotating Tubes", *J. Engrg. Phys.*, **12**, 1967.

[8] D. g. Kroger, "Air-Cooled Heat Exchanger and Cooling Towers-Thermal Flow Performance Evaluation and Design", *Dept of Mechanical Engineering*, South Africa, 1998.

[9] K. Kikuyama, M. Murakami, K. Nishibori and K. Kaeda, "Flow in Axially Rotating Pipes", *J. Fluids Energy. JSME*, **26**, 1983.

[10] A.S. Berman, "Laminar Flow in Channel with Porous Walls", *J, Appl. Phys*, **24**, 1953, pp 1232-1235.

[11] D.D. Ganji, A. Sadighi, "Application of Homotopy-perturbation and Variational Iteration methods to Nonlinear Heat Transfer and Porous Media Equations", *Journal of Computational and Applied Mathematics*, **207**, 1, 2007, PP 24-34.

[12] A.A. Joneidi, D.D. Ganji, M. Babaelahi, "Micropolar Flow in a Porous Channel with High Mass Transfer", *International Communications in Heat and Mass Transfer*, **36(10)**, 2009, PP 1082-1088.

[13] J. Majdalani, C. Zhou, C.A. Dawson, "Two-dimensional Viscous Flow between Slowly Expanding or Contracting Walls with Weak Permeability", *J. Biomech*, **35(10)**, 2002, pp1399–1403.

[14] Y.Z. Boutros, M.B. Abdul-Malek, N.A. Badran, H.S. Hassan, "Group method Solution for Two-dimensional Viscous Flow between Slowly Expanding or Contracting Walls with Weak Permeability", *Applied Math Model*, **31 (6)**, 2007, pp1092–1108.

[15] Rong-Fang Lu, Jaw-Ren Lin, "A Theoretical Study of Combined Effects of Non Newtonian Rheology and Viscosity–pressure Dependence in the Sphere-plate Squeeze-film System", *Tribol. Int.*, **40,** 2007, pp 125–131.

[16] T. Hayat, R. Naz, S. Asghar, "Hall Effects on Unsteady Duct Flow of a Non-Newtonian Fluid in a Porous Medium", *Applied Mathematics and Computation*, **157**, 2004, pp103–114.

[17] K. Vafai, C.L. Tien, "Boundary and Inertial Effect on Flow and Heat Transfer in Porous Media", *Int. J. Heat Mass Transfer*, **24**, 1981, pp195–203.

[18] I. Pop, D.B. Ingham, "Convective Heat Transfer: Mathematical and Computational Modeling of Viscous Fluids and Porous Media", Pergamon, Oxford, 2001.

[19] S. Kim, K. Vafai, "Analysis of Natural Convection about a Vertical Plate Embedded in a Porous Medium", *Int. J. Heat Mass Transfer*, **32**, 1989, pp 665–677.

[20] T. Toya, Y. Kameshima, A. Yasumori, K. Okada, "Gas Permeability of the Porous Alumina Ceramics with Uni-directionally Aligned Pores by Extrusion Method", *Journal of European Ceramics Society*, **27(1)**, 2007.

[21] P.A. Donald, "On Exact Solution of Flow Problems of a Second Grade Fluid through Two Parallel Porous Walls", *International Journal of Engineering Science*, **40**, 2002, pp 913-941.